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Symmetric Besov-Bessel Spaces

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Abstract

In this paper we introduce the symmetric Besov-Bessel spaces. Next, we give a Sonine formula for generalized Bessel functions. Finally, we give a characterization of these spaces using the Bochner-Riesz means.

1 Introduction

Let $\mathbb{F}$ be the skew field $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. For q be a positive integer consider Π_q the set of positive matrices over $\mathbb{F}$ and the closed Weyl chamber

$$\Xi_q = \{\xi = (\xi_1, \dots \xi_q) \in \mathbb{R}^q,\ \xi_1 \geq \cdots \geq \xi_q \geq 0\}$$

of the hyperoctahedarl group B_q, which acts on $\mathbb{R}^q$ by permutations of the basis vectors and sign changes.

In [15, Section 3], the author has shown that the cone Π_q carries a continuously parameterized family of commutative hypergroup structure $*_\mu$ with μ a real number satisfying $\mu > d(q - \dfrac{1}{2})$, where $d = \dim_{\mathbb{R}}\mathbb{F}$, which interpolate those occuring as orbit hypergroup for indices $\mu = \frac{pd}{2}$; $p \geq q$ an integer; with neutral element 0 and the identity mapping as the involution.

Each convolution $*_\mu$, in the set

$$\mathcal{M}_q := \left\{\frac{pd}{2}, p = q, q+1, \dots\right\} \cup]d(q - \frac{1}{2}), \infty[$$

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, induces a commutative hypergroup convolution $\circ_\mu$ on Ξ_q which is obtained by the technique of orbital hypergroup morphisms [11].

The Fourier transform on Ξ_q is defined for suitable functions f by

$$\hat{f}(\eta) = \int_{\Xi_q} f(\xi) J_k^{B_q}(\xi, i\eta) d\tilde{\omega}_\mu(\xi),$$

where $J_k^{B_q}(\xi, i\eta)$ represents the generalized Bessel function associated to root system of type B_q and $\tilde{\omega}_\mu$ is the Haar measure on the hypergroup Ξ_q. The functions $J_k^{B_q}$ admit a product formula which permits to define a translation operator τ_ξ, $\xi \in \Xi_q$.

This paper deals with new spaces that we will call symmetric Besov-Bessel spaces as follows. Let $0 < \alpha < q$ and $1 \leq p, r < \infty$. Let $u \in \Xi_q$ such that $\|u\| = \max_{i=1,\cdots,q} u_i = 1$ and put for $t > 0$, $\Lambda_p(f,t) = \|\tau_{tu} f - f\|_{p,\mu}$.
We say that a function f on Ξ_q is in $BB^{p,r}_{\alpha,\mu}$ if $f \in L^p(\tilde{\omega}_\mu)$ (the Lebesgue space with respect to the measure $\tilde{\omega}_\mu$) and

$$\int_0^\infty \left(\frac{\Lambda_p(f,t)}{t^\alpha}\right)^r \frac{dt}{t} < \infty$$

where $\|.\|_{p,\alpha}$ is the usual norm of $L^p(\tilde{\omega}_\mu)$.

The goal of this paper is to characterize these spaces by means of the Bochner Riesz means: For $T > 0$, $\beta \geq 0$ and $f \in L^1(\tilde{\omega}_\mu)$

$$\sigma_T^\beta(f)(\xi) = C_{\mu,q} \int_{B_T} \hat{f}(\eta) J_k^{B_q}(\eta, i\xi) \prod_{i=1}^q (1 - \eta_i^2 T^{-2})^\beta d\tilde{\omega}_\mu(\eta), \quad \xi \in \Xi_q.$$

where $B_T = \{\xi = (\xi_1, \ldots, \xi_q) \in \Xi_q \mid T \geq \xi_1 \geq \cdots \geq \xi_q \geq 0\}$ and $C_{\mu,q}$ a positive constant which depend only on μ and q.

To establish this result we shall generalize the Sonine formula corresponding to Bessel functions and give asymptotic behavior of the Bessel function.

Analogous results have been obtained by Giang and Moricz in [6] for the classical Fourier transform on $\mathbb{R}$. Later, Betancor and Rodriguez-Mesa in [1], [2], [3] have established similar results, in the framework of Hankel transform on $(0, +\infty)$. In [13] Kamoun proves an analogous result for the Dunkl transform in one dimensional case.

Let us now describe the organization of our paper. In section 2, we recall some notions about Bessel functions on the cone Π_q and Bessel functions of two arguments. Next, we develop the basic Fourier analysis on the hypergroup Ξ_q.

Section 3 is devoted to the proof of generalized Sonine formula and to the study of the asymptotic behavior of the matrix Bessel functions in the neighborhood of 0 and infinity.

In section 4, we define the Bochner-Riesz mean σ_T^β where $T > 0$ and $\beta \geq 0$, as an operator on $L^1(\tilde{\omega}_\mu)$. Next, we express differently σ_T^β in terms of convolution operator on $L^1(\tilde{\omega_\mu})$: $\sigma_T^\beta = \phi_{T,\beta} \circ_\mu f$ where $\phi_{T,\beta}$ is up to a constant factor equals to a Bessel function. Therefore, thanks to properties of symmetric Bessel convolution we extend the definition of the operator σ_T^β to the spaces $L^p(\tilde{\omega}_\mu)$, $1 \leq p \leq +\infty$.

Next, we introduce symmetric Besov-Bessel spaces $B.B^{p,r}_{\alpha,\mu}$, $0 < \alpha < q$ and $1 \leq p, r < +\infty$ and then provide their characterizations using Bochner-Riesz means.

Throughout this paper we denote by

- $\mathcal{C}_c(\Xi_q)$ (resp. $\mathcal{C}_0(\Xi_q)$ the space of continuous compactly supported functions on Ξ_q ((resp. those continous on Ξ_q and going to 0 at infinity).
- $\theta = (q-1)\dfrac{d}{2} + 1$.
- C will denote a suitable positive constant not necessarily the same in each occurrence.

2 Preliminaries

2.1 Bessel function on the symmetric cone

In this subsection, we provide some relevant background on symmetric cone, in particular matrix cones, and about Bessel functions on such cone.

Consider $M_{p,q} = M_{p,q}(\mathbb{F})$ the space of $p \times q$ matrices over $\mathbb{F}$. Let $M_q = M_{q,q}$. It is a real algebra with the involution $x \to x^* = \overline{x}^t$. Let $H_q = H_q(\mathbb{F})$ the set of Hermitian $q \times q$ matrices over $\mathbb{F}$. It is a Euclidean vector space, its dimension over $\mathbb{R}$ is $n = q + \frac{d}{2}q(q-1)$. Endowed with the following Jordan product $x \circ y = \frac{1}{2}(xy + yx)$, $H_q(\mathbb{F})$ becomes a Euclidean Jordan algebra with unit $1 = I_q$, the unit matrix. The rank of H_q is q.

The set $\Omega_q = \Omega_q(\mathbb{F})$ of those matrices from H_q which are positive definite is a symmetric cone (see [4]). Let $G_q = GL(q, \mathbb{F})$ the group of all invertible $q \times q$ matrices over $\mathbb{F}$ and K_q the maximal subgroup of G_q which consists of all matrices k in M_q such that $k^*k = 1$. Finally let Π_q the set of positive matrices over $\mathbb{F}$.

A function or a measure on $M_{p,q}$ is said to be radial if it is invariant under the action of the group U_p from the left $U_p \times M_{p,q} \to M_{p,q}, (u, x) \mapsto ux$.

The mapping $U_p.x \mapsto \sqrt{x^*x}$ establishes a homeomorphism between the space of U_p-orbits in $M_{p,q}$ and the cone Π_q. Radial functions on $M_{p,q}$ can thus be considered as functions on the cone Π_q. Polar coordinates in $M_{p,q}$ are given as follows: Let

$$\Sigma_{p,q} = \{x \in M_{p,q},\, x^*x = 1\}$$

be the Stiefel manifold. Any matrix $x \in G_q$ has a unique decomposition $x = \sigma\sqrt{r}$ into polar coordinates where $\sigma \in \Sigma_{p,q}$ and $\sqrt{r}$ is the unique positive square root of $r = x^*x \in \Pi_q$. The maximal subgroup K_q acts on Π_q via conjugation $(k, r) \mapsto krk^{-1}$, and the orbits under this action are parameterized by the set Ξ_q of possible spectra $\sigma(r)$ of matrices $r \in \Pi_q$.

The following integration formula is a special case of [4, Theorem VI.2.3]. For integrable function $g : \Pi_q \to \mathbb{C}$,

$$\int_{\Pi_q} g(r)dr = \kappa_q \int_{\Xi_q} \int_{K_q} g(u\xi u^{-1})du \prod_{i<j}(\xi_i - \xi_j)^d d\xi \tag{1}$$

here $\kappa_q > 0$ a normalization constant, du the normalized Haar measure on K_q and $\xi \in \Xi_q$ is identified with the diagonal matrix $\mathrm{diag}(\xi_1, \ldots \xi_q) \in \Pi_q$.

Hypergeometric functions of matrix argument are certain real-analytic functions on H_q which are invariant under the maximal compact subgroup K_q of G_q, these functions can be expanded in terms of the zonal polynomials.

Let us recall some notations

Notations. We denote

- Δ the function defined on $M_q(\mathbb{F})$, by

$$\Delta(x) = (\det x)^\epsilon, \quad \text{and} \quad \epsilon = \begin{cases} 1, & \mathbb{F} = \mathbb{R}, \mathbb{C}, \\ 1/2, & \mathbb{F} = \mathbb{H}. \end{cases}$$

- For $1 \leq j \leq q$ and $s \in H_q$, $\Delta_j(s)$ is the principal minors of $\Delta(s)$ with respect to a fixed Jordan frame $\{e_1, \ldots e_q\}$ of H_q.
- For $\lambda \geq 0$ is a $q-$tuple $\lambda = (\lambda_1, \ldots, \lambda_q)$ of integers such that $\lambda_1 \geq \cdots \geq \lambda_q \geq 0$ and $|\lambda| = \lambda_1 + \ldots, +\lambda_q$ the weight of λ.
- For $\lambda = (\lambda_1, \cdots, \lambda_q) \in \mathbb{C}^q$, $\Delta_\lambda(s) = \Delta(s)^{\lambda_q} \prod_{j=1}^{q-1} \Delta_j(s)^{\lambda_j - \lambda_{j+1}}$, the power function. For $\lambda \geq 0$, Δ_λ is a homogeneous polynomial of degree $|\lambda|$, positive on Ω_q.
- $\mathcal{P}$ the space of all polynomials on ${H_q}^{\mathbb{C}}$, where ${H_q}^{\mathbb{C}}$ is the complexification of the simple euclidean Jordan algebra H_q.
- For $\lambda \geq 0$, let $\mathcal{P}_\lambda$ be the subspace of $\mathcal{P}$ generated by the polynomials $z \mapsto \Delta_\lambda(g^{-1}z)$, $g \in G_q$. The polynomials belonging to $\mathcal{P}_\lambda$ are homogeneous of degree $|\lambda|$, hence $\mathcal{P}_\lambda$ is finite dimensional; let $d_\lambda = dim\mathcal{P}_\lambda$.

- $d_*r = \Delta(r)^{-\theta}dr$, where dr is the restriction of the Lebesgue measure on H_q to Ω_q. Notice that d_*r is a G_q-invariant measure on Ω_q.

Let us recall some notions

Definition 2.1. 1. The gamma function of the symmetric cone Ω_q :

$$\Gamma_{\Omega_q}(z) = \int_{\Omega_q} e^{-tr\,r}\Delta_z(r)d_*r, \quad z \in \mathbb{C}^q,\ \Re z_j > \frac{d}{2}(j-1). \tag{2}$$

2. For $\lambda \geq 0$, the generalized Pochammer symbol :

$$(\mu)^\alpha_\lambda = \prod_{j=1}^{q} \big(\mu - \frac{1}{\alpha}(j-1)\big)_{\lambda_j}, \quad \mu \in \mathbb{C},\ \alpha \in \mathbb{R}^\star_+$$

where $(a)_j = a(a+1)\dots(a+j-1)$ is the standard Pochammer symbol.

3. The beta function of the symmetric cone Ω_q is defined for $u, v \in \mathbb{C}^q$ satisfying $\Re u_j,\ \Re v_j > (j-1)\frac{d}{2}$, by

$$\beta_{\Omega_q}(u, v) = \int_{0<r<1} \Delta_u(r)\Delta_{v-\theta}(1-r)d_*r.$$

4. The zonal polynomial ϕ_λ of weight λ :

$$\phi_\lambda(s) = \int_{K_q} \Delta_\lambda(ks)dk, \quad s \in H_q.$$

where dk is the normalized Haar measure on K_q.

5. The normalized zonal polynomial Z_λ of weight λ :

$$Z_\lambda(s) = d_\lambda \frac{|\lambda|!}{(\frac{n}{q})_\lambda}\phi_\lambda(s).$$

6. For arbitrary $\alpha > 0$ and a parameter $\mu \in \mathbb{C}$ with $\Re\mu > \frac{1}{\alpha}(q-1)$, the hypergeometric function ${}_0\mathrm{F}^\alpha_1(\mu; .)$ *on* $\mathbb{R}^q$ is defined by

$${}_0\mathrm{F}^\alpha_1(\mu;\xi) = \sum_{\lambda\geq 0} \frac{1}{|\lambda|!}\frac{1}{(\mu)^\alpha_\lambda}C^\alpha_\lambda(\xi).$$

where C^α_λ refer to Jack polynomial of index $\alpha > 0$ (see [12]).

Properties. (See [4] and [7]).

1. For $\alpha = \frac{2}{d}$, we note $(\mu)_\lambda^{\frac{2}{d}} = (\mu)_\lambda$. Then
$$(\mu)_\lambda = \frac{\Gamma_{\Omega_q}(\mu+\lambda)}{\Gamma_{\Omega_q}(\mu)}. \tag{3}$$
2. The following relation relies the gamma and beta functions :
$$\beta_{\Omega_q}(u,v) = \frac{\Gamma_{\Omega_q}(u)\Gamma_{\Omega_q}(v)}{\Gamma_{\Omega_q}(u+v)}. \tag{4}$$
3. The zonal polynomials ϕ_λ is the unique K_q-invariant function satisfying $\phi_\lambda(1) = 1$, $s \in H_q$, $k \in K_q$.
4. The zonal polynomials satisfy the product formula
$$\int_{K_q} Z_\lambda(\sqrt{r}ksk^{-1}\sqrt{r})dk = \frac{Z_\lambda(s)Z_\lambda(r)}{Z_\lambda(1)}, \quad r, s \in \Pi_q. \tag{5}$$
5. The value of Z_λ at $s \in H_q$ depend uniquely on the eigenvalues of s,
$$Z_\lambda(s) = Z_\lambda(\xi) = C_\lambda^{\frac{2}{d}}(\xi), \tag{6}$$
ξ is a diagonal matrix with the diagonal entries the eigenvalues of s.

Remarks.

1. For $m \in \mathbb{C}^q$ and $r \in \mathbb{C}$ we will write $m + r = (m_1 + r, \dots, m_q + r)$; with this notation $\Delta_r(x) = \Delta(x)^r$. Therefore as special case of (2), we obtain

$$\Gamma_{\Omega_q}(z) = \int_{\Omega_q} e^{-tr\, r} \Delta(r)^z d_* r, \quad z \in \mathbb{C},\ \Re z > \frac{d}{2}(q-1) = \theta - 1. \tag{7}$$

2. The notion $x < y$ for $x, y \in M_q(\mathbb{F})$ means that $y - x$ is (strictly) positive-definite.

3. The normalization of the zonal polynomial is such that

$$(tr\, s)^m = \sum_{|\lambda|=m} Z_\lambda(s), \quad s \in H_q. \tag{8}$$

4. In the statistical literature the symbol C_λ^α, refereing to the Jack polynomial of index $\alpha > 0$, is used rather than Z_λ for the zonal polynomial normalized by (8).

Definition 2.2. 1. For a complex number μ such that $(\mu)_\lambda \neq 0$ for all $\lambda \geq 0$, the Bessel function $\mathcal{J}_\mu$ associated with Ω_q in the sense of [4], is defined by

$$\mathcal{J}_\mu(x) = \sum_{\lambda \geq 0} (-1)^{|\lambda|} \frac{1}{|\lambda|!} \frac{1}{(\mu)_\lambda} Z_\lambda(x), \quad x \in H_q. \tag{9}$$

2. The Bessel functions of two arguments $x, y \in H_q$, is defined by

$$\mathcal{J}_\mu(x, y) = {}_0\mathrm{F}_1(\mu; i\xi, i\eta) = \sum_{\lambda \geq 0} (-1)^{|\lambda|} \frac{1}{|\lambda|!} \frac{1}{(\mu)_\lambda} \frac{Z_\lambda(x) Z_\lambda(y)}{Z_\lambda(1)}. \tag{10}$$

Properties.

1. For $x \in H_q$ with eigenvalues $\xi = (\xi_1, \dots, \xi_q)$, one has $\mathcal{J}_\mu(x) = {}_0\mathrm{F}_1^{2/\mathrm{d}}(\mu; -\xi)$.

2. If $q = 1$ then $\Pi_1 = \mathbb{R}_+$ and $\mathcal{J}_\mu$ is given by a usual one-variable Bessel function: $\mathcal{J}_\mu(\frac{x^2}{4}) = j_{\mu-1}(x)$, where $j_{\mu-1}(x) = {}_0\mathrm{F}_1(\mu; -\frac{x^2}{4})$.

3. The product formula (5) gives an integral representation for the Bessel function of two arguments

$$\mathcal{J}_\mu(r, s) = \int_{K_q} \mathcal{J}_\mu(\sqrt{r} k s k^{-1} \sqrt{r}) dk, \quad r, s \in \Pi_q. \tag{11}$$

4. For $\xi, \eta \in \Xi_q$, we have

$$\mathcal{J}_\mu(\frac{\xi^2}{2}, \frac{\eta^2}{2}) = \int_{K_q} \mathcal{J}_\mu(\frac{1}{4}\xi k \eta^2 k^{-1} \xi) dk. \tag{12}$$

2.2 Bessel function associated with root system B_q

Bessel functions associated with root systems are part of the theory of rational Dunkl operators which are initiated by C.F. Dunkl in the late nineteen-eighties. Let W be a finite reflection group on $\mathbb{R}^q$ with the usual euclidian scalar product $\langle ., . \rangle$ and let R be its reduced root system. A W-invariant function $k : R \to \mathbb{C}$ is called a multiplicity function on R. In the present paper, we shall be concerned with root system $B_q = \{\pm e_i,\ 1 \le i \le q\} \cup \{\pm e_i \pm e_j, 1 \le i < j \le q\}$. Each multiplicity on B_q is of the form $k = (k_1, k_2)$ where k_1 is the value on the roots $\pm e_i$ and k_2 is the value on the roots $\pm e_i \pm e_j$.

For a fixed multiplicity k, the associated (rational) Dunkl operators are given by

$$T_\xi(k) = \partial_\xi + \sum_{\alpha \in R_+} k(\alpha) \langle \alpha, \xi \rangle \frac{1 - \sigma_\alpha}{\langle \alpha, . \rangle}, \ \ \xi \in \mathbb{R}^q.$$

Here R_+ is a positive subsystem of R, σ_α denotes the reflection in the hyperplane perpendicular to α and the action of W is extended to functions on $\mathbb{R}^q$ in the usual way. The operators $T_\xi(k)$ commute and therefore generate a commutative algebra of differential-reflection operators on $\mathbb{R}^q$. For $k \ge 0$ and spectral parameter $\eta \in \mathbb{C}^q$, consider the so-called Bessel system

$$p(T(k))f = p(\eta) f \quad\ p \in \mathcal{P}^W; \quad f(0) = 1.$$

$\mathcal{P}^W$ denotes the subalgebra of W-invariant polynomials in $\mathcal{P}$, and $p(T(k))$ is the Dunkl operator associated with the polynomial $p(x) = p(x_1, \dots, x_q)$ which is obtained by replacing x_i by $T_{e_i}(k)$. As proven in [14], the Bessel system has a unique analytic W-invariant solution $\xi \mapsto J_k^W(\xi, \eta)$ which is called the symmetric Bessel function associated with R. In rank one, one obtains the one-variable Bessel functions $J_k(\xi, \eta) = j_{k-1/2}(i\xi\eta)$.

In the general case, J_k^W satisfies

$$J_k^W(\xi, \eta) = J_k^W(\eta, \xi) \tag{13}$$

and is W-invariant in both arguments.

Proposition 2.3. *(See [15, Proposition 4.5]) Let $k = (k_1, k_2) \ge 0$ and $k_2 > 0$. Let $J_k^{B_q}$ denote the Dunkl type Bessel function of type B_q and with multiplicity*

k. For $\xi = (\xi_1, \ldots, \xi_q) \in \mathbb{C}^q$ put $\xi^2 = (\xi_1^2, \ldots, \xi_q^2)$. Then for all $\xi, \eta \in \mathbb{C}^q$, we have

$$J_k^{B_q}(\xi, \eta) = {}_0\mathrm{F}_1^{\alpha}(\mu; \frac{\xi^2}{2}, \frac{\eta^2}{2}), \quad \alpha = \frac{1}{k_2},\ \mu = k_1 + (q-1)k_2 + \frac{1}{2}.$$

According to this proposition and (10), we can say that for $r, s \in \Pi_q$ with eigenvalues $\xi = (\xi_1, \ldots, \xi_q)$ and $\eta = (\eta_1, \ldots, \eta_q)$ respectively, we have

$$\mathcal{J}_\mu(\frac{r^2}{2}, \frac{s^2}{2}) = J_k^{B_q}(\xi, i\eta) \tag{14}$$

where k is given by $k = k_{\mu,d} = (k_1, k_2) = (\mu - \frac{d}{2}(q-1) - \frac{1}{2}, \frac{d}{2})$; see [15, Corollary 4.6].

2.3 Harmonic analysis on Ξ_q.

As we recall in the introduction, Rösler in [15] proves that Ξ_q was equipped with a hypergroup structure and we have :

• The Haar measure of the commutative hypergroup $(\Xi_q, \circ_\mu)$ is given by

$$d\tilde{\omega}_\mu(\xi) = d_\mu h_\mu(\xi) d\xi = d_\mu \prod_{i=1}^{q} \xi_i^{2\delta+1} \prod_{i<j} (\xi_i^2 - \xi_j^2)^d d\xi, \tag{15}$$

where $\delta = \mu - \theta$ and the constant $d_\mu > 0$ given by

$$d_\mu = \left(\int_{\Xi_q} h_\mu(\xi) e^{-|x|^2} dx \right)^{-1}.$$

• The dual space of $(\Xi_q, \circ_\mu)$ is parameterized by Ξ_q and consists of the functions

$$\psi_\xi^\mu(\eta) = \int_K \mathcal{J}_\mu(\frac{1}{4}\xi k \eta^2 k^{-1} \xi) dk = J_k^{B_q}(\xi, i\eta)$$

where the multiplicity k is given by $k = k_{\mu,d}$.

• The Bessel functions $J_k^{B_q}$ with $k = k_{\mu,d}$ satisfies the positive product formula

$$J_k^{B_q}(\xi, z) J_k^{B_q}(\eta, z) = \int_{\Xi_q} J_k^{B_q}(\zeta, z) d(\delta_\xi \circ \delta_\eta)(\zeta), \quad \xi, \eta \in \Xi_q,\ z \in \mathbb{C}^q. \tag{16}$$

• The symmetric Bessel translation is defined on $L^p(\tilde{\omega}_\mu)$ by

$$\tau_\eta(f)(\xi) = \int_{\Xi_q} f(\zeta) d(\delta_\xi \circ_\mu \delta_\eta)(\zeta), \tag{17}$$

where $d(\delta_\xi \circ_\mu \delta_\eta)(\zeta)$ is the convolution on the hyprergroup Ξ_q. (See [9] for details).

• If f and g are two measurable functions on Ξ_q, the symmetric Bessel convolution $f \circ_\mu g$ of f and g is defined in [9] by

$$f \circ_\mu g(\xi) = \int_{\Xi_q} \tau_\xi f(\eta) g(\eta) d\tilde{\omega}_\mu(\eta),\ a.e.\xi \in \Xi_q, \tag{18}$$

when the last integral has a sense.

• The symmetric Bessel transform on Ξ_q is defined by

$$\hat{f}(\eta) = \int_{\Xi_q} f(\xi) J_k^{B_q}(\xi, i\eta) d\tilde{\omega}_\mu(\xi).$$

We collect some properties from [9] that we need in this paper :

1. For all $\xi \in \Xi_q$, the operator τ_ξ can be extended to $L^p(\tilde{\omega}_\mu)$ $(p \geq 1)$ and for $f \in L^p(\tilde{\omega}_\mu)$ we have
$$\|\tau_\xi(f)\|_{p,\mu} \leq \|f\|_{p,\mu}\,. \tag{19}$$
2. Let f, g two measurable functions on Ξ_q and let $\xi \in \Xi_q$, then
$$\int_{\Xi_q} (\tau_\xi f)(\eta) g(\eta) d\tilde{\omega}_\mu(\eta) = \int_{\Xi_q} f(\eta)(\tau_\xi g)(\eta) d\tilde{\omega}_\mu(\eta) \tag{20}$$
3. For all $f \in L^1(\tilde{\omega}_\mu)$ and $g \in L^p(\tilde{\omega}_\mu)$, $1 \leq p < \infty$, we have
$$\tau_\eta(f \circ_\mu g) = \tau_\eta(f) \circ_\mu g = f \circ_\mu \tau_\eta(g),\ \eta \in \Xi_q. \tag{21}$$
4. For $p, r, s \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{s} = \frac{1}{r}$, the map $(f, g) \mapsto f \circ_\mu g$, defined on $C_c(\Xi_q) \times C_c(\Xi_q)$, extends to a continuous map from $L^p(\tilde{\omega}_\mu) \times L^s(\tilde{\omega}_\mu)$ to $L^r(\tilde{\omega}_\mu)$ and
$$\|f \circ_\mu g\|_{r,\mu} \leq \|f\|_{p,\mu} \|g\|_{s,\mu} \tag{22}$$
5. For all $f \in L^1(\tilde{\omega}_\mu)$ such that $\hat{f} \in L^1(\tilde{\omega}_\mu)$ we have the inversion formula
$$f(\eta) = \int_{\Xi_q} \hat{f}(\xi) J_k^{B_q}(\xi, i\eta) d\tilde{\omega}_\mu(\xi).$$

3 Generalized Sonine's formula and asymptotic behaviour for the symmetric Bessel function.

3.1 Generalized Sonine's formula

Theorem 3.1. *Let $\mu, \nu \in \mathbb{C}$, such hat $\Re\mu > (q-1)\frac{d}{2}$, $\Re\nu > (q-1)\frac{d}{2}$. Then*

$$\int_{0<r<1} \mathcal{J}_\mu(\sqrt{s}r\sqrt{s})\Delta(1-r)^{\nu-\theta}\Delta(r)^\mu d_*r = \beta_{\Omega_q}(\mu,\nu)\mathcal{J}_{\mu+\nu}(s),\ s \in \Pi_q. \tag{23}$$

Remark. The proof of the following Theorem was communicated to the authors by Prof. Jacques Faraut.

Proof. We shall use the same proof like in [4, Proposition XV1.4]. Using the relation (9) we shall compute the integral

$$\int_{0<r<1} Z_\lambda(\sqrt{s}r\sqrt{s})\Delta(1-r)^{\nu-\theta}\Delta(r)^\mu d_*r.$$

Substituting r by krk^{-1}, we can write

$$\int_{0<r<1} Z_\lambda(\sqrt{s}r\sqrt{s})\Delta(1-r)^{\nu-\theta}\Delta(r)^\mu d_*r$$
$$= \int_{0<r<1} Z_\lambda(\sqrt{s}krk^{-1}\sqrt{s})\Delta(1-r)^{\nu-\theta}\Delta(r)^\mu d_*r$$

Now integrating over K_q, using the invariance under K_q and the product formula (5), one obtains

$$\int_{0<r<1} Z_\lambda(\sqrt{s}r\sqrt{s})\Delta(1-r)^{\nu-\theta}\Delta(r)^\mu d_*r$$
$$= Z_\lambda(s)\int_{0<r<1} \frac{Z_\lambda(r)}{Z_\lambda(1)}\Delta(1-r)^{\nu-\theta}\Delta(r)^\mu d_*r.$$

From the definition of the beta function and the relations (3) and (4), the left hand side of the above equality is equals to

$$Z_\lambda(s)\beta_{\Omega_q}(\lambda+\mu,\nu) = Z_\lambda(s)\frac{(\mu)_\lambda}{(\mu+\nu)_\lambda}\frac{\Gamma_{\Omega_q}(\mu)\Gamma_{\Omega_q}(\nu)}{\Gamma_{\Omega_q}(\mu+\nu)}.$$

Finally to obtain (23) we use relation (9) and integrate term by term. □

Corollary 3.2. *Let $\mu, \nu \in \mathbb{C}$, such that $\Re\mu > (q-1)\frac{d}{2}$, $\Re\nu > (q-1)\frac{d}{2}$. Then*

$$\beta_{\Omega_q}(\mu,\nu)\mathcal{J}_{\mu+\nu}(\frac{\eta^2}{4}) = 2^q \frac{\kappa_q}{d_\mu} \int_{B_1} J_k^{B_q}(\xi, i\eta) \prod_{i=1}^{q} (1-\xi_i^2)^{\nu-\theta} d\tilde{\omega}_\mu(\xi). \tag{24}$$

Proof. Using relation (1) and the K_q-invariance of the determinant we obtain

$$\int_{0<r<1} \mathcal{J}_\mu(\sqrt{s}r\sqrt{s})\Delta(1-r)^{\nu-\theta}\Delta(r)^\mu d_* r =$$

$$\kappa_q \int_{B_1} \Delta(1-\xi)^{\nu-\theta}\Delta(\xi)^{\mu-\theta} \left(\int_{K_q} \mathcal{J}_\mu(\sqrt{\eta}k\xi k^{-1}\sqrt{\eta})dk \right) \prod_{i<j} (\xi_i - \xi_j)^d d\xi$$

where $\xi, \eta \in \Xi_q$ are the eigenvalues of r (resp. of s) identified with the diagonal matrix $diag(\xi_1, \ldots, \xi_q)$, (resp. $diag(\eta_1, \ldots, \eta_q)$ in Π_q.
Now using (11), we obtain

$$\int_{0<r<1} \mathcal{J}_\mu(\sqrt{s}r\sqrt{s})\Delta(1-r)^{\nu-\theta}\Delta(r)^\mu d_* r$$

$$= \kappa_q \int_{B_1} \mathcal{J}_\mu(\xi,\eta)\Delta(1-\xi)^{\nu-\theta} \prod_{i=1}^{q} \xi_i^\delta \prod_{i<j}(\xi_i - \xi_j)^d d\xi.$$

Finally, replacing ξ by ξ^2 and using relations (12),(14) and (15) we obtain the desired result. □

3.2 Asymptotic behavior for the Bessel function $\mathcal{J}_\mu$

We come back to polar coordinates on $M_{p,q}$: Let $f \in L^1(M_{p,q})$, and $\mu = \frac{pd}{2}$ then

$$\int_{M_{p,q}} f(x)dx = \frac{\pi^{\mu q}}{\Gamma_{\Omega_q}(\mu)} \int_{\Omega_q} \int_{\Sigma_{p,q}} f(\sigma\sqrt{r})\Delta^\mu(r) d_* r d\sigma$$

where $d\sigma$ denotes the unique U_p-invariant measure on $\Sigma_{p,q}$ normalized according to $\sigma(\Sigma_{p,q}) = 1$. Let ω_μ denote the measure on Π_q wich is obtained as the image measure of the normalized Lebesgue measure $(2\pi)^{-\mu q}dx$ on $M_{p,q}$ under the mapping $x \mapsto \sqrt{x^* x}$. Calculation in polar coordinates gives

$$\omega_\mu(f) = \frac{2^{-\mu q}}{\Gamma_{\Omega_q}(\mu)} \int_{\Omega_q} f(\sqrt{r})\Delta^\mu(r) d_* r. \tag{25}$$

Remark. If we consider the canonical mapping $\sigma : \Pi_q \to \Xi_q$, $r \to \sigma(r)$, where $\sigma(r) = (\xi_1, \ldots, \xi_q) \in \mathbb{R}^q$ is the set of eigenvalues of r ordered by size according to $\xi_1 \geq \cdots \geq \xi_q \geq 0$. Then the image measure of ω_μ under σ is $\tilde{\omega}_\mu = d_\mu h_\mu(\xi) d\xi$.

Now suppose that $F \in L^1(M_{p,q})$ is radial with $F(x) = f(\sqrt{x^* x})$, then the Fourier transform of F is also radial and given by

$$\hat{F}(t) = \frac{1}{(2\pi)^{\mu q}} \int_{M_{p,q}} F(x) e^{-i(t,x)} dx = \int_{\Pi_q} f(r) \left(\int_{\Sigma_{p,q}} e^{-i(t,\sigma r)} d\sigma \right) d\omega_\mu(r).$$

The inner integral over the Stiefel manifold can be expressed in terms of the Bessel function $\mathcal{J}_\mu$ on Ω_q with parameter $\mu = \frac{pd}{2}$. According to [4, Proposition XVI2.2], we have for all $x \in M_{p,q}$

$$\int_{\Sigma_{p,q}} e^{-i(\sigma|x)} d\sigma = \mathcal{J}_\mu(\frac{1}{4} x^* x), \ \mu = \frac{pd}{2}. \tag{26}$$

An asymptotic formula for the Bessel function $\mathcal{J}_\mu$ for $\mu = \frac{pd}{2}$ was given in [5] : Let $r = \sum_{j=1}^{q} \xi_j e_j$ be an element in Ω_q with distinct eigenvalues $\xi_1 > \xi_2 > \cdots > \xi_q (> 0)$, then as $t \to +\infty$,

$$\begin{aligned} \mathcal{J}_\mu(tr^2) &= \frac{\Gamma_{\Omega_q}(\mu)}{(4\pi)^{\frac{n}{2}}} \left(\frac{2}{t}\right)^{q(\mu - \frac{\theta}{2})} \sum_{\omega \in \mathbb{Z}_2^q} \left(|H(\sigma_\omega)|^{-\frac{1}{2}} e^{i(\frac{\pi}{4} s(\sigma_\omega) + it(\sigma_\omega r|\sigma_0))} \right) \\ &+ O(t^{-(q(\mu - \frac{\theta}{2})+1)}), \end{aligned}$$

where $\sigma_0 = \begin{pmatrix} I_q \\ 0 \end{pmatrix} \in M_{p,q}(\mathbb{F})$, $\sigma_\omega = \sigma_0 r_\omega$ with $r_\omega = \sum_{j=1}^q \omega_j e_j$, $H(\sigma_\omega)$ denotes the Hessian of the function $g(\sigma) = (r\sigma \mid \sigma_0)$ and takes the value

$$H(\sigma_\omega) = (-1)^{2\mu q - n} \prod_{i<j} \left(\frac{1}{2} (\omega_i \xi_i + \omega_j \xi_j) \right)^d \left(\prod_{i=1}^q \omega_i \xi_i \right)^{(2\mu - (q-1)d - 1)}$$

while $s(\sigma_\omega)$ denotes the signature of the Hessian matrix $H(\sigma_\omega)$ and is equal to

$$s(\sigma_\omega) = -\sum_{i=1}^q (2\mu - (i-1)d - 1)\omega_i.$$

For $\mu = \frac{pd}{2}$ with an integer $p \geq q$, we obtain from (26) that for all $x \in M_q$,

$$\mathcal{J}_\mu(x^* x) = \int_{\Sigma_{p,q}} e^{-2i(\sigma|\sigma_0 x)} d\sigma = \int_{\Sigma_{p,q}} e^{-2i(\tilde{\sigma}|x)} d\sigma$$

where $\tilde{\sigma} = \sigma_0^*\sigma$. If $p \geq 2q$, then according to [15, Corollary 3.2] this can be written as

$$\mathcal{J}_\mu(x^*x) = \frac{1}{\kappa_\mu}\int_{D_q} e^{-2i(v|x)}\Delta(1-v^*v)^{\mu-\rho}dv. \tag{27}$$

where $D_q = \{v \in M_q,\ v^*v < I\}$ and for $\mu \in \mathbb{C}$ with $\Re\mu > \rho - 1$,

$$\kappa_\mu = \int_{D_q}\Delta(1-v^*v)^{\mu-\rho}dv.$$

Analytic continuation with respect to μ shows that (27) remains valid for all $\mu \in \mathbb{C}$ with $\Re\mu > \rho - 1$.

If $x \neq 0$, the function $v \mapsto 2(v \mid x)$ has no critical points, so it follows from the Riemann-Lebesgue lemma for the additive group $(M_q, +)$ that $\mathcal{J}_\mu$ is in $C_0(\Pi_q)$. When $\mathbb{F} = \mathbb{R}$ the result goes back to Herz, see[8].

Proposition 3.3. *1. For $x \to 0$, we have for $x \in H_q$*

$$\mathcal{J}_\mu(x) = 1 - \frac{1}{\mu}tr(x) + O(|x|^2).$$

where $|x|^2 = (x \mid x)$.

2. *Let $t > 0$, then for all $r, s \in \Omega_q$ with distinct eigenvalues $\xi = (\xi_1, \cdots, \xi_q)$ with $\xi_1 > \xi_2 > \cdots > \xi_q (> 0)$ and $\eta = (\eta_1, \cdots, \eta_q)$ with $\eta_1 > \eta_2 > \cdots > \eta_q (> 0)$ respectively we have:*

$$\sqrt{\tilde{\omega}_\mu(\xi)\tilde{\omega}_\mu(\eta)}\,|J_k(\xi, it\eta)| \leq Ct^{-(q\mu - \frac{q}{2})}$$

where C is a constant not depending on t and $k = (k_1, k_2)$ is given in Proposition 2.3.

Proof. 1) It follows immediately from (8) and (9).
2) According to [10, Corollary 1], for a reflection group W and a corresponding Weyl chamber C attached with the positive subsystem R_+, we have : There exists a constant non-zero vector $(v_g)_{g\in W} \in \mathbb{C}^{|W|}$ such that for all $x, y \in C$ and $g \in W$,

$$\lim_{t\to\infty} t^\gamma e^{-it\langle x, gy\rangle}E_k(itx, gy) = \frac{v_g}{\sqrt{\omega_k(x)\omega_k(y)}}$$

where $\omega_k(x)$ is the weight function defined by

$$\omega_k(x) = \prod_{\alpha\in R_+}|\langle\alpha, x\rangle|^{2k(\alpha)}$$

which is $W-$invariant and homogenous of degree 2γ, with the index $\gamma := \gamma(k) = \sum_{\alpha\in R_+} k(\alpha) \geq 0$. Applying this result for our case $W = B_q$, we obtain according to Proposition 2.3 and (15) that $\omega_k(x) = \frac{1}{d_\mu}\tilde{\omega}_\mu(x)$, $\gamma = q\mu - \frac{q}{2}$ and so we can write

$$\sqrt{\omega_k(x)\omega_k(y)}E_k(itx, gy) \sim t^{-(q\mu-\frac{q}{2})}e^{it\langle x,gy\rangle}v_g, \quad \text{as} \quad t\to\infty$$

The following relation :

$$J_k(x, ity) = \frac{1}{|W|}\sum_{g\in W} E_k(itx, gy),$$

gives the result. □

4 Characterization of symmetric Besov-Bessel spaces.

4.1 The Bochner-Riesz means

In this section we define the Bochner-Riesz means σ_T^β, $T > 0$ and $\beta \geq 0$, as operators on $L^1(\tilde{\omega}_\mu)$. We prove that we may define σ_T^β on $L^p(\tilde{\omega}_\mu)$.

Definition 4.1. Let $T > 0$ be a real number, $\beta \geq 0$ and $\mu \in \mathcal{M}_q$. We define the Bochner-Riesz mean $\sigma_T^\beta f$ of a function $f \in L^1(\tilde{\omega}_\mu)$ by

$$\sigma_T^\beta f(\xi) = \frac{2^n}{\Gamma_{\Omega_q}(\mu)}\frac{\kappa_q}{d_\mu}\int_{B_T} J_k^{B_q}(\eta, i\xi)\hat{f}(\eta)\prod_{i=1}^{q}(1-\eta_i^2T^{-2})^\beta d\tilde{\omega}_\mu(\eta),\ \xi\in\Xi_q \quad (28)$$

For $T > 0$ and $\beta \in \mathbb{R}_+$, we consider the function

$$\Phi_{T,\beta}(\alpha) = 2^{n-q}T^{2\mu q}\frac{\beta_{\Omega_q}(\mu, \beta+\theta)}{\Gamma_{\Omega_q}(\mu)}\mathcal{J}_{\beta+\mu+\theta}(T^2\frac{\alpha^2}{4}), \quad (29)$$

where $\alpha = (\alpha_1, \ldots, \alpha_q) \in \Xi_q$, usually identified with $diag((\alpha_1, \ldots, \alpha_q) \in \Pi_q$.

According to (7) and (4), the function $\Phi_{T,\beta}(\alpha)$ is well defined for $\mu > \frac{d}{2}(q-1)$ and $\beta + \theta > \frac{d}{2}(q-1)$.

Proposition 4.2. *Let $f \in L^1(\tilde{\omega}_\mu)$, $\mu \in \mathcal{M}_q$ verifying $\mu > \frac{d}{2}(q-1)$. For $T > 0$ and $\beta+\theta > \frac{d}{2}(q-1)$, the Bochner-Riesz mean $\sigma_T^\beta f$ verifies the convolution relation*

$$\sigma_T^\beta f = \Phi_{T,\beta} \circ_\mu f. \quad (30)$$

Proof. From (28) and Fubini's theorem we get for $\xi \in \Xi_q$,

$$\sigma_T^\beta f(\xi) = \frac{2^n}{\Gamma_\Omega(\mu)}\frac{\kappa_q}{d_\mu}\int_{\Xi_q}\int_{B_T} J_k^{B_q}(\eta, i\xi)J_k^{B_q}(\lambda, i\eta)\prod_{i=1}^q(1-\frac{\eta_i^2}{T^2})^\beta d\tilde{\omega}_\mu(\eta)f(\lambda)d\tilde{\omega}_\mu(\lambda).$$

To make concise the formula, we introduce

$$I_{T,\xi}(\lambda) = \frac{2^n}{\Gamma_\Omega(\mu)}\frac{\kappa_q}{d_\mu}\int_{B_T} J_k^{B_q}(\xi, i\eta)J_k^{B_q}(\lambda, i\eta)\prod_{i=1}^q(1-\frac{\eta_i^2}{T^2})^\beta d\tilde{\omega}_\mu(\eta).$$

It follows from (13) that

$$\sigma_T^\beta f(\xi) = \int_{\Xi_q} I_{T,\xi}(\lambda)f(\lambda)d\tilde{\omega}_\mu(\lambda).$$

Using the change of variable $Tz = \eta$, we obtain

$$I_{T,\xi}(\lambda) = \frac{2^n}{\Gamma_\Omega(\mu)}\frac{\kappa_q}{d_\mu}T^{N_q}\int_{B_1} J_k(\xi, iTz)J_k(\lambda, iTz)\prod_{i=1}^q(1-z_i^2)^\beta d\tilde{\omega}_\mu(z),$$

where $N_q = dq^2 + (2\gamma + 2 - d)q = 2\mu q$.
Now (16) and again Fubini's theorem give

$$I_{T,\xi}(\lambda) = \frac{2^n}{\Gamma_\Omega(\mu)}\frac{\kappa_q}{d_\mu}T^{2\mu q}\int_{\Xi_q}\int_{B_1} J_k(\alpha, iTz)\prod_{i=1}^q(1-z_i^2)^\beta d\tilde{\omega}_\mu(z)d(\delta_\xi \circ_\mu \delta_\lambda)(\alpha).$$

Thanks to (24) and (29), we obtain

$$\Phi_{T,\beta}(\alpha) = \frac{2^n}{\Gamma_\Omega(\mu)}\frac{\kappa_q}{d_\mu}T^{2\mu q}\int_{B_1} J_k(\alpha, iTz)\prod_{i=1}^q(1-z_i^2)^\beta d\tilde{\omega}_\mu(z).$$

So from (17)

$$I_{T,\xi}(\lambda) = \int_{\Xi_q}\Phi_{T,\beta}(\alpha)d(\delta_\xi \circ_\mu \delta_\lambda)(\alpha) = \tau_\xi\Phi_{T,\beta}(\lambda).$$

Make use of (20) and (18) we easily get $\sigma_T^\beta f(\xi) = \Phi_{T,\beta} \circ_\mu f(\xi)$ as desired. $\square$

Lemma 4.3. *For $\mu \in \mathcal{M}_q$ such that $\mu + \beta + \theta > d(q-1) + 1$, we have*

$$\int_{\Xi_q}\Phi_{T,\beta}(\alpha)d\tilde{\omega}_\mu(\alpha) = 1.$$

Proof. It follows from (29), (6) and (25) that

$$\begin{aligned}
&\int_{\Xi_q} \Phi_{T,\beta}(\alpha) d\tilde{\omega}_\mu(\alpha) \\
&= 2^{n-q} T^{2\mu q} \frac{\beta_{\Omega_q}(\mu, \beta+\theta)}{\Gamma_{\Omega_q}(\mu)} \int_{\Xi_q} \mathcal{J}_{\beta+\mu+\theta}(\frac{\alpha^2}{4}) d\tilde{\omega}_\mu(\alpha) \\
&= 2^{n-q} \frac{\beta_{\Omega_q}(\mu, \beta+\theta)}{\Gamma_{\Omega_q}(\mu)} \int_{\Pi_q} \mathcal{J}_{\beta+\mu+\theta} \circ \sigma(\frac{r^2}{4}) d\omega_\mu(r) \\
&= 2^{n-q(\mu+1)} \frac{\Gamma_{\Omega_q}(\beta+\theta)}{\Gamma_{\Omega_q}(\beta+\mu+\theta)\Gamma_{\Omega_q}(\mu)} \int_{\Pi_q} \mathcal{J}_{\beta+\mu+\theta} \circ \sigma(\frac{r}{2}) \Delta^\mu(r) d_* r \\
&= \frac{\Gamma_{\Omega_q}(\beta+\theta)}{\Gamma_{\Omega_q}(\mu)\Gamma_{\Omega_q}(\beta+\mu+\theta)} \int_{\Pi_q} \mathcal{J}_{\beta+\mu+\theta}(r) \Delta^\mu(r) d_* r
\end{aligned}$$

Applying [4, Proposition XV4.5], we obtain

$$\int_{\Omega_q} \mathcal{J}_{\beta+\mu+\theta}(s) \Delta^\mu(s) d_* s = \frac{\Gamma_{\Omega_q}(\mu+2\rho)\Gamma_{\Omega_q}(\beta+\mu+\theta)}{\Gamma_{\Omega_q}(\beta+\theta)},$$

here $\rho = (\rho_1, \dots, \rho_q)$ where $\rho_i = \frac{d}{4}(2i-q-1)$.

From [4, Proposition XIV5.1], we get

$$\Gamma_{\Omega_q}(s+2\rho) = \Gamma_{\Omega_q}(s^*), \quad s = (s_1, \cdots, s_q) \in \mathbb{C}^q; \text{and} \quad s^* = (s_q, \cdots, s_1).$$

As μ is a real number so $\mu^* = \mu$ and then $\Gamma_{\Omega_q}(\mu+2\rho) = \Gamma_{\Omega_q}(\mu)$. Which complete the proof. $\square$

Let $f \in L^p(\tilde{\omega}_\mu)$, $1 \le p \le +\infty$ and $\mu \in \mathcal{M}_q$ such that $\mu+\beta+\theta > d(q-1)+1$. Since $\Phi_{T,\beta} \in L^1(\tilde{\omega}_\mu)$ we have by virtue of (22)

$$\|\Phi_{T,\beta} \circ_\mu f\|_{p,\mu} \le \|\Phi_{T,\beta}\|_{1,\mu} \|f\|_{p,\mu}$$

That suggest us to extend the definition of the operator σ_T^β to $L^p(\tilde{\omega}_\mu)$, $p \ge 1$, by the relation (30).

Lemma 4.4. *Let $f \in L^p(\tilde{\omega}_\mu)$ for some $1 \le p < \infty$, and $\mu \in \mathcal{M}_q$ such that $\mu+\beta+\theta > d(q-1)+1$. Then*

1. $\sigma_T^\beta f(\xi) \longrightarrow f(\xi)$, *as* $T \to +\infty$, *a. e.* $\xi \in \Xi_q$.
2. $\sigma_T^\beta f(\xi) \longrightarrow 0$, *as* $T \to 0^+$.

Proof. 1) It follows by an analog proof as in the case of ordinary Fourier transform.(See [16]).
2) By virtue of relationship (22) we have

$$\left|\sigma_T^{\beta} f(\xi)\right| \leq \|\Phi_{T,\beta}\|_{r,\mu} \|f\|_{p,\mu}$$

with $\frac{1}{r} + \frac{1}{p} = 1$. But $\|\Phi_{T,\beta}\|_{r,\mu} = C.T^{\frac{2\mu q}{p}}$ where C is a positive constant not depending on T , consequently $\sigma_T^{\beta} f(\xi) \longrightarrow 0$, uniformly in $\xi \in \Xi_q$. □

Lemma 4.5. *Let $T > 0$, $\mu \in \mathcal{M}_q$ such that $\mu + \beta > (q-1)\frac{d}{2}$ and $1 \leq p < \infty$. For every function $f \in L^p(\tilde{\omega}_\mu)$, we have*

$$f(\xi)(Log2) = \int_0^\infty \left[\sigma_{2T}^{\beta} f(\xi) - \sigma_T^{\beta} f(\xi)\right] \frac{dT}{T} \ a.e.\ \xi \in \Xi_q. \tag{31}$$

Proof. Let $T > 0$ we can write

$$\sigma_{2T}^{\beta} f(\xi) - \sigma_T^{\beta} f(\xi) \quad = \quad \int_T^{2T} \frac{d}{dt} \sigma_t^{\beta} f(\xi) dt.$$

Integrating both sides and using Fubini's theorem we obtain

$$\begin{aligned}\int_0^\infty \left[\sigma_{2T}^{\beta} f(\xi) - \sigma_T^{\beta} f(\xi)\right] \frac{dT}{T} \quad &= \quad \int_0^\infty \frac{d}{dt}\left\{\sigma_t^{\beta} f(\xi)\right\} \left(\int_{t/2}^{t} \frac{dT}{T}\right) dt \\ &= \quad (Log2)_0^\infty \frac{d}{dt}\left\{\sigma_t^{\beta} f(\xi)\right\} dt.\end{aligned}$$

Applying Lemma 4.4, we get

$$\int_0^\infty \frac{d}{dt}\left\{\sigma_t^{\beta} f(\xi)\right\} dt = f(\xi), \quad a.e.\ \xi \in \Xi_q.$$

Our proof is now complete. □

4.2 Symmetric Besov-Bessel spaces

We are going to establish an analogous of [1, Theorem 2.1].

Theorem 4.6. *Let $T > 0$, $0 < \alpha < q$, $1 \leq p, r < \infty$, $\mu \in \mathcal{M}_q$, $-\frac{q}{2} < \mu q < (\beta + \theta - \frac{1}{2})q - \alpha$ and $f \in L^p(\tilde{\omega}_\mu)$. The following three properties are equivalent*

1. $f \in BB_{\alpha,\mu}^{p,r}$.
2. $T^{\alpha} \left\|\sigma_T^{\beta}(f) - f\right\|_{p,\mu} \in L^r((0,\infty), \frac{dT}{T})$.

3. $T^{\alpha}\left\|\sigma^{\beta}_{2T}(f)-\sigma^{\beta}_{T}(f)\right\|_{p,\mu}\in L^{r}((0,\infty),\frac{dT}{T})$.

Proof. 1) $\Rightarrow$ 2) Let $T>0$, by Lemma 4.3 together with (30), we can write

$$\sigma^{\beta}_{T}f(\xi)-f(\xi)=\int_{\Xi_q}\Phi_{T,\beta}(\eta)(\tau_{\eta}f(\xi)-f(\xi))d\tilde{\omega}_{\mu}(\eta),\ \xi\in\Xi_q.$$

Using the generalized Minkowski inequality, we spilt

$$\left\|\sigma^{\beta}_{T}(f)-f\right\|_{p,\mu}\leq\int_{\Xi_q}|\Phi_{T,\beta}(\eta)|\,\Lambda_p(f,\|\eta\|)d\tilde{\omega}_{\mu}(\eta)=I_1+I_2,$$

where

$$I_1=\int_0^{\frac{1}{T}}|\Phi_{T,\beta}(\eta)|\,\Lambda_p(f,\|\eta\|)d\tilde{\omega}_{\mu}(\eta)\ ;\quad I_2=\int_{\frac{1}{T}}^{+\infty}|\Phi_{T,\beta}(\eta)|\,\Lambda_p(f,\|\eta\|)d\tilde{\omega}_{\mu}(\eta).$$

Now according Proposition 3.3.(1) and (29), we get

$$\begin{aligned}I_1&\leq CT^{2\mu q}\int_0^{\frac{1}{T}}\Lambda_p(f,\|\eta\|)d\tilde{\omega}_{\mu}(\eta)\\&\leq CT^{2\mu q}\int_0^{\frac{1}{T}}\Lambda_p(f,\eta_1)\eta_1^{(2\delta+1)q+dq(q-1)}d\eta_1\\&\leq CT^{q}\int_0^{\frac{1}{T}}\Lambda_p(f,\eta_1)d\eta_1.\end{aligned}$$

To estimate I_2, we may use Proposition 3.3.(2), (9), (10), (14) and (29) to obtain

$$I_2\leq CT^{(\mu-\beta-\theta)q+\frac{q}{2}}\int_{\frac{1}{T}}^{+\infty}\Lambda_p(f,\eta_1)\eta_1^{(\mu-\beta-\theta)q-\frac{q}{2}}d\eta_1.$$

Arguing as in [6, Lemma 6] and [1, Lemma 2.2], we deduce that

$$\begin{aligned}&\left[\int_0^{\infty}\left(T^{\alpha}\|\sigma^{\beta}_{T}(f)-f\|_{p,\mu}\right)^{r}\frac{dT}{T}\right]^{\frac{1}{r}}\\&\leq\ C\left[\left|T^{\alpha+q}\int_0^{\frac{1}{T}}\Lambda_p(f,\eta_1)d\eta_1\right|^{r}\frac{dT}{T}\right]^{\frac{1}{r}}\\&+\ C\left[\left|T^{\alpha+(\mu-\beta-\theta)q+\frac{q}{2}}\int_{\frac{1}{T}}^{\infty}\Lambda_p(f,\eta_1)\eta_1^{(\mu-\beta-\theta)q-\frac{q}{2}}d\eta_1\right|^{r}\frac{dT}{T}\right]^{\frac{1}{r}}\\&\leq\ C\left[\int_0^{\infty}(\frac{\Lambda_p(f,t)}{t^{\alpha}})^{r}\frac{dt}{t}\right]^{\frac{1}{r}}.\end{aligned}$$

2) $\Rightarrow$ 3) is a consequence of the following inequality

$$\left\|\sigma_{2T}^{\beta}(f)-\sigma_{T}^{\beta}(f)\right\|_{p,\mu}\leq\left\|\sigma_{2T}^{\beta}(f)-f\right\|_{p,\mu}+\left\|\sigma_{T}^{\beta}(f)-f\right\|_{p,\mu}.$$

3) $\Rightarrow$ 1) We set for a function $f\in L^{p}(\tilde{\omega}_{\mu})$

$$\delta(f,\xi,t)=\tau_{tu}f(\xi)-f(\xi),\ \xi,\in\Xi_{q},t>0,\ u=(1,0,\cdots,0).$$

Since τ_{ξ} is a bounded operator in $L^{p}(\tilde{\omega}_{\mu})$ for all $\xi\in\Xi_{q}$ then according to (31), we can write for all $t>0$ and almost every where $\xi\in\Xi_{q}$,

$$\delta(f,\xi,t)(Log2)=\int_{0}^{\infty}\left[\sigma_{2T}^{\beta}\delta(f,.,t)(\xi)-\sigma_{T}^{\beta}\delta(f,.,t)(\xi)\right]\frac{dT}{T}.$$

Now thanks to the relation (30), we can write

$$\delta(f,\xi,t)(Log2)=\int_{0}^{\infty}(\Phi_{2T,\beta}-\Phi_{T,\beta})\circ_{\mu}\delta(f,.,t)(\xi)\frac{dT}{T}.$$

Hence (21) gives

$$\delta(f,\xi,t)(Log2)=\int_{0}^{\infty}\delta(\sigma_{2T}^{\beta}(f)-\sigma_{T}^{\beta}(f),\xi,t)\frac{dT}{T}.$$

By the generalized Minkowski inequality we have

$$\Lambda_{p}(f,t)(Log2)\leq\int_{0}^{\infty}\left\|\delta(\sigma_{2T}^{\beta}(f)-\sigma_{T}^{\beta}(f),.,t)\right\|_{p,\mu}\frac{dT}{T}.$$

From (19), we get obviously

$$\left\|\delta(\sigma_{2T}^{\beta}(f)-\sigma_{T}^{\beta}(f),.,t)\right\|_{p,\mu}\leq 2\left\|\sigma_{2T}^{\beta}(f)-\sigma_{T}^{\beta}(f)\right\|_{p,\mu}. \tag{32}$$

On the other hand, using the same techniques used in Bernstein's inequality in [9, Lemma 3.6], we can write

$$\left\|\delta(\sigma_{2T}^{\alpha}(f)-\sigma_{T}^{\beta}(f),.,t)\right\|_{p,\mu}\leq CtT\left\|\sigma_{2T}^{\beta}(f)-\sigma_{T}^{\beta}(f)\right\|_{p,\mu}. \tag{33}$$

Combining (32), (33) and the generalized Minkowski inequality it follows that

$$\Lambda_{p}(f,t)\leq C\left\{\int_{0}^{\frac{1}{t}}t\left\|\sigma_{2T}^{\beta}(f)-\sigma_{T}^{\beta}(f)\right\|_{p,\mu}dT+\int_{\frac{1}{t}}^{\infty}\left\|\sigma_{2T}^{\beta}(f)-\sigma_{T}^{\beta}(f)\right\|_{p,\mu}\frac{dT}{T}\right\},t>0$$

From [6, Lemma 4] it deduces

$$\left\{\int_{0}^{\infty}(\frac{\Lambda_{p}(f)(t)}{t^{\alpha}})^{r}\frac{dt}{t}\right\}^{\frac{1}{r}}\leq C\left\{\int_{0}^{\infty}(T^{\alpha}\left\|\sigma_{2T}^{\beta}(f)-\sigma_{T}^{\beta}(f)\right\|_{p,\mu})^{r}\frac{dT}{T}\right\}^{\frac{1}{r}},$$

so $f\in BB_{\alpha,\mu}^{p,r}$. $\square$

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