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Equations with Solution in Terms of Fibonacci and Lucas Sequences

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Dedicated to the memory of Ion Cucurezeanu (1936-2012)

Abstract

The main results characterize the equations (2.1) and (2.10) whose solutions are linear combinations with rational coefficients of at most two terms of classical Fibonacci and Lucas sequences.

1 Special Pell's equations

The Diophantine equations

$$u^2 - Dv^2 = \pm 4 \tag{1.1}$$

are called **special Pell's equations**. In some ways, solutions to (1.1) are more fundamental than solutions to Pell's equations $u^2 - Dv^2 = \pm 1$. For general aspects concerning the theory of the positive and negative Pell's equation we refer to the authors book [2]. As with the ± 1 equations, all solutions to (1.1) can be generated from the minimal positive solution.

If (u_1, v_1) is the minimal positive solution to Pell's special equation

$$u^2 - Dv^2 = 4,$$

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then for the n^{th} solution (u_n, v_n) we have

$$\frac{1}{2}(u_n + v_n\sqrt{D}) = \left(\frac{u_1 + v_1\sqrt{D}}{2}\right)^n. \tag{1.2}$$

The general solution (1.2) is completed with the trivial solution $(u_0, v_0) = (2, 0)$, obtained for $n = 0$. Also, it can be extended to negative integers n.

We can express all integer solutions to equation (1.1) by the following formula

$$\frac{1}{2}(u_n + v_n\sqrt{D}) = \varepsilon_n \left(\frac{u_1 + v_1\sqrt{D}}{2}\right)^n, \quad n \in \mathbb{Z}, \tag{1.3}$$

where ε_n is 1 or -1. Indeed, for $n > 0$ and $\varepsilon_n = 1$ we get all negative solutions. For $n > 0$ and $\varepsilon_n = -1$ we obtain all solutions (u_n, v_n) with u_n and v_n negative. For $n < 0$ and $\varepsilon_n = 1$ we have (u_n, v_n) with $u_n > 0$ and $v_n < 0$, while $n < 0$ and $\varepsilon_n = -1$ gives $u_n < 0$ and $v_n > 0$. The trivial solutions $(2, 0)$ and $(-2, 0)$ are obtained for $n = 0$.

Formula (1.3) captures all symmetries of equation $(u, v) \to (-u, -v)$, $(u, v) \to (u, -v)$, $(u, v) \to (-u, v)$. Therefore, in 2D the points (u_n, v_n) represents the orbits of the action of the Klein four-group $\mathbb{Z}_2 \times \mathbb{Z}_2$, i.e. points obtained by the 180 degree rotation, the vertical reflection, and by the horizontal reflection.

Now suppose the "negative" special Pell's equation $u^2 - Dv^2 = -4$ has solutions and let (u_1, v_1) be its minimal positive solution. Define the integers u_n, v_n by (1.3). Then, for n odd, (u_n, v_n) is a solution to $u^2 - Dv^2 = -4$, and for n even, (u_n, v_n) is a solution to $u^2 - Dv^2 = 4$. All integer solutions to these equations are generated in this way.

2 The main results

In this section we find all integers a for which the solutions to the quadratic equations

$$x^2 + axy + ay^2 = \pm 1, \tag{2.1}$$

are linear combinations with rational coefficients of the classical Fibonacci and Lucas numbers: $F_0 = 0$, $F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$, $n \geq 1$ and $L_0 = 2$, $L_1 = 1$, $L_{n+1} = L_n + L_{n-1}$, $n \geq 1$.

This problem is related to the so-called Diophantine representation of a sequence of integers, and for some results we refer to the papers of M.J. DeLeon [3], V.E. Hoggatt, M. Bicknell-Johnson [6], J.P. Jones [7], [8], [9], and W.L. McDaniel [11]. Also, it is connected to the Y.V. Matiyasevich and J. Robertson way to solve the Hilbert's Tenth Problem, and it has applications to the problem of singlefold Diophantine representation of recursively enumerable sets. In the recent paper of R. Keskin, N. Demiturk [10] the equations $x^2 - kxy + y^2 = 1, x^2 - kxy - y^2 = 1$ are solved in terms of generalized Fibonacci and Lucas numbers. Let us mention that S. Halici [4] defines Hankel matrices involving the Pell, Pell-Lucas and modified Pell sequences, computes their Frobenius norm, and investigate some spectral properties of them.

Recall the Binet's formulas for F_n and L_n:

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right], \tag{2.2}$$

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n. \tag{2.3}$$

These formulas can be extended to negative integers n in a natural way. We have $F_{-n} = (-1)^{n-1}F_n$ and $L_{-n} = (-1)^n L_n$, for all n.

Theorem 2.1. *The solutions to the positive equation (2.1) are linear combinations with rational coefficients of at most two Fibonacci and Lucas numbers if and only if* $a = a_n = \pm L_{2n} + 2$, $n \geq 1$.

For each n*, all of its integer solutions* (x_k, y_k) *are given by*

$$\begin{cases} x_k = \dfrac{\varepsilon_k}{2} L_{2kn} \mp \dfrac{a_n}{2F_{2n}} F_{2kn} \\ y_k = \pm \dfrac{1}{F_{2n}} F_{2kn}, \end{cases} \tag{2.4}$$

where $k \geq 1$*, signs* $+$ *and* $-$ *depend on* k *and correspond, while* $\varepsilon_k = \pm 1$.

Proof. The equation $x^2 + axy + ay^2 = 1$ is equivalent to the positive special Pell's equation

$$(2x + ay)^2 - (a^2 - 4a)y^2 = 4 \tag{2.5}$$

From (1.3) it follows that

$$2x_m + ay_m = \varepsilon_m \left[\left(\frac{u_1 + v_1\sqrt{D}}{2}\right)^m + \left(\frac{u_1 - v_1\sqrt{D}}{2}\right)^m\right]$$

and

$$y_m = \frac{\varepsilon_m}{\sqrt{D}}\left[\left(\frac{u_1 + v_1\sqrt{D}}{2}\right)^m - \left(\frac{u_1 - v_1\sqrt{D}}{2}\right)^m\right],$$

where $m \in \mathbb{Z}$, $\varepsilon_m = \pm 1$, $D = a^2 - 4a$, and (u_1, v_1) is the minimal positive solution to $u^2 - Dv^2 = 4$. we have $(u_1, v_1) = (a-2, 1)$, and combining the above relations it follows

$$x_m = \frac{\varepsilon_m}{2}\left[\left(1 - \frac{a}{\sqrt{a^2-4a}}\right)\left(\frac{a-2+\sqrt{a^2-4a}}{2}\right)^m + \left(1 + \frac{a}{\sqrt{a^2-4a}}\right)\left(\frac{a-2-\sqrt{a^2-4a}}{2}\right)^m\right] \tag{2.6}$$

and

$$y_m = \frac{\varepsilon_m}{\sqrt{a^2-4a}}\left[\left(\frac{a-2+\sqrt{a^2-4a}}{2}\right)^m - \left(\frac{a-2-\sqrt{a^2-4a}}{2}\right)^m\right]. \tag{2.7}$$

Taking into account Binet's formulas, solution (x_m, y_m) is representable in terms of F_m and L_m only if $a^2 - 4a = 5s^2$, for some positive integer s. This is equivalent to the special Pell's equation

$$(a-2)^2 - 5s^2 = 4, \tag{2.8}$$

whose minimal solution is $(a_1 - 2, s_1) = (3, 1)$. The general integer solution to (2.8) is

$$a_n - 2 = \varepsilon_n\left[\left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n\right] = \varepsilon_n L_{2n},$$

and

$$s_n = \frac{\varepsilon_n}{\sqrt{5}}\left[\left(\frac{3+\sqrt{5}}{2}\right)^n - \left(\frac{3-\sqrt{5}}{2}\right)^n\right] = \varepsilon_n F_{2n},$$

where n is an integer and $\varepsilon_n = \pm 1$. The equation (2.8) was also used by D. Savin [12] in the study of the equation $x^4 - 6x^2y^2 + 5y^4 = 16F_{n-1}F_{n+1}$, and R. Keskin, N. Demiturk [10].

From $(2x + ay)^2 - (a^2 - 4a)y^2 = 4$ we find $(2x + a_n y)^2 - 5(s_n y)^2 = 4$, with integer solution (x_m, y_m) given by

$$2x_m + a_n y_m = \varepsilon_{2m} L_{2m} \quad \text{and} \quad s_n y_m = \pm F_{2m}.$$

Hence

$$x_m = \frac{1}{2}\left[\varepsilon_{2m} L_{2m} \mp a_n \frac{F_{2m}}{F_{2n}}\right], \quad y_m = \pm \frac{F_{2m}}{F_{2n}}, \tag{2.9}$$

where signs + and − correspond, and $\varepsilon_{2m} = \pm 1$.

Taking into account that F_{2n} divides F_{2m} if and only if n divides m (see [1, pp. 180] and [4, pp. 39]), it is necessary that $m = kn$, for some positive integer k. Formulas (2.9) become (2.4).

A parity argument shows that in the equation

$$(x + ay)^2 - (a^2 - 4a)y^2 = 4,$$

x is even, so x_k in (2.4) is always an integer. □

The following two tables give the integer solutions to equation (2.1) at level k, including the trivial solution obtained for $k = 0$.

n	$a_n = L_{2n} + 2$	Equation (2.1)	Solutions
1	5	$x^2 + 5xy + 5y^2 = 1$	$x = \frac{\varepsilon_k}{2} L_{2k} \mp \frac{5}{2} F_{2k},\ y = \pm F_{2k}$
2	9	$x^2 + 9xy + 9y^2 = 1$	$x = \frac{\varepsilon_k}{2} L_{4k} \mp \frac{5}{6} F_{4k},\ y = \pm \frac{1}{3} F_{4k}$
3	20	$x^2 + 20xy + 20y^2 = 1$	$x = \frac{\varepsilon_k}{2} L_{6k} \mp \frac{5}{4} F_{6k},\ y = \pm \frac{1}{8} F_{6k}$
4	49	$x^2 + 49xy + 49y^2 = 1$	$x = \frac{\varepsilon_k}{2} L_{8k} \mp \frac{7}{6} F_{8k},\ y = \pm \frac{1}{21} F_{8k}$
5	125	$x^2 + 125xy + 125y^2 = 1$	$x = \frac{\varepsilon_k}{2} L_{10k} \mp \frac{25}{22} F_{10k},\ y = \pm \frac{1}{55} F_{10k}$
6	324	$x^2 + 324xy + 324y^2 = 1$	$x = \frac{\varepsilon_k}{2} L_{12k} \mp \frac{9}{8} F_{12k},\ y = \pm \frac{1}{144} F_{12k}$

n	$a_n = -L_{2n} + 2$	Equation (2.1)	Solutions
1	-1	$x^2 - xy - y^2 = 1$	$x = \frac{\varepsilon_k}{2} L_{2k} \pm \frac{1}{2} F_{2k},\ y = \pm F_{2k}$
2	-5	$x^2 - 5xy - 5y^2 = 1$	$x = \frac{\varepsilon_k}{2} L_{4k} \pm \frac{5}{6} F_{4k},\ y = \pm \frac{1}{3} F_{4k}$
3	-16	$x^2 - 16xy - 16y^2 = 1$	$x = \frac{\varepsilon_k}{2} L_{6k} \pm F_{6k},\ y = \pm \frac{1}{8} F_{6k}$
4	-45	$x^2 - 45xy - 45y^2 = 1$	$x = \frac{\varepsilon_k}{2} L_{8k} \pm \frac{15}{14} F_{8k},\ y = \pm \frac{1}{21} F_{8k}$
5	-121	$x^2 - 121xy - 121y^2 = 1$	$x = \frac{\varepsilon_k}{2} L_{10k} \mp \frac{11}{10} F_{10k},\ y = \pm \frac{1}{55} F_{10k}$
6	-320	$x^2 - 320xy - 320y^2 = 1$	$x = \frac{\varepsilon_k}{2} L_{12k} \mp \frac{10}{9} F_{12k},\ y = \pm \frac{1}{144} F_{12k}$

Next we will consider the "negative" equation of the type (2.1):

$$x^2 + axy + ay^2 = -1 \tag{2.10}$$

Similar "negative" type equations

$$x^2 - kxy - y^2 = -1,\ x^2 - kxy - y^2 = \pm(k^2 + 4)$$

are considered by R. Keskin, N. Demiturk [10].

Unlike the result in Theorem 2.1, there are only two values of a for which the corresponding property holds.

Theorem 2.2. *The solutions to the negative equation (2.10) are linear combinations with rational coefficients of at most two Fibonacci and Lucas numbers if and only if $a = -1$ or $a = 5$.*

If $a=-1$, all of its integer solutions (x_m, y_m) are given by

$$x_m = \frac{\varepsilon_m}{2} L_{2m+1} \pm \frac{1}{2} F_{2m+1}, \quad y_m = \pm F_{2m+1}, \quad m \geq 0. \tag{2.11}$$

If $a=5$, all integer solutions (x_m, y_m) are

$$x_m = \frac{\varepsilon_m}{2} L_{2m+1} \mp 5F_{2m+1}, \quad y_m = \pm F_{2m+1}, \quad m \geq 0. \tag{2.12}$$

The signs $+$ and $-$ depend on m and correspond, while $\varepsilon_m = \pm 1$.

Proof. As in the proof of Theorem 2.1 the equation is equivalent to

$$(2x+ay)^2 - (a^2-4a)y^2 = -4.$$

Suppose that this negative special Pell's equation is solvable. Its solution (x_m, y_m) is representable in terms of Fibonacci and Lucas numbers as a linear combination with rational coefficients only if $a^2-4a=5s^2$. As in the proof of Theorem 2.1 we obtain $a_n = \pm L_{2n}+2$ and $s_n = \pm F_{2n}$, $n \geq 1$.

The equation $(2x+ay)^2-(a^2-4a)y^2=-4$ becomes

$$(2x+ay)^2 - 5(s_n y)^2 = -4,$$

whose integer solutions are $2x_m + ay_m = \varepsilon_m L_{2m+1}$ and $s_n y_m = \pm F_{2m+1}$. It follows that

$$y_m = \pm \frac{F_{2m+1}}{F_{2n}}, \quad m \geq 1.$$

If $n \geq 2$, then $F_{2n} \geq 2$, and since $2n$ does not divide $2m+1$, it follows that F_{2n} does not divide F_{2m+1} (see [1], pp. 180 and [5], pp. 39), hence y_m is not an integer.

Thus $n=1$ and so $a = \pm L_2 + 2$, i.e. $a=-1$ or $a=5$.

For $a=-1$, it follows $y_m = \pm F_{2m+1}$ and $2x_m - y_m = \varepsilon_m L_{2m+1}$, and we obtain solutions (2.11).

If $a=5$, then $y_m = \pm F_{2m+1}$ and $2x_m + 5y_m = \varepsilon_m L_{2m+1}$, yielding the solutions (2.12). □

Remark. On the other hand, it is more or less known Zeckendorf's theorem in [13], which states that every positive integer can be represented uniquely as the sum of one or more distinct Fibonacci numbers in such a way that the sum does not include two consecutive Fibonacci numbers. Such a sum is called *Zeckendorf representation* and it is related to the Fibonacci coding of a positive integer. Our results are completely different, because the number of terms is reduced to at most two, and the sum in the representation of solutions is a linear combination with rational coefficients.

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