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Tripled and coincidence fixed point theorems for contractive mappings satisfying Φ-maps in partially ordered metric spaces

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Abstract

In this paper, we study some tripled fixed and coincidence point theorems for two mappings $F\colon X \times X \times X \to X$ and $g\colon X \to X$ satisfying a nonlinear contraction based on ϕ-maps. Our results extend and improve many existing results in the literature. Also, we introduce an example to support the validity of our results.

1 Introduction and Preliminaries

The notion of a coupled fixed point for a mapping $F\colon X \times X \to X$ was initiated by Gnana Bhaskar and Lakshmikantham [1] and they proved some interesting coupled fixed point theorems, while Ćirić and Lakshmikantham [2] introduced the concept of the coupled fixed point for two mappings $F\colon X \times X \to X$ and $g\colon X \to X$ and established many existing theorems. Vasile Berinde and Marin Borcut [3, 4] initiated the concept of a tripled fixed point of the mapping $F\colon X \times X \times X \to X$ and the concept of a tripled coincidence point of the two mappings $F\colon X \times X \times X \to X$ and $g\colon X \to X$ and extended the results of Gnana Bhaskar and Lakshmikantham [1] and Ćirić and Lakshmikantham [2] to the interesting tripled fixed and coincidence point theorems. For some

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coupled fixed point results in different metric spaces we refer the reader to [5]-[25].

Consistent with Berinde and Borcut [3, 4], we give the following definitions and preliminaries.

Definition 1.1 ([3]). Let X be a nonempty set and $F\colon X \times X \times X \to X$ be a given mapping. An element $(x, y, z) \in X \times X \times X$ is called a *tripled fixed point* of F if

$$F(x, y, z) = x,\ F(y, x, y) = y \quad \text{and} \quad F(z, y, x) = z.$$

Let (X, d) be a metric space. The mapping

$$\overline{d}\colon X \times X \times X \to \mathbb{R}, \quad \overline{d}((x, y, z), (u, v, w)) = d(x, u) + d(y, v) + d(z, w),$$

defines a metric on $X \times X \times X$, which will be denoted for convenience by d.

Definition 1.2 ([3]). Let $(X, \leq)$ be a partially ordered set and $F\colon X \times X \times X \to X$ be a mapping. We say that F has the *mixed monotone property* if $F(x, y, z)$ is monotone non-decreasing in x and z and is monotone non-increasing in y; that is, for any $x, y, z \in X$,

$$x_1, x_2 \in X, \quad x_1 \leq x_2 \quad \text{implies} \quad F(x_1, y, z) \leq F(x_2, y, z),$$
$$y_1, y_2 \in X, \quad y_1 \leq y_2 \quad \text{implies} \quad F(x, y_1, z) \geq F(x, y_2, z),$$

and

$$z_1, z_2 \in X, \quad z_1 \leq z_2 \quad \text{implies} \quad F(x, y, z_1) \leq F(x, y, z_2).$$

The main results of [3] are:

Theorem 1.1 ([3]). *Let $(X, \leq)$ be a partially ordered set and (X, d) be a complete metric space. Let $F\colon X \times X \times X \to X$ be a continuous mapping such that F has the mixed monotone property. Assume that there exist $j, k, l \in [0, 1)$ with $j + k + l < 1$ such that*

$$d(F(x, y, z), F(u, v, w)) \leq jd(x, u) + kd(y, v) + ld(z, w) \tag{1}$$

for all $x, y, z, u, v, w \in X$ with $x \geq u$, $y \leq v$, and $z \geq w$. If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \leq F(x_0, y_0, z_0)$, $y_0 \geq F(y_0, x_0, y_0)$, and $z_0 \leq F(z_0, y_0, x_0)$, then F has a tripled fixed point.

Theorem 1.2 ([3]). *Let $(X, \leq)$ be a partially ordered set and (X, d) be a complete metric space. Let $F\colon X \times X \times X \to X$ be a mapping having the mixed monotone property. Assume that there exist $j, k, l \in [0, 1)$ with $j + k + l < 1$ such that*

$$d(F(x, y, z), F(u, v, w)) \leq jd(x, u) + kd(y, v) + ld(z, w)$$

for all $x, y, z, u, v, w \in X$ with $x \geq u$, $y \leq v$, and $z \geq w$. Assume that X has the following properties:

i. if a non-decreasing sequence $x_n \to x$, then $x_n \leq x$ for all $n \in \mathbb{N}$,

ii. if a non-increasing sequence $y_n \to y$, then $y_n \geq y$ for all $n \in \mathbb{N}$.

If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \leq F(x_0, y_0, z_0)$, $y_0 \geq F(y_0, x_0, y_0)$, and $z_0 \leq F(z_0, y_0, x_0)$, then F has a tripled fixed point.

Also, Borcut and Berinde [4] introduced the concept of a tripled coincidence point of mappings $F \colon X \times X \times X \to X$ and $g \colon X \to X$.

Definition 1.3 ([4])**.** Let X be a nonempty set. Let $F \colon X \times X \times X \to X$ and $g \colon X \to X$ be two mappings. An element $(x, y, z) \in X \times X \times X$ is called a *tripled coincidence point* of F and g if

$$F(x, y, z) = gx, \ F(y, x, y) = gy \quad \text{and} \quad F(z, y, x) = gz.$$

Definition 1.4 ([4])**.** Let $(X, \leq)$ be a partially ordered set and $F \colon X \times X \times X \to X$ be a mapping. We say that F has the *mixed g-monotone property* if $F(x, y, z)$ is monotone non-decreasing in x and z and is monotone non-increasing in y; that is, for any $x, y, z \in X$,

$$x_1, x_2 \in X, \quad gx_1 \leq gx_2 \quad \text{implies} \quad F(x_1, y, z) \leq F(x_2, y, z),$$
$$y_1, y_2 \in X, \quad gy_1 \leq gy_2 \quad \text{implies} \quad F(x, y_1, z) \geq F(x, y_2, z),$$

and

$$z_1, z_2 \in X, \quad gz_1 \leq gz_2 \quad \text{implies} \quad F(x, y, z_1) \leq F(x, y, z_2).$$

The main results of [4] is:

Theorem 1.3 ([4])**.** *Let $(X, \leq)$ be a partially ordered set and (X, d) be a complete metric space. Let $F \colon X \times X \times X \to X$ and $g \colon X \to X$ such that F has the mixed g-monotone property. Assume that there exist $j, k, l \in [0, 1)$ with $j + k + l < 1$ such that*

$$d(F(x, y, z), F(u, v, w)) \leq jd(gx, gu) + kd(gy, gv) + ld(gz, gw) \tag{2}$$

for all $x, y, z, u, v, w \in X$ with $gx \geq gu$, $gy \leq gv$, and $gz \geq gw$. Syppose that $F(X \times X \times X \subseteq gX$, g is continuous and commute with F and also suppose either

(a) F is continuous or

(b) X has the following property:

(i) if a non-decreasing sequence $x_n \to x$, then $x_n \leq x$ for all n,

(ii) if a non-increasing sequence $y_n \to y$, then $y_n \geq y$ for all n.

If there exist $x_0, y_0, z_0 \in X$ such that $gx_0 \leq F(x_0, y_0, z_0)$, $gy_0 \geq F(y_0, x_0, y_0)$, and $gz_0 \leq F(z_0, y_0, x_0)$, then F and g a tripled coincidence point.

By following Matkowski [26], we let Φ be the set of all non-decreasing functions $\phi\colon [0, +\infty) \to [0, +\infty)$ such that $\lim_{n\to+\infty} \phi^n(t) = 0$, for all $t > 0$, where ϕ^n denotes the n-th iterate of ϕ. Then its an easy matter to show that:

(1) $\phi(t) < t$ for all $t > 0$,

(2) $\phi(0) = 0$.

If $\phi \in \Phi$, then we call ϕ to be a ϕ-map.

In this paper, we utilize the notion of ϕ-map to prove a number of tripled fixed and coincidence point results for two mapping $F\colon X \times X \times X \to X$ and $g\colon X \to X$ in ordered metric spaces. Our results generalize Theorems 1.1 and 1.3. Also, we support our results by introducing a nontrivial example satisfying our main results and doesn't satisfy the conditions 1 and 2 of Theorems 1.1 and 1.3 respectively.

2 Main Results

Our first result is the following.

Theorem 2.1. *Let $(X, \leq)$ be a partially ordered set and (X, d) be a metric space. Let $F\colon X \times X \times X \to X$, $g\colon X \to X$ be two mappings. Suppose the following*

1. *gX is a complete subspace of X,*
2. *$F(X \times X \times X) \subseteq gX$,*
3. *F has the mixed g-monotone property,*
4. *F is continuous and*
5. *g is continuous and commute with F.*

Assume that there exists $\phi \in \Phi$ such that

$$d(F(x,y,z), F(u,v,w)) \leq \phi\Big(\max\Big\{d(gx,gu), d(gy,gv), d(gz,gw), d(F(x,y,z),gx), d(F(z,y,x),gz), d(F(u,v,w),gu), d(F(w,v,u),gw)\Big\}\Big), \tag{3}$$

for all $x, y, z, u, v, w \in X$ with $x \geq u$, $y \leq v$, and $z \geq w$. If there exist $x_0, y_0, z_0 \in X$ such that $gx_0 \leq F(x_0, y_0, z_0)$, $gy_0 \geq F(y_0, x_0, y_0)$, and $gz_0 \leq F(z_0, y_0, x_0)$, then F and g have a tripled coincidence point.

Proof. Suppose there exist $x_0, y_0, z_0 \in X$ such that $gx_0 \leq F(x_0, y_0, z_0)$, $gy_0 \geq F(y_0, x_0, y_0)$, and $gz_0 \leq F(z_0, y_0, x_0)$. Define $gx_1 = F(x_0, y_0, z_0)$, $gy_1 = F(y_0, x_0, y_0)$ and $gz_1 = F(z_0, y_0, x_0)$. Then $gx_0 \leq gx_1$, $gy_0 \geq gy_1$ and $gz_0 \leq gz_1$. Again, define $gx_2 = F(x_1, y_1, z_1)$, $gy_2 = F(y_1, x_1, y_1)$ and $gz_2 = F(z_1, y_1, x_1)$. Since F has the mixed g-monotone property, we have $gx_0 \leq gx_1 \leq gx_2$, $gy_2 \leq gy_1 \leq gy_0$ and $gz_0 \leq gz_1 \leq gz_2$. Continuing this process, we can construct three sequences (gx_n), (gy_n) and (gz_n) in X such that

$$gx_n = F(x_{n-1}, y_{n-1}, z_{n-1}) \leq gx_{n+1} = F(x_n, y_n, z_n),$$

$$gy_{n+1} = F(y_n, x_n, y_n) \leq gy_n = F(y_{n-1}, x_{n-1}, y_{n-1}),$$

and

$$gz_n = F(z_{n-1}, y_{n-1}, x_{n-1}) \leq gz_{n+1} = F(z_n, y_n, x_n).$$

If, for some integer n, we have $(gx_{n+1}, gy_{n+1}, gz_{n+1}) = (gx_n, gy_n, gz_n)$, then $F(x_n, y_n, z_n) = gx_n$, $F(y_n, x_n, y_n) = gy_n$, and $F(z_n, y_n, x_n) = gz_n$; that is, (x_n, y_n, z_n) is a tripled coincidence point of F. Thus we shall assume that $(gx_{n+1}, gy_{n+1}, gz_{n+1}) \neq (gx_n, gy_n, gz_n)$ for all $n \in \mathbb{N}$; that is, we assume that either $gx_{n+1} \neq gx_n$ or $gy_{n+1} \neq gy_n$ or $gz_{n+1} \neq gz_n$. For any $n \in \mathbb{N}$, we have

$$\begin{aligned}
& d(gx_{n+1}, gx_n) \\
= \;& d(F(x_n, y_n, z_n), F(x_{n-1}, y_{n-1}, z_{n-1}) \\
\leq \;& \phi\Big(\max\Big\{ d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1}), d(gz_n, gz_{n-1}), \\
& d(F(x_n, y_n, z_n), gx_n), d(F(z_n, y_n, x_n), gz_n), \\
& d(F(x_{n-1}, y_{n-1}, z_{n-1}), gx_{n-1})), d(F(z_{n-1}, y_{n-1}, x_{n-1}), gz_{n-1}))\Big\}\Big) \\
= \;& \phi\Big(\max\Big\{ d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1}), d(gz_n, gz_{n-1}), \\
& d(gx_{n+1}, gx_n), d(gz_{n+1}, gz_n)\Big\}\Big) \\
\leq \;& \phi\Big(\max\Big\{ d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1}), d(gz_n, gz_{n-1}), \\
& d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n), d(gz_{n+1}, gz_n)\Big\}\Big),
\end{aligned} \tag{4}$$

$$\begin{aligned}
& d(gy_n, gy_{n+1}) \\
= \;& d(F(y_{n-1}, x_{n-1}, y_{n-1}), F(y_n, x_n, y_n) \\
\leq \;& \phi\Big(\max\Big\{ d(gy_{n-1}, gy_n), d(gx_{n-1}, gx_n), d(F(y_{n-1}, x_{n-1}, y_{n-1}), gy_{n-1}), \\
& d(F(y_n, x_n, y_n), gy_n) \Big\}\Big) \\
= \;& \phi(\max\{d(gy_{n-1}, gy_n), d(gx_{n-1}, gx_n), d(y_{n+1}, y_n)\}) \\
\leq \;& \phi\Big(\max\Big\{ d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1}), d(gz_n, gz_{n-1}), d(gx_{n+1}, gx_n), \\
& d(gy_{n+1}, gy_n), d(gz_{n+1}, gz_n) \Big\}\Big), \qquad (5)
\end{aligned}$$

and

$$\begin{aligned}
& d(gz_{n+1}, gz_n) \\
= \;& d(F(z_n, y_n, x_n), F(z_{n-1}, y_{n-1}, x_{n-1}) \\
\leq \;& \phi\Big(\max\Big\{ d(gz_n, gz_{n-1}), d(gy_n, gy_{n-1}), d(gx_n, gx_{n-1}), \\
& d(F(z_n, y_n, x_n), gz_n), d(F(x_n, y_n, z_n), gx_n), \\
& d(F(z_{n-1}, y_{n-1}, x_{n-1}), gz_{n-1})), d(F(x_{n-1}, y_{n-1}, z_{n-1}), gx_{n-1})) \Big\}\Big) \\
= \;& \phi\Big(\max\Big\{ d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1}), d(gz_n, gz_{n-1}), \\
& d(gx_{n+1}, gx_n), d(gz_{n+1}, gz_n) \Big\}\Big) \\
\leq \;& \phi\Big(\max\Big\{ d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1}), d(gz_n, gz_{n-1}), \\
& d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n), d(gz_{n+1}, gz_n) \Big\}\Big), \qquad (6)
\end{aligned}$$

From (4), (5), and (6), it follows that

$$\begin{aligned}
& \max\{d(gx_{n+1}, gx_n), d(gy_n, gy_{n+1}), d(gz_{n+1}, gz_n)\} \\
\leq \;& \phi\Big(\max\Big\{ d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1}), d(gz_n, gz_{n-1}), \\
& d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n), d(gz_{n+1}, gz_n) \Big\}\Big). \qquad (7)
\end{aligned}$$

If

$$\begin{gathered}
\max\big\{ d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1}), d(gz_n, gz_{n-1}), \\
d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n), d(gz_{n+1}, gz_n) \big\} \\
= \max\big\{ d(gx_{n+1}, gx_n), d(gy_n, gy_{n+1}), d(gz_{n+1}, gz_n) \big\},
\end{gathered}$$

then from (7) we have

$$
\begin{aligned}
& \max\{d(gx_{n+1}, gx_n), d(gy_n, gy_{n+1}), d(gz_{n+1}, gz_n)\} \\
\leq \ & \phi\big(\max\big\{d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1}), d(gz_n, gz_{n-1})\big\}\big) \\
< \ & \max\{d(gx_{n+1}, gx_n), d(gy_n, gy_{n+1}), d(gz_{n+1}, gz_n)\},
\end{aligned}
$$

a contradiction. Thus (7) becomes

$$
\begin{aligned}
& \max\{d(gx_{n+1}, gx_n), d(gy_n, gy_{n+1}), d(gz_{n+1}, gz_n)\} \\
\leq \ & \phi\big(\max\big\{d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1}), d(gz_n, gz_{n-1})\big\}\big).
\end{aligned} \tag{8}
$$

By repeating (8) n-times, we get that

$$
\begin{aligned}
& \max\{d(gx_{n+1}, gx_n), d(gy_n, gy_{n+1}), d(gz_{n+1}, gz_n)\} \\
\leq \ & \phi(\max\{d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1}), d(gz_n, gz_{n-1})\}) \\
\leq \ & \phi^2(\max\{d(gx_{n-1}, gx_{n-2}), d(gy_{n-1}, gy_{n-2}), d(gz_{n-1}, gz_{n-2})\}) \\
\vdots \ & \\
\leq \ & \phi^n(\max\{d(gx_1, gx_0), d(gy_1, gy_0), d(gz_1, gz_0)\}).
\end{aligned} \tag{9}
$$

Now, we shall show that (gx_n), (gy_n), and (gz_n) are Cauchy sequence in X. Let $\epsilon > 0$. Since

$$\lim_{n\to+\infty} \phi^n(\max\{d(gx_1, gx_0), d(gy_1, gy_0), d(gz_1, gz_0)\}) = 0$$

and $\epsilon > \phi(\epsilon)$, there exist $n_0 \in \mathbb{N}$ such that

$$\phi^n(\max\{d(gx_1, gx_0), d(gy_1, gy_0), d(gz_1, gz_0)\}) < \epsilon - \phi(\epsilon),$$

for all $n \geq n_0$.

This implies that,

$$\max\{d(gx_{n+1}, gx_n), d(gy_n, gy_{n+1}), d(gz_{n+1}, gz_n)\} < \epsilon - \phi(\epsilon), \tag{10}$$

for all $n \geq n_0$.

For $m, n \in \mathbb{N}$, we will prove by induction on m that

$$\max\{d(gx_n, gx_m), d(gy_n, gy_m), d(gz_n, gz_m)\} < \epsilon, \tag{11}$$

for all $m \geq n \geq n_0$.

For $m = n + 1$, (11) is true by using (10) and noting that $\epsilon - \phi(\epsilon) < \epsilon$. Now suppose that (11) holds for $m = k$; that is,

$$\max\{d(gx_n, gx_k), d(gy_n, gy_k), d(gz_n, gz_k)\} < \epsilon. \tag{12}$$

For $m = k+1$, we have

$$\begin{aligned}
& d(gx_n, gx_{k+1}) \\
\leq\ & d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{k+1}) \\
\leq\ & \epsilon - \phi(\epsilon) + d(F(x_k, y_k, z_k), F(x_n, y_n, z_n)) \\
\leq\ & \epsilon - \phi(\epsilon) + \phi\Big(\max\Big\{ d(gx_k, gx_n), d(gy_k, gy_n), d(gz_k, gz_n), \\
& d(F(x_k, y_k, z_k), gx_k), d(F(z_k, y_k, x_k), gz_k), \\
& d(F(x_n, y_n, z_n), gx_n), d(F(z_n, y_n, x_n), gz_n) \Big\} \Big) \\
=\ & \epsilon - \phi(\epsilon) + \phi\Big(\max\Big\{ d(gx_n, gx_k), d(gy_n, gy_k), d(gz_n, gz_k), d(gx_{n+1}, gx_n), \\
& d(gz_{n+1}, gz_n), d(gx_{k+1}, gx_k), d(gz_{k+1}, gz_k) \Big\} \Big) \\
\leq\ & \epsilon - \phi(\epsilon) + \phi\Big(\max\Big\{ d(gx_n, gx_k), d(gy_n, gy_k), d(gz_n, gz_k), d(gx_{n+1}, gx_n), \\
& d(gy_{n+1}, gy_n), d(gz_{n+1}, gz_n), d(gx_{k+1}, gx_k), \\
& d(gy_{k+1}, gy_k), d(gz_{k+1}, gz_k) \Big\} \Big). \qquad (13)
\end{aligned}$$

Using (10) and (12), we have

$$\begin{aligned}
& \max\Big\{ d(gx_n, gx_k), d(gy_n, gy_k), d(gz_n, gz_k), d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n), \\
& d(gz_{n+1}, gz_n), d(gx_{k+1}, gx_k), d(gy_{k+1}, gy_k), d(gz_{k+1}, gz_k) \Big\} \\
& \leq \max\{\epsilon, \epsilon - \phi(\epsilon)\} = \epsilon. \qquad (14)
\end{aligned}$$

From (13), (14) and the properties of ϕ, we obtain

$$d(gx_n, gx_{n+1}) < \epsilon - \phi(\epsilon) + \phi(\epsilon) = \epsilon.$$

Similarly, we show that

$$d(gy_n, gy_{k+1}) < \epsilon,$$

and

$$d(gz_n, gz_{k+1}) < \epsilon.$$

Hence, we have

$$\max\{d(gx_n, gx_{k+1}), d(gy_n, gy_{k+1}), d(gz_n, gz_{k+1})\} < \epsilon.$$

Thus (11) holds for all $m \geq n \geq n_0$. Hence (gx_n), (gy_n) and (gz_n) are Cauchy sequences in gX.

Since gX is complete, there exist $p, q, r \in gX$ such that (gx_n), (gy_n) and (gz_n) converge to p, q, and r respectively. Choose $x, y, z \in X$ such that $p = gx, q = gy$ and $r = gz$. Now, we show that (p, q, r) is a coincidence point of F. Since F, g are commute and F is continuous, we have

$$ggx_{n+1} = g(F(x_n, y_n, z_n)) = F(gx_n, gy_n, gz_n) \to F(p, q, r).$$

Also, since g is continuous and $gx_n \to p$, we have $ggx_n \to gp$. By uniqueness of limit, we get $F(p, q, r) = gp$. Similarly, we show that $gq = F(q, p, q)$ and $gr = F(r, q, p)$. So (p, q, r) is a tripled coincidence point of F and g. □

By taking $\phi(t) = kt$, $k \in [0, 1)$ in Theorem 2.1, we have the following:

Corollary 2.1. *Let $(X, \leq)$ be a partially ordered set and (X, d) be a metric space. Let $F\colon X \times X \times X \to X$, $g\colon X \to X$ be two mappings. Suppose the following:*

1. *gX is a complete subspace of X,*
2. *$F(X \times X \times X) \subseteq gX$,*
3. *F has the mixed g-monotone property,*
4. *F is continuous and*
5. *g is continuous and commute with F.*

Assume that there exists $k \in [0, 1)$ such that

$$\begin{gathered} d(F(x,y,z), F(u,v,w)) \leq k \max\big\{d(gx,gu), d(gy,gv), d(gz,gw), \\ d(F(x,y,z),gx), d(F(z,y,x),gz), d(F(u,v,w),gu), d(F(w,v,u),gw)\big\}, \end{gathered}$$

for all $x, y, z, u, v, w \in X$ with $x \geq u$, $y \leq v$, and $z \geq w$. If there exist $x_0, y_0, z_0 \in X$ such that $gx_0 \leq F(x_0, y_0, z_0)$, $gy_0 \geq F(y_0, x_0, y_0)$, and $gz_0 \leq F(z_0, y_0, x_0)$, then F and g have a tripled coincidence point.

As a consequence of Corollary 2.1, we have the following:

Corollary 2.2. *Let $(X, \leq)$ be a partially ordered set and (X, d) be a metric space. Let $F\colon X \times X \times X \to X$, $g\colon X \to X$ be two mappings. Suppose the following*

1. *gX is a complete subspace of X,*
2. *$F(X \times X \times X) \subseteq gX$,*
3. *F has the mixed g-monotone property,*
4. *F is continuous and*
5. *g is continuous and commute with F.*

Assume that there exist $a_1, a_2, a_3, a_4, a_5, a_6, a_7 \in [0, 1)$ with $\sum_{i=1}^{7} a_i < 1$ such that

$$\begin{gathered} d(F(x,y,z), F(u,v,w)) \leq a_1 d(gx,gu) + a_2 d(gy,gv) + a_3 d(gz,gw) \\ +a_4 d(F(x,y,z),gx) + a_5 d(F(z,y,x),gz) \\ +a_6 d(F(u,v,w),gu) + a_7 d(F(w,v,u),gw), \end{gathered}$$

for all $x, y, z, u, v, w \in X$ *with* $x \geq u$, $y \leq v$, *and* $z \geq w$. *If there exist* $x_0, y_0, z_0 \in X$ *such that* $gx_0 \leq F(x_0, y_0, z_0)$, $gy_0 \geq F(y_0, x_0, y_0)$, *and* $gz_0 \leq F(z_0, y_0, x_0)$, *then* F *and* g *have a tripled coincidence point.*

By taking $g = i_X$ (the identity mapping on X) in Theorem 2.1, Corollary 2.1 and Corollary 2.2, we have the following:

Corollary 2.3. *Let* $(X, \leq)$ *be a partially ordered set and* (X, d) *be a complete metric space. Let* $F\colon X \times X \times X \to X$ *be a continuous mapping having the mixed monotone property. Assume that there exists* $\phi \in \Phi$ *such that*

$$d(F(x,y,z), F(u,v,w)) \leq \phi\big(\max\big\{d(x,u), d(y,v), d(z,w), d(F(x,y,z),x), \\ d(F(z,y,x),z), d(F(u,v,w),u), d(F(w,v,u),w)\big\}\big),$$

for all $x, y, z, u, v, w \in X$ *with* $x \geq u$, $y \leq v$, *and* $z \leq w$. *If there exist* $x_0, y_0, z_0 \in X$ *such that* $x_0 \leq F(x_0, y_0, z_0)$, $y_0 \geq F(y_0, x_0, y_0)$, *and* $z_0 \leq F(z_0, y_0, x_0)$, *then* F *has a tripled fixed point.*

Corollary 2.4. *Let* $(X, \leq)$ *be a partially ordered set and* (X, d) *be a complete metric space. Let* $F\colon X \times X \times X \to X$ *be a continuous mapping having the mixed monotone property. Assume that there exists* $k \in [0, 1)$ *such that*

$$d(F(x,y,z), F(u,v,w)) \leq k \max\big\{d(x,u), d(y,v), d(z,w), d(F(x,y,z),x), \\ d(F(z,y,x),z), d(F(u,v,w),u), d(F(w,v,u),w)\big\},$$

for all $x, y, z, u, v, w \in X$ *with* $x \geq u$, $y \leq v$, *and* $z \leq w$. *If there exist* $x_0, y_0, z_0 \in X$ *such that* $x_0 \leq F(x_0, y_0, z_0)$, $y_0 \geq F(y_0, x_0, y_0)$, *and* $z_0 \leq F(z_0, y_0, x_0)$, *then* F *has a tripled fixed point.*

Corollary 2.5. *Let* $(X, \leq)$ *be a partiallyy ordered set and* (X, d) *be a complete metric space. Let* $F\colon X \times X \times X \to X$ *be a continuous mapping having the mixed monotone property. Assume that there exists* $a_1, a_2, a_3, a_4, a_5, a_6, a_7$ *in* $[0, 1)$ *with* $\sum_{i=1}^{7} a_i < 1$ *such that*

$$d(F(x,y,z), F(u,v,w)) \leq a_1 d(x,u) + a_2 d(y,v) + a_3 d(z,w) + \\ a_4 d(F(x,y,z),x) + a_5 d(F(z,y,x),z) + a_6 d(F(u,v,w),u) + a_7 d(F(w,v,u),w),$$

for all $x, y, z, u, v, w \in X$ *with* $x \geq u$, $y \leq v$, *and* $z \leq w$. *If there exist* $x_0, y_0, z_0 \in X$ *such that* $x_0 \leq F(x_0, y_0, z_0)$, $y_0 \geq F(y_0, x_0, y_0)$, *and* $z_0 \leq F(z_0, y_0, x_0)$, *then* F *has a tripled fixed point.*

By adding an additional hypotheses, the continuity of F and g in Theorem 2.1 can be dropped.

Theorem 2.2. *Let $(X, \leq)$ be a partially ordered set and (X, d) be a complete metric space. Let $F\colon X \times X \times X \to X$ and $g\colon X \to X$ be two mappings. Suppose that there exists $\phi \in \Phi$ such that*

$$\begin{aligned} d(F(x,y,z),F(u,v,w)) \leq \phi\big(\max\{d(gx,gu), d(gy,gv), d(gz,gw), \\ d(F(x,y,z),gx), d(F(z,y,x),gz), d(F(u,v,w),gu), d(F(w,v,u),gw)\}\big), \end{aligned} \tag{15}$$

for all $x, y, z, u, v, w \in X$ with $gx \geq gu$, $gy \leq gv$, and $gz \leq gw$. Suppose the following:

1. *gX is a complete subspace of X,*
2. *$F(X \times X \times X) \subseteq gX$,*
3. *F has the mixed g-monotone property,*
4. *F and g are commute.*

Also, assume that X has the following properties:

i. if a non-decreasing sequence $x_n \to x$, then $x_n \leq x$ for all $n \in \mathbb{N}$,

ii. if a non-increasing sequence $y_n \to y$, then $y_n \geq y$ for all $n \in \mathbb{N}$.

If there exist $x_0, y_0, z_0 \in X$ such that $gx_0 \leq F(x_0, y_0, z_0)$, $gy_0 \geq F(y_0, x_0, y_0)$, and $gz_0 \leq F(z_0, y_0, x_0)$, then F and g have a tripled coincidence point.

Proof. By following the same process in Theorem 2.1, we construct three Cauchy sequences (gx_n), (gy_n) and (gz_n) in gX with

$$gx_1 \leq gx_2 \leq \ldots \leq gx_n \leq \ldots,$$

$$gy_1 \geq gy_2 \geq \ldots \geq gy_n \geq \ldots,$$

and

$$gz_1 \leq gz_2 \leq \ldots \leq gz_n \leq \ldots$$

such that $gx_n \to p = gx \in gX$, $gy_n \to q = gy \in gX$, and $gz_n \to r = gz \in gX$, where $x, y, z \in X$. By the hypotheses on X, we have $gx_n \leq gx$, $gy_n \geq gy$ and $gz_n \leq gz$ for all $n \in \mathbb{N}$. From (15), we have

$$\begin{aligned} & d(F(x,y,z), x_{n+1}) \\ = \;& d(F(x,y,z), F(x_n, y_n, z_n)) \\ \leq \;& \phi\Big(\max\Big\{ d(x,x_n), d(y,y_n), d(z,z_n), d(F(x,y,z),gx), d(F(z,y,x),gz), \\ & d(F(x_n,y_n,z_n),gx_n), d(F(z_n,y_n,x_n),gz_n)\Big\}\Big), \end{aligned}$$

$$
\begin{aligned}
&= \phi\Big(\max\Big\{d(gx,gx_n),d(gy,gy_n),d(gz,gz_n),d(F(x,y,z),gx),\\
&\quad d(F(z,y,x),gz),d(gx_{n+1},gx_n),d(gz_{n+1},gz_n)\Big\}\Big),\\
&\le \phi\Big(\max\Big\{d(gx,gx_n),d(gy,gy_n),d(gz,gz_n),d(F(x,y,z),gx),\\
&\quad d(F(z,y,x),gz),d(F(y,x,y),gy),d(gx_{n+1},gx_n),\\
&\quad d(gy_{n+1},gy_n),d(gz_{n+1},gz_n)\Big\}\Big), \qquad (16)
\end{aligned}
$$

$$
\begin{aligned}
&d(y_{n+1},F(y,x,y))\\
&= d(F(y_n,x_n,y_n),F(y,x,y))\\
&\le \phi\Big(\max\Big\{d(y_n,y),d(x_n,x),d(F(y_n,x_n,y_n),gy_n),d(F(y,x,y),gy)\Big\}\Big)\\
&= \phi\Big(\max\Big\{d(gy_n,gy),d(gx_n,gx),d(gy_{n+1},gy_n),d(F(y,x,y),gy)\Big\}\Big)\\
&\le \phi\Big(\max\Big\{d(gx,gx_n),d(gy,gy_n),d(gz,gz_n),\\
&\quad d(F(x,y,z),gx),d(F(z,y,x),gz),d(F(y,x,y),gy),\\
&\quad d(gx_{n+1},gx_n),d(gy_{n+1},gy_n),d(gz_{n+1},gz_n)\Big\}\Big) \qquad (17)
\end{aligned}
$$

and

$$
\begin{aligned}
&d(F(z,y,x),z_{n+1})\\
&= d(F(z,y,x),F(z_n,y_n,x_n))\\
&\le \phi\Big(\max\Big\{d(z,z_n),d(y,y_n),d(x,x_n),d(F(z,y,x),gz),d(F(x,y,z),gx),\\
&\quad d(F(z_n,y_n,x_n),gz_n),d(F(x_n,y_n,z_n),gx_n)\Big\}\Big),\\
&\le \phi\Big(\max\Big\{d(gz,gz_n),d(gy,gy_n),d(gx,gx_n),d(F(z,y,x),gz),\\
&\quad d(F(x,y,z),gx),d(gz_{n+1},gz_n),d(gx_{n+1},gx_n)\Big\}\Big),\\
&\le \phi\Big(\max\Big\{d(gx,gx_n),d(gy,gy_n),d(gz,gz_n),d(F(x,y,z),gx),\\
&\quad d(F(z,y,x),gz),d(F(y,x,y),gy),d(gx_{n+1},gx_n),\\
&\quad d(gy_{n+1},gy_n),d(gz_{n+1},gz_n)\Big\}\Big). \qquad (18)
\end{aligned}
$$

From (16)-(18), we have

$$\begin{aligned}
&\max\{d(F(x,y,z),x_{n+1}),d(y_{n+1},F(y,x,y)),d(F(z,y,x),z_{n+1})\}\\
\leq\ &\phi\Big(\max\Big\{d(gx,gx_n),d(gy,gy_n),d(gz,gz_n),d(F(x,y,z),gx),\\
&d(F(z,y,x),gz),d(F(y,x,y),gy),d(gx_{n+1},gx_n),\\
&d(gy_{n+1},gy_n),d(gz_{n+1},gz_n)\Big\}\Big). \qquad (19)
\end{aligned}$$

Our claim is:

$$\max\{d(F(x,y,z),gx),d(gy,F(y,x,y)),d(F(z,y,x),gz)\}=0.$$

To prove our claim, suppose that

$$\max\{d(F(x,y,z),gx),d(gy,F(y,x,y)),d(F(z,y,x),gz)\}\neq 0.$$

Let

$$\epsilon=\max\{d(F(x,y,z),gx),d(gy,F(y,x,y)),d(F(z,y,x),gz)\}.$$

Since $\epsilon>0$, $d(gx,gx_n)\to 0$, $d(gy,gy_n)\to 0$, $d(gz,gz_n)\to 0$, $d(gx_n,gx_{n+1})\to 0$, $d(gy_n,gy_{n+1})\to 0$, and $d(gz_n,gz_{n+1})\to 0$, we choose $n_0\in\mathbb{N}$ such that

$$d(x,x_n)<\frac{\epsilon}{2} \text{ for all } n\geq n_0,$$

$$d(y,y_n)<\frac{\epsilon}{2} \text{ for all } n\geq n_0,$$

$$d(z,z_n)<\frac{\epsilon}{2} \text{ for all } n\geq n_0,$$

$$d(x_n,x_{n+1})<\frac{\epsilon}{2} \text{ for all } n\geq n_0,$$

$$d(y_n,y_{n+1})<e\frac{\epsilon}{2} \text{ for all } n\geq n_0,$$

and

$$d(z_n,z_{n+1})<\frac{\epsilon}{2} \text{ for all } n\geq n_0.$$

Thus (19) becomes

$$\begin{aligned}
&\max\{d(F(x,y,z),x_{n+1}),d(y_{n+1},F(y,x,y)),d(F(z,y,x),z_{n+1})\}\\
\leq\ &\phi\Big(\max\Big\{\frac{\epsilon}{2},d(F(x,y,z),gx),d(F(z,y,x),gz),d(F(y,x,y),gy)\Big\}\Big)\\
=\ &\phi\Big(\max\Big\{d(F(x,y,z),gx),d(F(z,y,x),gz),d(F(y,x,y),gy)\Big\}\Big) \qquad (20)
\end{aligned}$$

for all $n\geq n_0$.

Letting $n \to +\infty$ in (20) it follows that

$$\begin{aligned} & \max\{d(F(x,y,z),gx), d(gy,F(y,x,y)), d(F(z,y,x),gz)\} \\ \leq\ & \phi\Big(\max\Big\{d(F(x,y,z),gx), d(F(z,y,x),gz), d(F(y,x,y),gy)\Big\}\Big) \\ <\ & \max\{d(F(x,y,z),gx), d(gy,F(y,x,y)), d(F(z,y,x),gz)\}, \end{aligned}$$

a contradiction. Therefore

$$\max\{d(F(x,y,z),gx), d(gy,F(y,x,y)), d(F(z,y,x),gz)\} = 0$$

and hence $d(F(x,y,z),gx) = 0, d(gy,F(y,x,y)) = 0$ and $d(F(z,y,x),gz) = 0$. Thus $F(x,y,z) = gx$, $F(y,x,y) = gy$ and $F(z,y,x) = gz$; that is (x,y,z) is a tripled fixed point of F and g. □

By taking $\phi(t) = kt$, $k \in [0,1)$ in Theorem 2.2, we have the following:

Corollary 2.6. *Let $(X,\leq)$ be a partially ordered set and (X,d) be a complete metric space. Let $F\colon X \times X \times X \to X$ and $g\colon X \to X$ be two mappings. Suppose that there exists $k \in [0,1)$ such that*

$$\begin{aligned} & d(F(x,y,z),F(u,v,w)) \\ \leq\ & k\max\big\{d(gx,gu), d(gy,gv), d(gz,gw), d(F(x,y,z),gx), \\ & d(F(z,y,x),gz), d(F(u,v,w),gu), d(F(w,v,u),gw)\big\}, \end{aligned}$$

for all $x,y,z,u,v,w \in X$ with $gx \geq gu$, $gy \leq gv$, and $gz \leq gw$. Suppose the following:

1. *gX is a complete subspace of X,*
2. *$F(X \times X \times X) \subseteq gX$,*
3. *F has the mixed g-monotone property,*
4. *F and g are commute.*

Also, assume that X has the following properties:

i. if a non-decreasing sequence $x_n \to x$, then $x_n \leq x$ for all $n \in \mathbb{N}$,

ii. if a non-increasing sequence $y_n \to y$, then $y_n \geq y$ for all $n \in \mathbb{N}$.

If there exist $x_0, y_0, z_0 \in X$ such that $gx_0 \leq F(x_0,y_0,z_0)$, $gy_0 \geq F(y_0,x_0,y_0)$, and $gz_0 \leq F(z_0,y_0,x_0)$, then F and g have a tripled coincidence point.

As a consequence result of Corollary 2.6, we have

Corollary 2.7. *Let $(X,\leq)$ be a partially ordered set and (X,d) be a complete metric space. Let $F\colon X \times X \times X \to X$ and $g\colon X \to X$ be two mappings.*

Suppose that there exist $a_1, a_2, a_3, a_4, a_5, a_6, a_7 \in [0, 1)$ *with* $\sum_{i=1}^{7} a_i < 1$ *such that*

$$\begin{aligned} & d(F(x,y,z), F(u,v,w)) \\ \leq \; & a_1 d(gx, gu) + a_2 d(gy, gv) + a_3 d(gz, gw) + a_4 d(F(x,y,z), gx) + \\ & a_5 d(F(z,y,x), gz) + d a_6 (F(u,v,w), gu) + a_7 d(F(w,v,u), gw), \end{aligned}$$

for all $x, y, z, u, v, w \in X$ *with* $gx \geq gu$, $gy \leq gv$, *and* $gz \leq gw$. *Suppose the following:*

1. gX *is a complete subspace of* X,
2. $F(X \times X \times X) \subseteq gX$,
3. F *has the mixed* g*-monotone property,*
4. F *and* g *are commute.*

Also, assume that X *has the following properties:*

i. if a non-decreasing sequence $x_n \to x$, *then* $x_n \leq x$ *for all* $n \in \mathbb{N}$,

ii. if a non-increasing sequence $y_n \to y$, *then* $y_n \geq y$ *for all* $n \in \mathbb{N}$.

If there exist $x_0, y_0, z_0 \in X$ *such that* $gx_0 \leq F(x_0, y_0, z_0)$, $gy_0 \geq F(y_0, x_0, y_0)$, *and* $gz_0 \leq F(z_0, y_0, x_0)$, *then* F *and* g *have a tripled coincidence point.*

By taking $g = i_X$ (the identity mapping on X) in Theorem 2.2, Corollary 2.6 and Corollary 2.7, we have the following:

Corollary 2.8. *Let* $(X, \leq)$ *be a partially ordered set and* (X, d) *be a complete metric space. Let* $F\colon X \times X \times X \to X$ *be a mapping having the mixed monotone property. Assume that there exists* $\phi \in \Phi$ *such that*

$$\begin{aligned} d(F(x,y,z), F(u,v,w)) \leq \phi\Big(\max \Big\{ & d(x,u), d(y,v), d(z,w), d(F(x,y,z), x), \\ & d(F(z,y,x), z), d(F(u,v,w), u), d(F(w,v,u), w) \Big\} \Big) \end{aligned}$$

for all $x, y, z, u, v, w \in X$ *with* $x \geq u$, $y \leq v$, *and* $z \leq w$. *Assume also that* X *has the following properties:*

i. if a non-decreasing sequence $x_n \to x$, *then* $x_n \leq x$ *for all* $n \in \mathbb{N}$,

ii. if a non-increasing sequence $y_n \to y$, *then* $y_n \geq y$ *for all* $n \in \mathbb{N}$.

If there exist $x_0, y_0, z_0 \in X$ *such that* $x_0 \leq F(x_0, y_0, z_0)$, $y_0 \geq F(y_0, x_0, y_0)$, *and* $z_0 \leq F(z_0, y_0, x_0)$, *then* F *has a tripled fixed point.*

Corollary 2.9. *Let* $(X, \leq)$ *be a partially ordered set and* (X, d) *be a complete metric space. Let* $F\colon X \times X \times X \to X$ *be a mapping having the mixed monotone property. Assume that there exists* $k \in [0, 1)$ *such that*

$$\begin{aligned} d(F(x,y,z), F(u,v,w)) \leq k \max \Big\{ & d(x,u), d(y,v), d(z,w), d(F(x,y,z), x), \\ & d(F(z,y,x), z), d(F(u,v,w), u), d(F(w,v,u), w) \Big\} \end{aligned}$$

for all $x, y, z, u, v, w \in X$ with $x \geq u$, $y \leq v$, and $z \leq w$. Assume also that X has the following properties:

i. if a non-decreasing sequence $x_n \to x$, then $x_n \leq x$ for all $n \in \mathbb{N}$,

ii. if a non-increasing sequence $y_n \to y$, then $y_n \geq y$ for all $n \in \mathbb{N}$.

If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \leq F(x_0, y_0, z_0)$, $y_0 \geq F(y_0, x_0, y_0)$, and $z_0 \leq F(z_0, y_0, x_0)$, then F has a tripled fixed point.

Corollary 2.10 ([3]). *Let $(X, \leq)$ be a partially ordered set and (X, d) be a complete metric space. Let $F\colon X \times X \times X \to X$ be a mapping having the mixed monotone property. Assume that there exist $a_1, a_2, a_3, a_4, a_5, a_6, a_7 \in [0, 1)$ with $\sum_{i=1}^{7} a_i < 1$ such that*

$$d(F(x,y,z), F(u,v,w)) \leq a_1 d(x,u) + a_2 d(y,v) + a_3 d(z,w) +$$
$$a_4 d(F(x,y,z),x) + a_5 d(F(z,y,x),z) + a_6 d(F(u,v,w),u) + a_7 d(F(w,v,u),w),$$

for all $x, y, z, u, v, w \in X$ with $x \geq u$, $y \leq v$, and $z \geq w$. Assume that X has the following properties:

i. if a non-decreasing sequence $x_n \to x$, then $x_n \leq x$ for all $n \in \mathbb{N}$,

ii. if a non-increasing sequence $y_n \to y$, then $y_n \geq y$ for all $n \in \mathbb{N}$.

If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \leq F(x_0, y_0, z_0)$, $y_0 \geq F(y_0, x_0, y_0)$, and $z_0 \leq F(z_0, y_0, x_0)$, then F has a tripled fixed point.

Now, we prove a uniqueness theorem for a tripled fixed point.

Theorem 2.3. *In addition to the hypotheses of Theorem 2.1 (respectfully Theorem 2.2) suppose that*

$$[(gx_0 \leq gy_0) \wedge (gz_0 \leq gy_0)] \vee [(gy_0 \leq gx_0) \wedge (gy_0 \leq gz_0)].$$

Then $gx = gy = gz$.

Proof. Suppose to the contrary, that is $gx \neq gy$ or $gy \neq gz$ or $gx \neq gz$. Let

$$\epsilon = \max\{d(gx, gy), d(gx, gz), d(gy, gz)\}.$$

Since $\epsilon > 0$, $d(gx_n, gx_{n+1}) \to 0$, $d(gy_n, gy_{n+1}) \to 0$ and $d(gz_n, gz_{n+1}) \to 0$ there exist $n_0 > 0$ such that

$$d(gx_n, gx_{n+1}) < \frac{\epsilon}{2} \text{ for all } n \geq n_0,$$
$$d(gy_n, gy_{n+1}) < \frac{\epsilon}{2} \text{ for all } n \geq n_0$$

and

$$d(gz_n, gz_{n+1}) < \frac{\epsilon}{2} \text{ for all } n \geq n_0.$$

Without loss of generality, we may assume that $gx_0 \le gy_0$ and $gz_0 \le gy_0$. By the mixed monotone property of F, we have $gx_n \le gy_n$ and $gz_n \le gy_n$ for all $n \in \mathbb{N}$. Thus by (3), we have

$$\begin{aligned} & d(gy_{n+1}, gx_{n+1}) \\ = \;& d(F(y_n, x_n, y_n), F(x_n, y_n, z_n)) \\ \le \;& \phi\Big(\max \Big\{ d(gy_n, gx_n), d(gy_n, gz_n), d(F(y_n, x_n, y_n), gy_n), \\ & d(F(x_n, y_n, z_n), gx_n), d(F(z_n, y_n, x_n), gz_n) \Big\}\Big) \\ \le \;& \phi\Big(\max \Big\{ d(gy_n, gx_n), d(gy_n, gz_n), d(gy_{n+1}), gy_n), \\ & d(gx_{n+1}, gx_n), d(gz_{n+1}, gz_n) \Big\}\Big) \end{aligned} \tag{21}$$

and

$$\begin{aligned} & d(gy_{n+1}, gz_{n+1}) \\ = \;& d(F(y_n, x_n, y_n), F(z_n, y_n, x_n)) \\ \le \;& \phi\Big(\max \Big\{ d(gy_n, gx_n), d(gy_n, gz_n), d(F(y_n, x_n, y_n), gy_n), \\ & d(F(x_n, y_n, z_n), gx_n), d(F(z_n, y_n, x_n), gz_n) \Big\}\Big) \\ \le \;& \phi\Big(\max \Big\{ d(gy_n, gx_n), d(gy_n, gz_n), d(gy_{n+1}), gy_n), \\ & d(gx_{n+1}, gx_n), d(gz_{n+1}, gz_n) \Big\}\Big) \end{aligned} \tag{22}$$

By (21) and (22), we have

$$\begin{aligned} & \max\{d(gy_{n+1}, gx_{n+1}), d(gy_{n+1}, gz_{n+1})\} \\ \le \;& \phi\Big(\max \Big\{ d(gy_n, gx_n), d(gy_n, gz_n), d(gy_{n+1}), gy_n), \\ & d(gx_{n+1}, gx_n), d(gz_{n+1}, gz_n) \Big\}\Big). \end{aligned}$$

For $n \ge n_0$, we have

$$\begin{aligned} & \max\{d(gy_{n+1}, gx_{n+1}), d(gy_{n+1}, gz_{n+1})\} \\ \le \;& \phi\Big(\max \Big\{ d(gy_n, gx_n), d(gy_n, gz_n), \frac{\epsilon}{2} \Big\}\Big) \\ \le \;& \phi^2\Big(\max \Big\{ d(gy_{n-1}, gx_{n-1}), d(gy_{n-1}, gz_{n-1}), \frac{\epsilon}{2} \Big\}\Big) \end{aligned}$$

$$
\begin{aligned}
&\le \quad \phi^3\Big(\max\Big\{d(gy_{n-2}, gx_{n-2}), d(gy_{n-2}, gz_{n-2}), \frac{\epsilon}{2}\Big\}\Big) \\
&\vdots \\
&\le \quad \phi^{n-n_0}\Big(\max\Big\{d(gy_{n_0}, gx_{n_0}), d(gy_{n_0}, gz_{n_0}), \frac{\epsilon}{2}\Big\}\Big). \qquad (23)
\end{aligned}
$$

On letting $n \to +\infty$ in (23) and using the property of ϕ and the fact that d is continuous on its variables, we get that $\max\{d(gy, gx), d(gy, gz)\} = 0$. Hence $gy = gz = gx$, a contradiction. □

Corollary 2.11. *In addition to the hypotheses of Corollary* 2.1 *(respectfully Corollary* 2.6*) suppose that*

$$[(gx_0 \le gy_0) \wedge (gz_0 \le gy_0)] \vee [(gy_0 \le gx_0) \wedge (gy_0 \le gz_0)].$$

Then $gx = gy = gz$.

Now, we introduce an example to support the useability of our results:

Example 2.1. Let $X = [0,1]$ with usual ordered. Define $d\colon X \times X \to X$ by $d(x,y) = |x-y|$. Define $g\colon X \to X$ and $F\colon X \times X \times X \to X$ by $gx = \frac{3}{4}x$, and

$$F(x,y,z) = \begin{cases} 0, & y \ge \min\{x,z\}; \\ \frac{1}{3}(\min\{x,z\} - y), & y < \min\{x,z\}. \end{cases}$$

Then:

1. gX is a complete subspace of X.
2. $F(X \times X \times X) \subseteq gX$.
3. F and g are commute.
4. F has the mixed g-monotone property.
5. For $x, y, z, u, v, w \in X$ we have

$$
\begin{aligned}
& d(F(x,y,z), F(u,v,w)) \\
\le \quad & \frac{8}{9}\max\big\{d(gx,gu), d(gy,gv), d(gz,gw), d(F(x,y,z),gx), \\
& d(F(z,y,x),gx), d(F(u,v,w),gu), d(F(w,v,u),gw)\big\}
\end{aligned}
$$

holds, for all $gx \ge gu$, $gy \le gv$ and $gz \ge gw$.

6. There are no $j, k, l \in [0,1)$ with $j + k + l < 1$ such that

$$d(F(x,y,z), F(u,v,w)) \le jd(x,u) + kd(y,v) + ld(z,w)$$

holds for all $x \ge u$, $y \le v$ and $z \ge w$.

7. There are no $j, k, l \in [0,1)$ with $j + k + l < 1$ such that

$$d(F(x,y,z), F(u,v,w)) \le jd(gx,gu) + kd(gy,gv) + ld(gz,gw)$$

holds for all $gx \ge gu$, $gy \le gv$ and $gz \ge gw$.

Proof. The proof of (1), (2) and (3) are clear. To prove (4), let $x, y, z \in X$. To show that $F(x, y, z)$ is monotone g-non-decreasing in x, let $x_1, x_2 \in X$ with $gx_1 \le gx_2$. Then $\frac{3}{4}x_1 \le \frac{3}{4}x_2$ and hence $x_1 \le x_2$.
If $y \ge \min\{x_1, z\}$, then $F(x_1, y, z) = 0 \le F(x_2, y, z)$.
If $y < \min\{x_1, z\}$, then

$$F(x_1, y, z) = \frac{1}{3}(\min\{x_1, z\} - y) \le \frac{1}{3}(\min\{x_2, z\} - y) = F(x_2, y, z).$$

Therefore, $F(x, y, z)$ is monotone g-non-decreasing in x. Similarly, we may show that $F(x, y, z)$ is monotone g-non-decreasing in z and monotone g-non-increasing in y. To prove (5), given $x, y, z, u, v, w \in X$ with $gx \ge gu$, $gy \le gv$ and $gz \ge gw$. Then $\frac{3}{4}x \ge \frac{3}{4}u$, $\frac{3}{4}y \le \frac{3}{4}v$ and $\frac{3}{4}z \ge \frac{3}{4}w$. Hence $x \ge u$, $y \le v$ and $z \ge w$. So, we have the following cases:

Case 1: $y \ge \min\{x, z\}$ and $v \ge \min\{u, w\}$. Here, we have

$$\begin{aligned} & d(F(x, y, z), F(u, v, w)) = 0 \\ \le\ & \frac{1}{2}\max\big\{d(gx, gu), d(gy, gv), d(gz, gw), d(F(x, y, z), gx), \\ & d(F(z, y, x), gx), d(F(u, v, w), gu), d(F(w, v, u), gw)\big\}. \end{aligned}$$

Case 2: $y \ge \min\{x, z\}$ and $v < \min\{u, w\}$. This case is impossible since

$$y \le v < \min\{u, w\} \le \min\{x, z\}.$$

Case 3: $y < \min\{x, z\}$ and $v \ge \min\{u, w\}$.
Suppose $w \le v$, then $w - y \le v - y$ and hence

$$\begin{aligned} & \min\{x, z\} - y \\ \le\ & z - y = z - w + w - y \\ \le\ & z - w + v - y \\ =\ & \frac{4}{3}\Big[\frac{3}{4}(z - w) + \frac{3}{4}(v - y)\Big] \\ =\ & \frac{4}{3}\Big[(gz - gw) + (gv - gy)\Big] \\ =\ & \frac{4}{3}[d(gz, gw) + d(gv, gy)] \\ \le\ & \frac{8}{3}\max\{d(gy, gv), d(gz, gw)\} \\ \le\ & \frac{8}{3}\max\big\{d(gx, gu), d(gy, gv), d(gz, gw), d(F(x, y, z), gx), \\ & d(F(z, y, x), gx), d(F(u, v, w), gu), d(F(w, v, u), gw)\big\}. \end{aligned}$$

Therefore,

$$\begin{aligned}
& d(F(x,y,z),F(u,v,w)) \\
= \;& d\left(\frac{1}{3}(\min\{x,z\}-y),0\right) \\
= \;& \frac{1}{3}(\min\{x,z\}-y) \\
\leq \;& \frac{8}{9}\max\big\{d(gx,gu),d(gy,gv),d(gz,gw),d(F(x,y,z),gx), \\
& d(F(z,y,x),gx),d(F(u,v,w),gu),d(F(w,v,u),gw)\big\}.
\end{aligned}$$

Suppose $u \leq v$, then $u-y \leq v-y$ and hence

$$\begin{aligned}
& \min\{x,z\}-y \\
\leq \;& x-y \\
= \;& x-u+u-y \\
\leq \;& x-u+v-y \\
= \;& \frac{4}{3}\Big[\frac{3}{4}(x-u)+\frac{3}{4}(u-y)\Big] \\
= \;& \frac{4}{3}\Big[(gx-gu)+(gu-gy)\Big] \\
= \;& \frac{4}{3}\Big[d(gx,gu)+d(gv,gy)\Big] \\
\leq \;& \frac{8}{3}\max\{d(gx,gu),d(gy,gv))\} \\
\leq \;& \frac{8}{3}\max\big\{d(gx,gu),d(gy,gv),d(gz,gw),d(F(x,y,z),gx), \\
& d(F(z,y,x),gx),d(F(u,v,w),gu),d(F(w,v,u),gw)\big\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& d(F(x,y,z),F(u,v,w)) \\
= \;& d\left(\frac{1}{3}(\min\{x,z\}-y),0\right) \\
= \;& \frac{1}{3}(\min\{x,z\}-y) \\
\leq \;& \frac{8}{9}\max\big\{d(gx,gu),d(gy,gv),d(gz,gw),d(F(x,y,z),gx), \\
& d(F(z,y,x),gx),d(F(u,v,w),gu),d(F(w,v,u),gw)\big\}.
\end{aligned}$$

Case 4: $y < \min\{x,z\}$ and $v < \min\{u,w\}$.

Since $x \geq u$ and $z \geq w$, then $\min\{x,z\} \geq \min\{u,w\}$. Thus

$$\begin{aligned} & d(F(x,y,z),F(u,v,w)) \\ = \;& d\left(\frac{1}{3}(\min\{x,z\}-y), \frac{1}{3}(\min\{u,w\}-v)\right) \\ = \;& \frac{1}{3}\Big|(\min\{x,z\}-\min\{u,w\})+(v-y)\Big| \\ = \;& \frac{1}{3}\Big[(\min\{x,z\}-\min\{u,w\})+(v-y)\Big] \end{aligned}$$

If $\min\{u,w\}=u$, then $\min\{x,z\}-\min\{u,w\} \leq x-u$, and hence

$$\begin{aligned} & d(F(x,y,z),F(u,v,w)) \\ \leq \;& \frac{1}{3}\Big[(x-u)+(v-y)\Big] \\ = \;& \left(\frac{1}{3}\right)\left(\frac{4}{3}\right)\Big[\frac{3}{4}(x-u)+\frac{3}{4}(v-y)\Big] \\ = \;& \frac{4}{9}\Big[(gx-gu)+(gv-gy)\Big] \\ = \;& \frac{4}{9}\Big[d(gx,gu)+d(gy,gv)\Big] \\ \leq \;& \frac{8}{9}\max\Big\{d(gx,gu),d(gy,gv)\Big\} \\ \leq \;& \frac{8}{9}\max\Big\{d(gx,gu),d(gy,gv),d(gz,gw),d(F(x,y,z),gx), \\ & d(F(z,y,x),gx),d(F(u,v,w),gu),d(F(w,v,u),gw)\Big\}. \end{aligned}$$

If $\min\{u,w\}=w$, then $\min\{x,z\}-\min\{u,w\} \leq z-w$, and hence

$$\begin{aligned} & d(F(x,y,z),F(u,v,w)) \\ \leq \;& \frac{1}{3}\Big[(z-w)+(v-y)\Big] \\ = \;& \left(\frac{1}{3}\right)\left(\frac{4}{3}\right)\Big[\frac{3}{4}(z-w)+\frac{3}{4}(v-y)\Big] \\ = \;& \frac{4}{9}\Big[(gz-gw)+(gv-gy)\Big] \\ = \;& \frac{4}{9}[d(gz,gw)+d(gy,gv)] \\ \leq \;& \frac{8}{9}\max\{d(gy,gv),d(gz,gw)\} \\ \leq \;& \frac{8}{9}\max\Big\{d(gx,gu),d(gy,gv),d(gz,gw),d(F(x,y,z),gx), \\ & d(F(z,y,x),gx),d(F(u,v,w),gu),d(F(w,v,u),gw)\Big\}. \end{aligned}$$

To prove (6), suppose there exist $j, k, l \in [0, 1)$ with $j+k+l < 1$ such that

$$d(F(x, y, z), F(u, v, w)) \leq jd(x, u) + kd(y, v) + ld(z, w)$$

holds for all $x \geq u$, $y \leq v$ and $z \geq w$. Then

$$d(F(1, 0, 1), F(0, 0, 1)) = d(\frac{1}{3}, 0) = \frac{1}{3} \leq j, \tag{24}$$

$$d(F(1, 0, 1), F(1, 1, 1)) = d(\frac{1}{3}, 0) = \frac{1}{3} \leq k, \tag{25}$$

and

$$d(F(1, 0, 1), F(1, 0, 0)) = d(\frac{1}{3}, 0) = \frac{1}{3} \leq l. \tag{26}$$

From (24), (25), and (26), we have $j + k + l \geq 1$, a contradiction.

To prove (7), suppose there exist $j, k, l \in [0, 1)$ with $j+k+l < 1$ such that

$$d(F(x, y, z), F(u, v, w)) \leq jd(gx, gu) + kd(gy, gv) + ld(gz, gw)$$

holds for all $gx \geq gu$, $gy \leq gv$ and $gz \geq gw$. Then

$$d(F(1, 0, 1), F(0, 0, 1)) = d(\frac{1}{3}, 0) = \frac{1}{3} \leq \frac{3}{4}j, \tag{27}$$

$$d(F(1, 0, 1), F(1, 1, 1)) = d(\frac{1}{3}, 0) = \frac{1}{3} \leq \frac{3}{4}k, \tag{28}$$

and

$$d(F(1, 0, 1), F(1, 0, 0)) = d(\frac{1}{3}, 0) = \frac{1}{3} \leq \frac{3}{4}l. \tag{29}$$

From (27), (28), and (29), we have $j + k + l \geq \frac{12}{9}$, a contradiction.

Thus by Theorems 2.1 and 2.3, F and g have a tripled coincidence point. Here, $(0, 0, 0)$ is the tripled coincidence point of F and g. □

Remarks:

1. Example 2.1 does not satisfy condition 1 of Theorem 1.1 (Theorem 7 of [3]).
2. Example 2.1 does not satisfy condition 2 of Theorem 1.3 (Theorem 4 of [4]).
3. Theorem 1.3 (Theorem 4 of [4]) is a special case of Corollaries 2.2 and 2.7.
4. Theorem 1.1 (Theorem 7 of [3]) is a special case of Corollary 2.5.
5. Theorem 1.2 (Theorem 8 of [3]) is a special case of Corollary 2.10.

References

[1] T. G. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal. 65 (2006), 1379–1393.

[2] V. Lakshmikantham, Lj. Ćirić, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal. 70 (2009), 4341–4349.

[3] V. Berinde, M. Borcut, Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces, Nonlinear Anal. 74 (2011), 4889–4897.

[4] M. Borcut, V. Berinde, Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces, Appl. Math. Comput. 218 (2012), 5929-5936.

[5] H. Aydi, M. Postolache, W. Shatanawi, Coupled fixed point results for (psi,phi)-weakly contractive mappings in ordered G-metric spaces, Comput. Math. Appl. 63 (2012), 298–309.

[6] H. Aydi, E. Karapinar, M. Postolache, Tripled coincidence point theorems for weak phi-contractions in partially ordered metric spaces, Fixed Point Theory Appl., Vol. 2012, ID: 2012:44, 12 pp.

[7] V. Berinde, Coupled coincidence point theorems for mixed monotone nonlinear operators, Comput. Math. Appl. 64 (2012), 1770–1777.

[8] V. Berinde, Coupled fixed point theorems for ϕ-contractive mixed monotone mappings in partially ordered metric spaces, Nonlinear Anal. 75 (2012), 3218-3228.

[9] S. Chandok, Z. Mustafa, M. Postolache, Coupled common fixed point theorems for mixed g-monotone mappings in partially ordered G-metric spaces, U. Politeh. Buch. Ser. A, 75 (2013), No. 4, 13-26.

[10] D. Dorić, Z. Kadelburg, S. Radenović, Coupled fixed point results for mappings without mixed monotone property, Appl. Math. Lett. 25 (2012), 1803-1808.

[11] B. S. Choudhury, N. Metiya, M. Postolache, A generalized weak contraction principle with applications to coupled coincidence point problems, Fixed Point Theory Appl., Vol. 2013, ID: 2013:152, 21 pp.

[12] N. V. Luong, N. X. Thuan, Coupled fixed points in partially ordered metric spaces and applications, Nonlinear Anal. 72 (2011), 983–992.

[13] Z. Golubović, Z. Kadelburg, S. Radenović, Coupled coincidence points of mappings in ordered partial metric spaces, Abstr. Appl. Anal. 2012 (2012), art. no. 192581.

[14] H. K. Nashine and W. Shatanawi, Coupled common fixed point theorems for a pair of commuting mappings in partially ordered complete metric spaces, Comput. Math. Appl. 62 (2011), 1984–1993.

[15] H. K. Nashine, Z. Kadelburg, S. Radenović, Coupled common fixed point theorems for w-compatible mappings in ordered cone metric spaces, Appl. Math. Comput. 218 (2012), 5422-5432.

[16] B. Samet, Coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces, Nonlinear Anal. 72 (2010), 4508–4517.

[17] B. Samet, C. Vetro, Coupled fixed point, F-invariant set and fixed point of N-order, Ann. Funct. Anal. 1 (2) (2010), 46–56.

[18] B. Samet, C. Vetro, Coupled fixed point theorems for multi-valued nonlinear contraction mappings in partially ordered metric spaces, Nonlinear Anal. 74 (2011), 4260–4268

[19] B. Samet, H. Yazidi, Coupled fixed point theorems in partially ordered ε-chainable metric spaces, TJMCS. 1 (3) (2010), 142–151.

[20] W. Shatanawi, A. Pitea, Fixed and coupled fixed point theorems of omega-distance for nonlinear contraction. Fixed Point Theory Appl., Vol. 2013, ID: 2013:275, 16 pp.

[21] W. Shatanawi, A. Pitea, Omega-distance and coupled fixed point in G-metric spaces. Fixed Point Theory Appl., Vol. 2013, Article ID: 2013:208, 15 pp.

[22] W. Shatanawi, A. Pitea, Some coupled fixed point theorems in quasi-partial metric spaces. Fixed Point Theory Appl., Vol. 2013, ID: 2013:153, 13 pp.

[23] W. Shatanawi, Z. Mustafa, On coupled random fixed point results in partially ordered metric spaces, Matematicki Vesnik, 64 (2012), 139–146.

[24] W. Shatanawi, Coupled fixed point theorems in generalized metric spaces, Hacettepe J. Mathematics and Statistics, Volume 40 (3) (2011), 441–447.

[25] W. Shatanawi, Fixed point theorems for nonlinear weakly C-contractive mappings in metric spaces, Math. Comput. Modelling 54 (2011), 2816–2826, doi:10.1016/j.mcm.2011.06.069

[26] J. Matkowski, Fixed point theorems for mappings with a contractive iterate at a point, Proceedings of the American Mathematical Society, 62 (2) (1977), pp. 344–348.

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