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A DUAL APPROACH TO SOLVING LINEAR INEQUALITIES BY UNCONSTRAINED QUADRATIC OPTIMIZATION

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To Professor Silviu Sburlan, at his 60's anniversary

Abstract

The method used to obtain the minimum-norm solution of a large-scale system of linear inequalities proceeds by solving a finite number of smaller unconstrained subproblems. Such a subproblem has the form of optimizing a quadratic function, which is easily solved. The case when the vector b is perturbed is also included.

1. INTRODUCTION

Consider the following problem:

$$\begin{cases} \min \frac{1}{2} \|x\|^2 \\ Ax \le b, \end{cases} \tag{1}$$

where $A \in M_{m\times n}$, $b \in \Re^m$, $x \in \Re^n$, $\|.\|$ stands for the Euclidean norm in $\Re^n$. The constraints of the problem may be also written as

$$a_i^T x \le b_i, i \in \{1, 2, ..., m\},$$

where a_i^T forms the i^{th} row of the matrix A and b_i is the i^{th} component of b. The Lagrangean associated to problem (1) is:

$$\psi(x, u) = \frac{1}{2} x^T x + u^T (Ax - b),\ u \in \Re^m. \tag{2}$$

By the Kuhn-Tucker (K-T) theorem, the necessary and sufficient condition for $\widehat{x} \in \Re^n$ to be the optimal solution of problem (1), is to exist $\widehat{u} \in \Re^m$ such that:

$$A\widehat{x} \le b \tag{3}$$

$$\begin{cases} \widehat{u} \geq 0 \\ \widehat{x} + A^T\widehat{u} = 0 \end{cases} \tag{3.1}$$

$$\widehat{u}^T(A\widehat{x} - b) = 0 \text{ (ecart conditions).} \tag{3.2}$$

We will write the dual problem to problem (1), which is also a quadratic programming problem, but the constraints can be brought to a simpler form, i. e. to conditions of nonnegativity for dual variables. The Lagrangean (2) is the objective function of the dual and the K-T conditions (3.1) are its constraints:

$$\begin{cases} \max_{(x,u)}\psi(x,u) \\ x + A^T u = 0 \\ u \geq 0. \end{cases} \tag{4}$$

By the duality theorem, if $(\widehat{x}, \widehat{u})$ is an optimal solution for the problem (4) then $\widehat{x}$ is the optimal solution of the problem (1).Writting the Lagrangean (2) under the form :

$$\psi(x,u) = -\frac{1}{2}x^T x + x^T(x + A^T u) - b^T u$$

and substituting

$$x = -A^T u, \tag{5}$$

it results that the dual problem (4) is transformed into the following :

$$\begin{cases} \max \varphi(u) \\ u \geq 0, \end{cases} \tag{6}$$

where

$$\varphi(u) = -\frac{1}{2}\left\langle u, AA^T u \right\rangle - b^T u.$$

(by $\langle .,. \rangle$ we denote the scalar product in $\Re^m$).

The $m \times m$ matrix $D = AA^T$ is symmetric and nonnegative definite. Indeed:

$$u^T D u = (A^T u)^T (A^T u) \geq 0.$$

It follows that :

$$\varphi(u) = -\frac{1}{2}u^T D u - b^T u \tag{7}$$

is a concave continuously differentiable function. The problem (6) with the objective function (7) may have more solutions, but no matter which one is replaced in (5), the unique solution of the problem (1) is obtained.

2. ELIMINATION OF CONSTRAINTS

For any $u \in \Re^m$, we use $N(u)$ to denote the set of indices for which the corresponding components of the point u are zero. That is

$$N(u) = \{i / u_i = 0\}.$$

PROPOSITION 1. *Let $U \subset \Re^m$ be a convex set and f be a concave function on the set U. If $\widehat{u}$ is an optimal point for problem*

$$\begin{cases} \max f(u) \\ u \geq 0, \end{cases} \tag{8}$$

then it is also optimal for "restraint" problem:

$$\begin{cases} \max f(u) \\ u_i \geq 0, i \in N(\widehat{u}) \end{cases} \tag{9}$$

(i.e. the i^{th} constraints for which $\widehat{u}_i > 0$ may be excluded with no change of solution).

Proof. We will assume that $\widehat{u}$ wouldn't be optimal for the problem (9), i. e. would exist $\overline{u}$ so that:

$$\overline{u}_i \geq 0, i \in N(\widehat{u}), \tag{10}$$

$$f(\overline{u}) > f(\widehat{u}). \tag{11}$$

Let $u^\tau = \tau\overline{u} + (1-\tau)\widehat{u} \in U$, with $\tau \in (0,1)$. We will show that for τ small enough, u^τ is a solution of problem (8) better than $\widehat{u}$ and this contradicts the optimality of $\widehat{u}$. For $i \in N(\widehat{u})$, $u_i^\tau = \tau\overline{u}_i + (1-\tau).0 \geq 0$, for any $\tau \in (0,1)$. For $i \notin N(\widehat{u})$ we can find τ_0 small enough to have

$$u_i^{\tau_0} = \tau_0\overline{u_i} + (1-\tau_0)\widehat{u}_i > 0.$$

Beside this, from (11) and from the fact that f is concave, it follows:

$$f(u^{\tau_0}) \geq \tau_0 f(\overline{u}) + (1-\tau_0) f(\widehat{u}) > f(\widehat{u}).\blacksquare$$

If we can determine the set $N(\widehat{u})$ of indices for which the corresponding components of the optimal solution $\widehat{u}$ of the problem (8) are zero, we can solve the "restraint" problem with constraints taken as equalities:

$$\begin{cases} \max f(u) \\ u_i = 0,\ i \in N(\widehat{u}). \end{cases} \tag{12}$$

The problem (12) has the form of maximizing a concave function f over subspace of the *free* variables $\Re^{m-q}$, where q is the number of indices from $N(\widehat{u})$ (the set of indices of *fixed* variables). It is a specialisation of the problem of maximizing a concave function, over a linear manifold [3].
Remark 1. From practical point of view, (12) is a unconstrained problem, with $m-q$ free variables, of form:

$$\begin{cases} \max g(z) \\ z \in \Re^{m-q}, \end{cases}$$

where z is the vector with components u_i, $i \notin N(\widehat{u})$ (the free part of u).

3. THE METHOD

Assume that f is a concave continuously differentiable function. By the K-T theorem, the necessary and sufficient conditions for $\widehat{u} \in \Re^m_+$ to be an optimal solution for problem the (8), is to exist multipliers $\widehat{\lambda}_i \geq 0, i = 1, ..., m$, such that:

$$\widehat{\lambda}_i \widehat{u}_i = 0, i = 1, ..., m, \tag{13}$$

$$\nabla f(\widehat{u}) = -\sum_{i=1}^{m} \widehat{\lambda}_i e^i, \tag{14}$$

where $\Re^m_+$ stands for the nonnegative orthant of $\Re^m$ and $e^i = (0, ..., \overset{i}{1}, ..., 0)^T$. If $i \notin N(\widehat{u})$, then $\widehat{u_i} > 0$ and from (13), $\widehat{\lambda}_i = 0$. Then from (14), it follows that the necessary and sufficient condition for $\widehat{u} \in \Re^m_+$ to be an optimal solution for the problem (8) is to exist:

$$\widehat{\lambda}_i \geq 0, i \in N(\widehat{u}) \text{ such that:} \tag{15}$$

$$\nabla f(\widehat{u}) = -\sum_{i \in N(\widehat{u})} \widehat{\lambda}_i e^i. \tag{16}$$

The gradient of f at the optimal solution $\widehat{u}$ has the components:

$$(\nabla f(\widehat{u}))_i = 0, i \notin N(\widehat{u}),$$

$$(\nabla f(\widehat{u}))_i = -\widehat{\lambda}_i, i \in N(\widehat{u}).$$

The inequality $-\widehat{\lambda}_i \leq 0$ shows the trend of function f to decrease in a neighbourhood of the optimal point $\widehat{u}$.

Let v be optimal for the problem (12). It follows that there are Lagrange multipliers λ_i, $i \in N(\widehat{u})$ such that :

$$v_i = 0,\ i \in N(\widehat{u}), \tag{17}$$

$$\nabla f(v) = -\sum_{i \in N(\widehat{u})} \lambda_i e^i. \tag{18}$$

If conditions

$$v_i \geq 0,\ i \notin N(\widehat{u}),$$

$$\lambda_i \geq 0,\ i \in N(\widehat{u})$$

are satisfied, then using (15) and (16), v is an optimal solution for the problem (8).
If, for some i_0, $\lambda_{i_0} < 0$, the function f has the trend to increase in neighborhood of the point v on direction i_0 and the greater is the value of $-\lambda_{i_0}$, the stronger is this trend.

PROPOSITION 2. *Let f be a continuously differentiable function on $\Re^m$. Let v be optimal for*

$$\begin{cases} \max f(u) \\ u_i = 0,\ i \in N \subset \{1, ..., m\} \end{cases} \tag{19}$$

and λ_i, $i \in N$, Lagrange multipliers such that:

$$\nabla f(v) = -\sum_{i \in N} \lambda_i e^i. \tag{20}$$

Assume for some $i_0 \in N$ that $\lambda_{i_0} < 0$. If v' is optimal for

$$\begin{cases} \max f(u) \\ u_i = 0,\ i \in N \setminus \{i_0\} \end{cases} \tag{21}$$

then v' is a better point, i. e.

$$f(v') > f(v). \tag{22}$$

Proof. Because the problem (21) has one constraint less than the problem (19), the inequality $f(v') \geq f(v)$ is obvious. We shall prove that v is not optimal for the problem

$$\begin{cases} \max f(u) \\ u_i = 0,\ i \in N \setminus \{i_0\} \\ u_{i_0} \geq 0. \end{cases} \tag{23}$$

Suppose on the contrary that v is optimal for the problem (23). Then, from K-T conditions, there are multipliers λ'_i, $i \in N \backslash \{i_0\}$ and $\lambda'_{i_0} \geq 0$ such that:

$$\nabla f(v) = -\sum_{i \in N \backslash \{i_0\}} \lambda'_i e^i - \lambda'_{i_0} e^{i_0}. \tag{24}$$

But $\nabla f(v)$ is expressed as a unique linear combination of the unit vectors e^i, $i \in N$. From (20) and (24), it follows that $\lambda_i = \lambda'_i$, $i \in N \backslash \{i_0\}$ and $\lambda_{i_0} = \lambda'_{i_0} \geq 0$ (contradiction). Therefore v is not optimal for the problem (23).

If $\overline{v}$ is optimal for the problem (23), then $f(\overline{v}) > f(v)$. On the other hand, the problem (21) has one constraint less than the problem (23). It follows:

$$f(v') \geq f(\overline{v}) > f(v) . \blacksquare$$

The method is briefly the following: suppose the starting point $u^1 \geq 0$ is given; then we solve the subproblem

$$\begin{cases} \max f(u) \\ \quad u_i = 0, \ i \in N(u^1) \end{cases} \tag{25}$$

and let v^1 be an optimal point for the problem (25). We distinguish between two possibilities.

The first case: $v^1 \geq 0$ (v^1 is feasible for the problem (8)), then $u^2 = v^1$.

If v^1 is also optimal for the problem (8) (i. e. $\lambda_i \geq 0$, $i \in N(u^1)$), the procedure stops. If there exists $\lambda_i < 0$, then a subspace of a large dimension is considered, by releasing a constraint (according to Proposition 2, a fixed variable becomes free) and we repeat this process.

The second case: if there exists $i \notin N(u^1)$ such that $v_i^1 < 0$ (v^1 is not feasible for the problem (8)), then let u^2 be the feasible point closest to v^1 on the line segment between u^1 and v^1 : $u^2 = u^1 + \tau(v^1 - u^1)$,where the scalar $\tau \in (0,1)$ is determined such that $u^2 \geq 0$ and u^2 is closest to v^1. At least one additional component $r \notin N(u^1)$ of u^2 is zero.
Then we solve the subproblem:

$$\begin{cases} \max f(u) \\ \quad u_i = 0, \ i \in N(u^2). \end{cases}$$

The dimension of the subspace used in this subproblem decreases due to the appearance of at least one constraint more: $N(u^2) \supset N(u^1) \cup \{r\}$.

Remark 2. The second case can occur at most a finite number of times in succession without an occurrence of the first case, namely at most until all components u_i are zero. But the null vector is a feasible point for the problem

(8), therefore the first case will appear.

ALGORITHM (suboptimization algorithm)

1. *Initialisation*: $u^1 \in \Re_+^m$ arbitrary and $N^1 = N(u^1)$.

2. *Typical step*: Determine v^p optimal for subproblem:

$$\begin{cases} \max f(u) \\ \quad u_i = 0, \ i \in N^p. \end{cases} \tag{26}$$

Case 1: $v^p \geq 0$. Determine:
$\lambda_{i_0} = \min_{i \in N^p} \{\lambda_i\}$
If $\lambda_{i_0} \geq 0$, STOP; v^p is optimal for problem (8).
If $\lambda_{i_0} < 0$, define

$$\begin{array}{rcl} N^{p+1} & = & N^p \backslash \{i_0\} \\ u^{p+1} & = & v^p. \end{array}$$

Go to 2 with p+1 replacing p.

Case 2: $v^p \ngeq 0$. Determine u^{p+1} the feasible point closest to v^p on the line between u^p and v^p. Define

$$N^{p+1} = N\left(u^{p+1}\right)$$

Go to 2 with p+1 replacing p.

Assumption 1: The optimal solutions of the subproblem (26) exist.

Remark 3. According to Remark 1, the subproblems (26) may be solved using a unconstrained maximization technique. Such a technique is applied only in the subspace of the free variables, while the fixed variables remain by definition, zero. By starting the unconstrained technique from u^p (as initial feasible point), the optimal point v^p of the subproblem (26) should be near u^p and therefore it may be easily obtained.

4. FINITE CONVERGENCE

The algorithm terminates in a finite number of steps. To prove this, it is enough, according to Remark 2, to establish that Case 1 can only occurs a finite number of times without termination.

PROPOSITION 3. *Let f be a concave continuously differentiable function. Under Assumption 1, the Case 1 from the statement of the Suboptimization Algorithm can only occur a finite number of times without termination.*

Proof. Suppose Case 1 occurs, i. e. $v^p \geq 0$, and v^p is not optimal for the problem (8). According to the algorithm, $u^{p+1} = v^p$ and the problem

$$\begin{cases} \max f(u) \\ u_i = 0,\ i \in N^p \setminus \{i_0\} \end{cases} \tag{27}$$

is solved, obtaining the optimal solution v^{p+1}. We distinguish between two posibilities:

a) $v^{p+1} \geq 0$; then $u^{p+2} = v^{p+1}$ and according to Proposition 2

$$f\left(u^{p+2}\right) > f\left(u^{p+1}\right).$$

b) $v^{p+1} \ngeq 0$; then $u^{p+2} = u^{p+1} + \tau\left(v^{p+1} - u^{p+1}\right)$. Proposition 2 does provide that $f\left(v^{p+1}\right) > f\left(u^{p+1}\right)$, but by concavity of f

$f\left(u^{p+2}\right) = f\left(\tau v^{p+1} + (1-\tau)\, u^{p+1}\right) \geq \tau f\left(v^{p+1}\right) + (1-\tau)\, f\left(u^{p+1}\right) >$

$> \tau f\left(u^{p+1}\right) + (1-\tau)\, f\left(u^{p+1}\right) = f\left(u^{p+1}\right)$.

Hence, when Case 1 occurs

$$f\left(u^{p+2}\right) > f\left(u^{p+1}\right) \tag{28}$$

and u^{p+2} is set equal to the solution v^{p+1} of the subproblem (27). From (28) the subproblem cannot repeat. Due to their form, there are only a finite number of them, so the conclusion follows.■

5. APPLICATION OF THE METHOD TO QUADRATIC PROGRAMMING PROBLEM

In what follows we apply the suboptimization algorithm to solve the quadratic programming problem:

$$\begin{cases} \max \varphi(u) \\ u \geq 0, \end{cases}$$

where $\varphi(u) = -\frac{1}{2}u^T Du - b^T u$ and $D = AA^T$.
The gradient of φ is $\nabla\varphi(u) = -AA^T u - b$.

Consider the subproblem:

$$\begin{cases} \max \varphi(u) \\ u_i = 0,\ i \in N^p \subset \{1, ..., m\}. \end{cases} \tag{29}$$

Assume that F^p is the complement of N^p, that is, the set of free variables at p iteration. We also assume without loss of generality, that

$$F^p = \{1, ..., m-q\},$$

where q is the number of indices from N^p. Then we have the following partition:

$$u = \begin{pmatrix} z \\ 0 \end{pmatrix},$$

where the vector z contains the first $m-q$ components of u (the free part). The subproblem then reduces to finding a vector $z^p \in \Re^{m-q}$ by solving

$$max_{z\in\Re^{m-q}} \left\{ -\frac{1}{2} z^T C z - dz \right\},$$

where $C \in M_{(m-q)\times(m-q)}, d \in \Re^{m-q}$. Then, returning to the m-dimensional space, the optimal point for the subproblem (29) is:

$$v^p = \begin{pmatrix} z^p \\ 0 \end{pmatrix}.$$

Let λ_i, $i \in N^p$, be Lagrange multipliers such that:

$$\nabla\varphi\left(v^p\right) = -\sum_{i\in N^p} \lambda_i e^i.$$

It results that

$$\lambda_i = -\left(\nabla\varphi\left(v^p\right)\right)_i, i \in N^p.$$

For $j \in N^p$ we have $v_j^p = 0$. It follows that

$$\lambda_i = \sum_{j\in F^p} d_{ij} v_j^p + b_i, i \in N^p,$$

where $d_{ij} = \sum_{k=1}^n a_{ik} a_{jk}$.

In view of having as few unconstrained subproblems to solve, as it is possible, it is very important how the feasible starting point u^1 is chosen.

In order to find this point, a number (not very large) of iterations from the primal-dual algorithm of Lent and Censor [1] can be used. This is an easily programmable and low storage demanding method.

Starting from initial values $u^{(0)} = 0$ and $x^{(0)} = 0$, the primal-dual algorithm generates simultaneously primal iterates $x^{(k)}$ which converge (unfortunately slow) to the solution of (1) and dual iterates $u^{(k)}$ with $u^{(k)} \geq 0$. The sequence of dual vectors $\left\{u^{(k)}\right\}$ is feasible for the problem (6) and the values of $\varphi\left(u^{(k)}\right)$ increase monotonically and converge:

$$a)\varphi\left(u^{(k+1)}\right) \geq \varphi\left(u^{(k)}\right),$$

$$b) lim_{k\to\infty}[\varphi\left(u^{(k+1)}\right) - \varphi\left(u^{(k)}\right)] = 0.$$

For dual variables the algorithm is analogous with the relaxation method. The transition from $u^{(k)}$ to $u^{(k+1)}$ consists in parallel displacement on axis Ou_{i_k} where $i_k = k\,(mod m) + 1$, such that for each displacement the greatest increase of function φ to be obtained, under the condition that $u^{(k+1)} \geq 0$ (see [2])

$$u^{(k+1)} = u^{(k)} + \alpha^{(k)} e_{i_k},$$

where $e_{i_k} = \left(0, ..., \overset{i_k}{1}, ..., 0\right)^T$. The step size $\alpha^{(k)}$ of displacement at iteration k is:

$$\alpha^{(k)} = -\min\left(u_{i_k}^{(k)}, \frac{b_{i_k} - < a_{i_k}, x^{(k)} >}{\|a_{i_k}\|^2}\right),$$

where

$$x^{(k)} = x^{(k-1)} - \alpha^{(k-1)} a_{i_{k-1}}.$$

6. b (θ) − LINEAR FUNCTION IN PARAMETER θ.

Since in real life the numeric data for the problem(1) are not exactly determined, they are known by approximation. Consider the following parametric programming problem:

$$\begin{cases} \min \frac{1}{2}\|x\|^2 \\ Ax \leq b(\theta), \end{cases} \tag{30}$$

where θ is a real valued parameter, $b(\theta) = b + \theta b^1$ and $b, b^1 \in \Re^m$ are given. The dual to the problem (30) is the parametric quadratic programming problem and will denote it by $P(\theta)$:

$$\begin{cases} \max\left\{-\frac{1}{2}u^T D u - b^T(\theta) u\right\} \\ \qquad u \geq 0. \end{cases} \tag{31}$$

For $\theta = 0$, the quadratic programming problem (6) is obtained.
We assume that, applying the suboptimization algorithm to the problem $P(0)$, we have been obtained an optimal solution $\widehat{u} \in \Re^m$ and the dual variables $\widehat{\lambda} \in \Re^m$ which verifies the K-T conditions:

$$\begin{cases} -D\widehat{u} + \widehat{\lambda} = b \\ \widehat{u} \geq 0,\ \widehat{\lambda} \geq 0 \\ (\widehat{u})^T \widehat{\lambda} = 0. \end{cases} \tag{32}$$

Consider the "restraint" problem:

$$\begin{cases} \max\left\{-\frac{1}{2}v^T Dv - (b^1)^T v\right\} \\ \widehat{\lambda}^T v = 0 \\ v_i \geq 0,\ i \in N(\widehat{u}). \end{cases} \tag{33}$$

We assume now that the "restraint" problem has an optimal solution, let's say $\widehat{v}$. Then using again the K-T conditions, it follows that there exist multipliers $\eta_i \geq 0$, $i \in N(\widehat{u})$ and $\eta_0 \in \Re$ such that

$$\begin{cases} -D\widehat{v} + \eta_0\widehat{\lambda} + \eta = b^1 \\ \widehat{\lambda}^T\widehat{v} = 0 \\ \widehat{v}_i \geq 0,\ i \in N(\widehat{u}) \\ \eta^T\widehat{v} = 0, \end{cases} \tag{34}$$

where for $i \notin N(\widehat{u})$ we have defined $\eta_i = 0$.

PROPOSITION 4. *If the problems (6) and (33) have the optimal solutions $\widehat{u}$ and respectively $\widehat{v}$, then there exists $\theta_0 > 0$ such that $u(\theta) = \widehat{u} + \theta\widehat{v}$ is an optimal solution of $P(\theta)$*, for any $0 \leq \theta \leq \theta_0$.

Proof. Let $\lambda(\theta) = \widehat{\lambda} + \theta\left(\eta + \eta_0\widehat{\lambda}\right)$. Now define

$$I_1 = \{i/\widehat{v}_i < 0\}$$
$$I_2 = \left\{i/\eta_i + \eta_0\widehat{\lambda}_i < 0\right\}$$

$$\theta_1 = \begin{cases} \min_{i\in I_1}\left\{-\frac{\widehat{u}_i}{\widehat{v}_i}\right\} & \text{if } I_1 \neq \phi \\ +\infty & \text{if } I_1 = \phi \end{cases}$$

$$\theta_2 = \begin{cases} \min_{i\in I_2}\left\{\frac{-\widehat{\lambda}_i}{\eta_i + \eta_0\widehat{\lambda}_i}\right\} & \text{if } I_2 \neq \phi \\ +\infty & \text{if } I_2 = \phi \end{cases}$$

$$\theta_0 = \min\{\theta_1, \theta_2\}.$$

By definition $\theta_1 \geq 0$. In fact $\theta_1 > 0$. Indeed, if $\widehat{u}_i = 0$, it results that $i \in N(\widehat{u})$ and therefore $\widehat{v} \geq 0$, i.e. $i \notin I_1$. Analogously, it is proved that $\theta_2 > 0$. Indeed, if $\widehat{\lambda}_i = 0$ then $\eta_i + \eta_0\widehat{\lambda}_i = \eta_i \geq 0$, i. e. $i \notin I_2$. It follows that $\theta_0 > 0$. We will prove now that $u(\theta)$ and $\lambda(\theta)$, $0 \leq \theta \leq \theta_0$, satisfy the K-T conditions for the problem $P(\theta)$, i. e.

$$\begin{cases} -D(\widehat{u} + \theta\widehat{v}) + \widehat{\lambda} + \theta(\eta + \eta_0\widehat{\lambda}) = b + \theta b^1 \\ \widehat{u} + \theta\widehat{v} \geq 0,\ \widehat{\lambda} + \theta(\eta + \eta_0\widehat{\lambda}) \geq 0 \\ (\widehat{u} + \theta\widehat{v})^T(\widehat{\lambda} + \theta(\eta + \eta_0\widehat{\lambda}) = 0. \end{cases}$$

From the definition of θ_0, the vector $u(\theta)$ is feasible for $P(\theta)$ and $\lambda(\theta) \geq 0$, for any $0 \leq \theta \leq \theta_0$. The other conditions result from (32) and (34) and from the relation

$$(\widehat{u})^T \eta = 0.$$

The last one is equivalent to $\widehat{u}_i \eta_i = 0, 1 \leq i \leq m$, what is true, because $i \in N(\widehat{u})$ we have $\widehat{u}_i = 0$ and for $i \notin N(\widehat{u})$, $\eta_i = 0$, by definition.■

If $u(\theta)$ is replaced in (5), the solution of the problem (30) is obtained:

$$x(\theta) = -A^T \widehat{u} - \theta A^T \widehat{v}$$

or

$$x(\theta) = \widehat{x} - \theta A^T \widehat{v},$$

where $\widehat{x}$ is the optimal solution of problem (1) and $\widehat{v}$ is a solution of "restraint" problem (33).

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