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ON AN INTEGRAL OPERATOR

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Abstract

In this paper we define a general integral operator for analytic functions in the open unit disk and we determine some conditions for univalence of this integral operator.

1 Introduction

Let $\mathcal{A}$ be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

normalized by $f(0) = f'(0) - 1 = 0$, which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$.

We consider $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions $f \in \mathcal{A}$, which are univalent in $\mathcal{U}$.

We denote by $\mathcal{P}$ the class of functions p of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} b_k z^k,$$

which are analytic in $\mathcal{U}$, with $Re\, p(z) > 0$, for all $z \in \mathcal{U}$.

In this work we introduce a new integral operator defined by

$$V_n(z) = \left\{ \beta \int_0^z u^{\beta-1} \left(p_1(u)\right)^{\gamma_1} \ldots \left(p_n(u)\right)^{\gamma_n} du \right\}^{\frac{1}{\beta}}, \tag{1}$$

for functions $p_j \in \mathcal{P}$ and $\beta,\ \gamma_j$ be complex numbers $\beta \neq 0$ and $j = \overline{1,n}$.

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2 Preliminary results

We shall use the following lemmas.

Lemma 2.1. *[3]. Let α be a complex number, $Re\ \alpha > 0$ and $f \in \mathcal{A}$. If*

$$\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le 1, \tag{1}$$

for all $z \in \mathcal{U}$, then for any complex number β, $Re\ \beta \ge Re\ \alpha$, the function

$$F_\beta(z) = \left[\beta\int_0^z u^{\beta-1}f'(u)\right]^{\frac{1}{\beta}} \tag{2}$$

is regular and univalent in $\mathcal{U}$.

Lemma 2.2. *(Schwarz [1]). Let f be the function regular in the disk $\mathcal{U}_R = \{z \in \mathbb{C} : |z| < R\}$, with $|f(z)| < M$, M fixed. If $f(z)$ has in $z = 0$ one zero with multiply $\ge m$, then*

$$|f(z)| \le \frac{M}{R^m}|z|^m,\ (z \in \mathcal{U}_R)\,, \tag{3}$$

the equality (in the inequality (3) for $z \neq 0$) can hold only if

$$f(z) = e^{i\theta}\frac{M}{R^m}z^m,$$

where θ is constant.

Lemma 2.3. *[2]. If the function f is regular in $\mathcal{U}$ and $|f(z)| < 1$ in $\mathcal{U}$, then for all $\xi \in \mathcal{U}$ and $z \in \mathcal{U}$ the following inequalities hold*

$$\left|\frac{f(\xi)-f(z)}{1-\overline{f(z)}f(\xi)}\right| \le \frac{|\xi - z|}{|1-\overline{z}\xi|}, \tag{4}$$

$$|f'(z)| \le \frac{1-|f(z)|^2}{1-|z|^2}, \tag{5}$$

the equalities hold only in the case $f(z) = \frac{\epsilon(z+u)}{1+\overline{u}z}$, where $|\epsilon| = 1$ and $|u| < 1$.

Remark 2.4. *[2]. For $z = 0$, from inequality (4)*

$$\left|\frac{f(\xi)-f(0)}{1-\overline{f(0)}f(\xi)}\right| \le |\xi| \tag{6}$$

and, hence

$$|f(\xi)| \leq \frac{|\xi| + |f(0)|}{1 + |f(0)||\xi|}. \tag{7}$$

Considering $f(0) = a$ and $\xi = z$, we have

$$|f(z)| \leq \frac{|z| + |a|}{1 + |a||z|}, \tag{8}$$

for all $z \in \mathcal{U}$.

3 Main results

Theorem 3.1. *Let α, β, γ_j be complex numbers, $j = \overline{1,n}$, $Re\ \alpha > 0$ and $p_j \in \mathcal{P}$, $p_j(z) = 1 + b_{1j}z + b_{2j}z^2 + \ldots$, $j = \overline{1,n}$.*

If

$$\left|\frac{zp_j'(z)}{p_j(z)}\right| \leq \frac{(2Re\ \alpha + 1)^{\frac{2Re\ \alpha+1}{2Re\ \alpha}}}{2}, \quad (z \in \mathcal{U};\ j = \overline{1,n}), \tag{1}$$

and

$$|\gamma_1| + |\gamma_2| + \ldots + |\gamma_n| \leq 1, \tag{2}$$

then for any complex number β, $Re\ \beta \geq Re\ \alpha$, the integral operator V_n defined by (1) is in the class $\mathcal{S}$.

Proof. Let's consider the function

$$g_n(z) = \int_0^z (p_1(u))^{\gamma_1} \ldots (p_n(u))^{\gamma_n}\, du, \quad (p_j \in \mathcal{P};\ j = \overline{1,n}). \tag{3}$$

The function g_n is regular in $\mathcal{U}$ and $g_n(0) = g_n'(0) - 1 = 0$.

From (3) we obtain

$$\frac{1 - |z|^{2Re\ \alpha}}{Re\ \alpha}\left|\frac{zg_n''(z)}{g_n'(z)}\right| \leq \frac{1 - |z|^{2Re\ \alpha}}{Re\ \alpha}\sum_{j=1}^{n}|\gamma_j|\left|\frac{zp_j'(z)}{p_j(z)}\right|, \tag{4}$$

for all $z \in \mathcal{U}$.

By (1), applying Lemma 2.2 we have

$$\left|\frac{zp_j'(z)}{p_j(z)}\right| \leq \frac{(2Re\ \alpha + 1)^{\frac{2Re\ \alpha+1}{2Re\ \alpha}}}{2}|z|, \quad (z \in \mathcal{U};\ j = \overline{1,n}), \tag{5}$$

and hence, by (4) we get

$$\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}\left|\frac{zg_n''(z)}{g_n'(z)}\right| \le \frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}|z|\frac{(2Re\ \alpha+1)^{\frac{2Re\ \alpha+1}{2Re\ \alpha}}}{2}\sum_{j=1}^{n}|\gamma_j|, \tag{6}$$

for all $z \in \mathcal{U}$.

Since

$$\max_{|z|\le 1}\left[\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}|z|\right] = \frac{2}{(2Re\ \alpha+1)^{\frac{2Re\ \alpha+1}{2Re\ \alpha}}},$$

from (2) and (6), we obtain

$$\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}\left|\frac{zg_n''(z)}{g_n'(z)}\right| \le 1, \quad (z \in \mathcal{U}). \tag{7}$$

From (7) and since $g_n'(z) = (p_1(z))^{\gamma_1}\dots(p_z(u))^{\gamma_n}$, by Lemma 2.1 it results that the integral operator V_n defined by (1) is in the class $\mathcal{S}$. □

Theorem 3.2. *Let α, β, γ_j be complex numbers, $j = \overline{1,n}$, $Re\ \alpha > 0$, M_j positive real numbers and $p_j \in \mathcal{P}$, $p_j(z) = 1 + b_{1j}z + b_{2j}z^2 + \dots$, $j = \overline{1,n}$.*

If

$$\left|\frac{p_j'(z)}{p_j(z)}\right| < M_j, \quad (z \in \mathcal{U};\ j = \overline{1,n}), \tag{8}$$

$$M_1|\gamma_1| + M_2|\gamma_2| + \dots + M_n|\gamma_n| \le \frac{1}{\max_{|z|\le 1}\left[\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}|z|\frac{|z|+|c|}{1+|c||z|}\right]}, \tag{9}$$

where

$$c = \frac{b_{11}\gamma_1 + b_{12}\gamma_2 + \dots + b_{1n}\gamma_n}{M_1|\gamma_1| + M_2|\gamma_2| + \dots + M_n|\gamma_n|}, \tag{10}$$

then for any complex number β, $Re\ \beta \ge Re\ \alpha$, the integral operator V_n given by (1) is in the class $\mathcal{S}$.

Proof. We consider the function

$$g_n(z) = \int_0^z (p_1(u))^{\gamma_1}\dots(p_n(u))^{\gamma_n}\,du, \tag{11}$$

which is regular in $\mathcal{U}$ and $g_n(0) = g_n'(0) - 1 = 0$.

Let's consider the function

$$h(z) = \frac{1}{M_1|\gamma_1| + M_2|\gamma_2| + \ldots + M_n|\gamma_n|} \frac{g_n''(z)}{g_n'(z)}, \quad (z \in \mathcal{U}) \tag{12}$$

and from (11) we get

$$\begin{aligned} h(z) &= \frac{\gamma_1}{M_1|\gamma_1| + \ldots + M_n|\gamma_2|} \frac{p_1'(z)}{p_1(z)} + \ldots + \\ &+ \frac{\gamma_n}{M_1|\gamma_1| + \ldots + M_n|\gamma_2|} \frac{p_n'(z)}{p_n(z)}, \end{aligned} \tag{13}$$

for all $z \in \mathcal{U}$.

From (8) and (13) we obtain $|h(z)| < 1$, $z \in \mathcal{U}$.

We have

$$h(0) = \frac{b_{11}\gamma_1 + b_{12}\gamma_2 + \ldots + b_{1n}\gamma_n}{M_1|\gamma_1| + M_2|\gamma_2| + \ldots + M_n|\gamma_n|} = c$$

and by applying the Remark 2.4 we get

$$|h(z)| \leq \frac{|z| + |c|}{1 + |c||z|}, \quad (z \in \mathcal{U}), \tag{14}$$

where

$$|c| = \frac{|b_{11}\gamma_1 + b_{12}\gamma_2 + \ldots + b_{1n}\gamma_n|}{M_1|\gamma_1| + M_2|\gamma_2| + \ldots + M_n|\gamma_n|},$$

From (12), (14) we obtain

$$\frac{1 - |z|^{2Re\ \alpha}}{Re\ \alpha} \left| \frac{z g_n''(z)}{g_n'(z)} \right| \leq$$

$$\leq (M_1|\gamma_1| + \ldots + M_n|\gamma_n|) \max_{|z| \leq 1} \left[\frac{1 - |z|^{2Re\ \alpha}}{Re\ \alpha} |z| \frac{|z| + |c|}{1 + |c||z|} \right] \tag{15}$$

for all $z \in \mathcal{U}$.

From (9) and (15) we have

$$\frac{1 - |z|^{2Re\ \alpha}}{Re\ \alpha} \left| \frac{z g_n''(z)}{g_n'(z)} \right| \leq 1, \quad (z \in \mathcal{U}). \tag{16}$$

From (16) and since $g_n'(z) = (p_1(z))^{\gamma_1} \ldots (p_n(z))^{\gamma_n}$, by Lemma 2.1, it results that the integral operator V_n given by (1) belongs to the class $\mathcal{S}$. $\square$

Theorem 3.3. *Let α, γ_j be complex numbers, $j = \overline{1,n}$, $Re\ \alpha > 0$ and $p_j \in \mathcal{P}$, $p_j(z) = 1 + b_{1j}(z) + b_{2j}(z) + \ldots$, $j = \overline{1,n}$.*

If

$$|\gamma_1| + |\gamma_2| + \ldots + |\gamma_n| \leq \frac{Re\ \alpha}{2}, \quad (0 < Re\ \alpha \leq 1), \tag{17}$$

or

$$|\gamma_1| + |\gamma_2| + \ldots + |\gamma_n| \leq \frac{1}{2}, \quad (Re\ \alpha > 1), \tag{18}$$

then for any complex number β, $Re\ \beta \geq Re\ \alpha$, the integral operator $V_n \in \mathcal{S}$.

Proof. Since $p_j \in \mathcal{P}$, $j = \overline{1,n}$, we have

$$\left| \frac{z p_j'(z)}{p_j(z)} \right| \leq \frac{2|z|}{1 - |z|^2}, \quad (z \in \mathcal{U}). \tag{19}$$

We consider the function

$$g_n(z) = \int_0^z (p_1(u))^{\gamma_1} \ldots (p_n(u))^{\gamma_n}\, du. \tag{20}$$

From (19) and (20) we obtain

$$\frac{1 - |z|^{2Re\ \alpha}}{Re\ \alpha} \left| \frac{z g_n''(z)}{g_n'(z)} \right| \leq$$

$$\leq \frac{1 - |z|^{2Re\ \alpha}}{Re\ \alpha} \cdot \frac{2|z|}{1 - |z|^2} \sum_{j=1}^{n} |\gamma_j|, \quad (z \in \mathcal{U}). \tag{21}$$

For $0 < Re\ \alpha \leq 1$ we have $1 - |z|^{2Re\ \alpha} \leq 1 - |z|^2$, for all $z \in \mathcal{U}$. By (17) and (21) we get

$$\frac{1 - |z|^{2Re\ \alpha}}{Re\ \alpha} \left| \frac{z g_n''(z)}{g_n'(z)} \right| \leq 1, \quad (z \in \mathcal{U};\ 0 < Re\ \alpha \leq 1). \tag{22}$$

For $Re\ \alpha > 1$ we obtain $\frac{1 - |z|^{2Re\ \alpha}}{Re\ \alpha} \leq 1 - |z|^2$, for all $z \in \mathcal{U}$. By (18) and (21) we get

$$\frac{1 - |z|^{2Re\ \alpha}}{Re\ \alpha} \left| \frac{z g_n''(z)}{g_n'(z)} \right| \leq 1, \quad (z \in \mathcal{U};\ Re\ \alpha > 1). \tag{23}$$

From (22), (23) and Lemma 2.1 we obtain that $V_n \in \mathcal{S}$. $\square$

4 Corollaries

Corollary 4.1. *Let α, γ_j be complex numbers, $j = \overline{1,n}$ $0 < Re\ \alpha \leq 1$ and $p_j \in \mathcal{P}$, $p_j(z) = 1 + b_{1j}(z) + b_{2j}(z) + \ldots$, $j = \overline{1,n}$.*

If

$$\left| \frac{zp_j'(z)}{p_j(z)} \right| \leq \frac{(2Re\ \alpha + 1)^{\frac{2Re\ \alpha+1}{2Re\ \alpha}}}{2}, \quad (z \in \mathcal{U};\ j = \overline{1,n}), \tag{1}$$

$$|\gamma_1| + |\gamma_2| + \ldots + |\gamma_n| \leq 1, \tag{2}$$

then the integral operator defined by

$$K_n(z) = \int_0^z (p_1(u))^{\gamma_1} \ldots (p_n(u))^{\gamma_n}\, du \tag{3}$$

is in the class $\mathcal{S}$.

Proof. For $\beta = 1$, from Theorem 3.1, we obtain the Corollary 4.1. □

Corollary 4.2. *Let α, γ_j be complex numbers, $j = \overline{1,n}$, $0 < Re\ \alpha \leq 1$, M_j positive real numbers and $p_j \in \mathcal{P}$, $p_j(z) = 1 + b_{1j}z + b_{2j}z^2 + \ldots$, $j = \overline{1,n}$.*

If

$$\left| \frac{p_j'(z)}{p_j(z)} \right| \leq M_j, \quad (z \in \mathcal{U};\ j = \overline{1,n}), \tag{4}$$

$$M_1|\gamma_1| + M_2|\gamma_2| + \ldots + M_n|\gamma_n| \leq \frac{1}{\max_{|z|\leq 1} \left[\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha} |z| \frac{|z|+|c|}{1+|c||z|} \right]}, \tag{5}$$

where

$$c = \frac{b_{11}\gamma_1 + b_{12}\gamma_2 + \ldots + b_{1n}\gamma_n}{M_1|\gamma_1| + M_2|\gamma_2| + \ldots + M_n|\gamma_n|}, \tag{6}$$

then the integral operator K_n defined by (3) belongs to the class $\mathcal{S}$.

Proof. We take $\beta = 1$ in Theorem 3.2. □

Corollary 4.3. *Let α, β, γ_j be complex numbers, $j = \overline{1,n}$, $Re\ \beta \geq Re\ \alpha > 0$, M_j positive real numbers and $p_j \in \mathcal{P}$, $p_j(z) = 1 + b_{1j}z + b_{2j}z^2 + \ldots$, $j = \overline{1,n}$.*

If

$$\left|\frac{p_j'(z)}{p_j(z)}\right| \le M_j, \quad (z \in \mathcal{U},\ j = \overline{1,n}), \tag{7}$$

$$|b_{11}\gamma_1 + b_{12}\gamma_2 + \ldots + b_{1n}\gamma_n| \le \frac{(2Re\ \alpha+1)^{\frac{2Re\ \alpha+1}{2Re\ \alpha}}}{2}, \tag{8}$$

$$|b_{11}\gamma_1 + b_{12}\gamma_2 + \ldots + b_{1n}\gamma_n| = M_1|\gamma_1| + M_2|\gamma_2| + \ldots + M_n|\gamma_n|, \tag{9}$$

then the integral operator $V_n \in \mathcal{S}$.

Proof. By (10), from Theorem 3.2 and (9) we have $|c| = 1$.
Using the inequality (9) we obtain

$$M_1|\gamma_1| + M_2|\gamma_2| + \ldots + M_n|\gamma_n| \le \frac{1}{\max_{|z|\le 1} \frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}|z|}. \tag{10}$$

We have

$$\max_{|z|\le 1}\left[\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}|z|\right] = \frac{2}{(2Re\ \alpha+1)^{\frac{2Re\ \alpha+1}{2Re\ \alpha}}} \tag{11}$$

and from (9), (10) and (11) we obtain (8). The conditions of Theorem 3.2 are satisfied. □

Corollary 4.4. *Let* $\alpha,\ \beta,\ \gamma_j$ *be complex numbers,* $j = \overline{1,n}$, $Re\ \beta \ge Re\ \alpha > 0$, M_j *positive real numbers and* $p_j \in \mathcal{P}$, $p_j(z) = 1 + b_{1j}z + b_{2j}z^2 + \ldots$, $j = \overline{1,n}$, $b_{11}\gamma_1 + b_{12}\gamma_2 + \ldots + b_{1n}\gamma_n = 0$.
If

$$\left|\frac{p_j'(z)}{p_j(z)}\right| \le M_j, \quad (z \in \mathcal{U},\ j = \overline{1,n}), \tag{12}$$

$$M_1|\gamma_1| + M_2|\gamma_2| + \ldots + M_n|\gamma_n| \le (Re\ \alpha+1)^{\frac{Re\ \alpha+1}{Re\ \alpha}}, \tag{13}$$

then the integral operator $V_n \in \mathcal{S}$.

Proof. From (10) we have $c = 0$ and using the inequality (9) we obtain

$$M_1|\gamma_1| + M_2|\gamma_2| + \ldots + M_n|\gamma_n| \le \frac{1}{\max_{|z|\le 1} \frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}|z|^2}. \tag{14}$$

Since

$$\max_{|z|\leq 1}\left[\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}|z|^2\right]=\frac{1}{(Re\ \alpha+1)^{\frac{Re\ \alpha+1}{Re\ \alpha}}}$$

from (14) we have the inequality (13).

The conditions of Theorem 3.2 are verified and hence, we obtain $V_n \in \mathcal{S}$. □

Corollary 4.5. *Let* α, β, γ_j *be complex numbers,* $j=\overline{1,n}$, $0 < Re\ \alpha \leq 1$ *and* $p_j \in \mathcal{P}$, $p_j(z) = 1 + b_{1j}z + b_{2j}z^2 + \ldots$, $j = \overline{1,n}$.

If

$$|\gamma_1| + |\gamma_2| + \ldots + |\gamma_n| \leq \frac{Re\ \alpha}{2}, \quad (0 < Re\ \alpha \leq 1), \tag{15}$$

then the integral operator K_n *given by (3) is in the class* $\mathcal{S}$.

Proof. We take $\beta = 1$ in Theorem 3.3. □

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