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STATISTICAL INFERENCE ON THE TRAFFIC INTENSITY FOR THE M/M/s QUEUEING SYSTEM

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To Professor Silviu Sburlan, at his 60's anniversary

Abstract

In the paper it is shown that, for the certain plans of observations on the M/M/s queueing system's performances, there exists a uniformly most powerful test of statistical hypothesis on the traffic intensity.

1. Preliminaries. Let us consider the $M/M/s$ queueing system. That means we have the system with poissonian input with parameter λ, $\lambda > 0$ and exponential output of customers with parameter μ, $\mu > 0$, s $(s \geq 1)$ being the number of servers.

Suppose that we observe functioning of our system in the time interval $[0, R]$, where $R = R(\omega)$ is a random variable. In addition, we suppose the outcome (n, m) of the random vector (N_R, M_R) is a point of a pseudomonotone frontier $(PM - frontier)$ in $\mathbb{R}^2$, where N_t, M_t are the numbers of up or down jumps, respectively, for birth-death process X_t with birth-death intensities, respectively,

$$\lambda_k = \lambda, k \geq 0, \mu_k = \left\{ \begin{array}{l} k \cdot \mu, pentru\ k \leq s\ , \\ s \cdot \mu, pentru\ k > s\ , \end{array} \right. \quad k \geq 1.$$

Note that the process X_t describes the $M/M/s$ system.

The notion of $PM-$ frontier was introduced in [1] an its definition will be reproduced in our paper.

We suppose also that statistical trials give us the full information about performance of the system. In this conditions for the parameter $\rho = \lambda/\mu$,

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which for $s = 1$ is known as a *traffic intensity*, we have to solve the problem of testing hypothesis $\mathbf{H}_0 : \rho \le 1$, $\mathbf{H}_1 : \rho > 1$.

2. Solving the problem. Since the observations give us full information about the $M/M/s$ system's performances, we have, as a result of sampling, the vector $\mathbf{A} = (\tau_{01}^1, ..., \tau_{01}^{k_{01}}, ..., \tau_{q1}^1, ..., \tau_{q1}^{k_{q1}}, \tau_{02}^2, ..., \tau_{02}^{k_{02}}, ..., \tau_{q2}^2, ..., \tau_{q2}^{k_{q2}})$, where

$q = \max\{\textit{lenghts of the registrated queues observing system's performances}\}$;

k_{i1} is the number which shows how many times the system (or the process X_t) has reached the state i before his jumping up in the state $i+1$, $i = 0, 1, ..., q-1$;

k_{i2} is the number which shows how many times the system (or the process X_t) has reached the state i before his jumping down in the state $i-1$, $i = 1, ..., q$;

τ_{i1}^j is the duration of the $j-th$ visit in the state i before system's jumping up in the state $i+1$, $i = 0, 1, ..., q-1$, $j = 1, 2, ..., k_{i1}$;

τ_{i2}^j is the duration of the $j-th$ visit in the state i before system's jumping down in the state $i-1$, $i = 1, ..., q$, $j = 1, 2, ..., k_{i2}$.

Total duration of system's visit in the state i, $i = 1, ..., q-1$, is equal to $t_i = \sum_{j=1}^{k_{i1}} \tau_{i1}^j + \sum_{j=1}^{k_{i2}} \tau_{i2}^j$, and, for $i \in \{0, q\}$, we have respectively $t_0 = \sum_{j=1}^{k_{01}} \tau_{01}^j$, $t_q = \sum_{j=1}^{k_{q2}} \tau_{q2}^j$.

Since the process X_t is a Markovian, from [2] we deduce that likelihood function (probability density function) of the vector $\mathbf{A}$ is equal to:

case a) $q \le s$

$$L(\tau_{01}^1, ..., \tau_{01}^{k_{01}}, ..., \tau_{q1}^1, ..., \tau_{q1}^{k_{q1}}, \tau_{02}^2, ..., \tau_{02}^{k_{02}}, ..., \tau_{q2}^2, ..., \tau_{q2}^{k_{q2}}; \lambda, \mu) = \\ = \lambda^n \mu^m \cdot 1^{k_{12}} \cdot 2^{k_{22}} \cdot ... \cdot q^{k_{q2}} \exp\{-\lambda T_1\} \exp\{-\mu T_2\},$$

where $n = \sum_{i=0}^{q-1} k_{i1}$, $m = \sum_{i=1}^{q} k_{i2}$, $T_1 = \sum_{i=0}^{q} t_i$, $T_2 = \sum_{i=0}^{q} it_i$;

case b) $q > s$

$$L(\tau_{01}^1, ..., \tau_{01}^{k_{01}}, ..., \tau_{q1}^1, ..., \tau_{q1}^{k_{q1}}, \tau_{02}^2, ..., \tau_{02}^{k_{02}}, ..., \tau_{q2}^2, ..., \tau_{q2}^{k_{q2}}; \lambda, \mu) = \\ = \lambda^n \mu^m \cdot 1^{k_{12}} \cdot 2^{k_{22}} \cdot ... \cdot (s-1)^{k_{(s-1)2}} \cdot s^{k_{s2} + ... + k_{q2}} \exp\{-\lambda T_1\} \exp\{-\mu T_2\},$$

but in this case $T_2 = \sum_{i=0}^{s-1} it_i + s \sum_{i=s}^{q} t_i$.

Looking over likelihood function in the cases a) and b) we deduce that the vector $(n, n+m, T_1, T_2)$ is a vector of sufficient statistics with its likelihood function

$$L(n, n+m, T_1, T_2; \lambda, \mu) = \lambda^n \mu^m \exp\{-(\lambda T_1 + \mu T_2)\} \cdot \mathbf{C}(n, m) =$$

$$= \mathbf{C}(n,m)\exp\{-(\lambda T_1+\mu T_2)+n\ln\frac{\lambda}{\mu}+(n+m)\ln\mu\},$$

where

$$\mathbf{C}(n,m)=\begin{cases} \sum_{q=1}^{s}\sum_{S_{qm}}2^{k_{22}}\cdot\ldots\cdot q^{k_{q2}}+ \\ +\sum_{q=s+1}^{s}\sum_{S_{qm}}2^{k_{22}}\cdot\ldots\cdot(s-1)^{k_{(s-1)2}}\cdot s^{k_{s2}+\ldots+\,k_{q2}},\ for\ n>s \\ \sum_{q=1}^{n}\sum_{S_{qm}}2^{k_{22}}\cdot\ldots\cdot q^{k_{q2}},\ for\ n\le s,\quad (1)\end{cases}$$

and $S_{qm}=\{(k_{12},...,k_{q2})\mid k_{12}+...+k_{q2}=m,\ k_{i2}=0,1,2,...,\ i=\overline{1,q}\ \}$.

From (1), we have that conditioned likelihood function

$$\mathbf{P}\{N_R=n,M_R=m,T_{1=}t_1,T_{2=}t_2/N_R+M_R=n+m\}=$$

$$=\frac{\mathbf{C}(n,k-n)\lambda^n\mu^{k-n}}{\sum_{i=[k/2]}^{k}\ \mathbf{C}(i,k-i)\lambda^i\mu^{k-i}}=\frac{\mathbf{C}(n,k-n)\rho^n}{\sum_{i=[k/2]}^{k}\ \mathbf{C}(i,k-i)\rho^i}=W_\rho(n,k-n),(2)$$

that means it is not dependant on $t_1$and t_2 , but that it depends only on ρ,n and $k=n+m$.

Let's consider $PM-$frontier $\Gamma_{\mathcal{D}}$ of the stopping points for the process of observations as a subset of the domain $\mathcal{D}=\{(N,M)\mid N,M=0,1,2,...\}$. According to [1], the points $Q\in\Gamma_{\mathcal{D}}$ may be ordered in this way: if $Q_1=(n_{Q_1},m_{Q_1})$, $Q_2=(n_{Q_2},m_{Q_2})$ belong to $\Gamma_{\mathcal{D}}$, then, by definition, Q_1 precedes Q_2 ($Q_1\preceq Q_2$) if $k_{Q_1=}n_{Q_1+}m_{Q_1}\le n_{Q_2+}m_{Q_2}=k_{Q_2}$, and Q_2 follows immediately after Q_1 if $k_{Q_2}=k_{Q_1}+1$ or $k_{Q_2}=k_{Q_1}$. In the subset $S_k=\{(n,m)=Q\mid n+m=k\}$, the numbering depends on n_Q. More exactly, for $Q_1,Q_2\in S_k$ we consider that $Q_1\preceq Q_2$ if $n_{Q_1}\le n_{Q_2}$.

From the main theorem proved in [1] we have the following

Proposition. *If the frontier $\Gamma_{\mathcal{D}}$ of the stopping points for the process of observations on the $M/M/s$ system's performances is a $PM-frontier$, then the family of the conditioned likelihood functions $\{W_\rho(n,k-n)\}$, which depend on the parameter ρ, possesses the monotony propriety for the likelihood ratio, on the $\Gamma_{\mathcal{D}}$.*

So, according to paper [3], on the base of *conditioned likelihood functions* $W_\rho(n,k-n)$ and of the given significance level α , $\alpha\in(0,1)$, we may construct the *uniformly most powerful criterion* to test hypothesis $\mathbf{H}_0:\rho\le 1$, $\mathbf{H}_1:\rho>1$.

The critical function which coresponds to the above mentioned (randomized) criterion will be

$$\varphi(Q) = \begin{cases} 1, & if\ Q \succ Q^*, \\ \gamma, & if\ Q = Q^*, \\ 0, & if\ Q \prec Q^*, \end{cases}$$

where Q^* and γ are such that the mean value of the random variable φ calculated for $\rho = 1$ coincides with α.

This result may be applied particularly to the stopping frontiers which correspond to the following stopping times $R = R(\omega)$: a) $R = T'_N$, where T'_N coincides with the moment when the $N - th$ customer arrives in the system; b) $R = T''_M$, where T''_M coincides with the moment when the $M - th$ customer finishes his service; c) $R = \min(T'_N, T''_M)$.

It may be verified directly that the above described frontiers are $PM - frontiers$.

References

[1] Tchepourin E.V., *Testing hypotheses on the intensities ratio of the birth-death process*, Bull. Acad. Sci. USSR, Ser. Engeneering Cibernetics, Nr.1, 1971, pp. 111-131 (rus.).

[2] Billingsley P., *Statistical inference for Markov processes*, Univ. Chicago Press, 1961.

[3] Lehmann E., *Testing statistical hypothesis* . John Willey, N.-Y, 1959.

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