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# ON A SINGULARLY PERTURBED, COUPLED ELLIPTIC-ELLIPTIC PROBLEM

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*To Professor Silviu Sburlan, at his 60' anniversary*

**Abstract**

The behavior of the solution of the below problem $(E_\epsilon)$, $(BC_\epsilon)$, $(TC_\epsilon)$ is studied when the small parameter $\epsilon$ tends to 0.

## 1. Introduction.

We consider the following coupled boundary value problem of elliptic-elliptic type, denoted by $P_\epsilon$:

$$\begin{cases} -\epsilon u''(x) + \alpha(x)u'(x) + \beta(x)u(x) = f(x), \ x \in (a,b), \\ -\ \big(\mu(x)v'(x)\big)' + \alpha(x)v'(x) + \beta(x)v(x) = g(x), \ x \in (b,c), \end{cases} \tag{$E_\epsilon$}$$

with homogeneous Dirichlet boundary conditions

$$u(a) = v(c) = 0 \tag{$BC_\epsilon$}$$

and transmission conditions at $x = b$

$$u(b) = v(b), \quad \epsilon u'(b) = (\mu v')(b). \tag{$TC_\epsilon$}$$

The transmission conditions at $x = b$ express the continuity of the solution and of the flux.

The following assumptions will be required in the following:

$(A_1)$ $a,\ b,\ c \in \mathbb{R},\ a < b < c,\ \epsilon > 0$ is a small parameter;

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$(A_2)$ $\alpha \in H^1(a,c),\ \beta \in L^\infty(a,c),\ \mu \in H^1(a,c)$;

$(A_3)$ $\alpha(x) \leq \alpha_0 < 0$ in $[a,c]$, $\mu(x) \geq \mu_0 > 0$ in $[b,c]$, $\beta - \frac{\alpha'}{2} \geq 0$ a.e. in $(a,c)$;

$(A_4) f \in L^2(a,b), g \in L^2(b,c)$.

The aim of this paper is to investigate the problem $P_\epsilon$ for $\epsilon$ going to zero from the view point of singular perturbation theory. This is a singularly perturbed problem with respect to the norm of uniform convergence and the boundary layer is the point $x = a$. To have an idea about this matter, let us consider the particular case when $\alpha, \beta, \mu$ are constant functions. If the solution $(u, v)$ of $(P_\epsilon)$ converges in $C[a,b] \times C[b,c]$ to $(U, V)$, then it can easily be seen that $(U, V)$ satisfies

$$\begin{cases} \alpha U' + \beta U = f, \quad \text{in} \quad (a,b), \\ -\mu V'' + \alpha V' + \beta V = g, \quad \text{in} \quad (b,c), \\ U(a) = 0, U(b) = V(b), \\ V'(b) = 0, V(c) = 0. \end{cases}$$

The condition $U(a) = 0$ is not satisfied in general (this exceeds the number of conditions allowed). This fact is not acceptable from a physical point of view. Actually we shall see that indeed, in the point $x = a$, the solution $(u, v)$ has a singular behaviour as $\epsilon$ goes to zero. The corresponding unperturbed (reduced) problem, denoted by $P_0$, is the following [6]:

$$\begin{cases} \alpha(x)U'(x) + \beta(x)U(x) = f(x), \quad x \in (a,b) \\ -\big(\mu(x)V'(x)\big)' + \alpha(x)V'(x) + \beta(x)V(x) = g(x), \quad x \in (b,c), \end{cases} \tag{$E_0$}$$

$$V(c) = 0, \tag{$BC_0$}$$

$$\begin{cases} U(b) = V(b), \\ V'(b) = 0. \end{cases} \tag{$TC_0$}$$

This problem will be reobtained below by using the Vishik-Lusternik method and it is the same as that derived in [6] by using a different way.

In [1], [2] we studied a similar transmission of type elliptic-elliptic but we considered that $\alpha \geq \alpha_0 > 0$. Here we are assuming $\alpha < 0$, hence the asymptotic behavior is different from the case $\alpha > 0$: actually we cannot have a convergence of $u(a)$ to $U(a)$, in general. A comment is needed about the conditions $(TC_0), (BC_0)$. Note that the problem $P_0$ has no conditions at all for $x = a$ and needs two conditions at $b$, in this case, when $\alpha < 0$.

In Section 2 we shall derive a formal zero-th order asymptotic expansion for the solution $(u, v)$ of the problem $(P_\epsilon)$. The interface point $x = a$ is a boundary layer and the expansion of $u$ contains a corresponding corrector (boundary layer function).

In Section 3 we shall investigate the existence and uniqueness of the solutions to the problems $(P_\epsilon)$ and $P_0$.

Finally, Section 4 is devoted to obtaining some estimates for the remainder terms of the expansion established in Section 2 with respect to the uniform convergence topology.

## 2. A formal asymptotic expansion for the solution of $P_\epsilon$

The classical perturbation theory (see [7] for details) can be adapted to our specific singular perturbation problem. Following this theory, we are going to derive formally an expansion of the solution $(u,\ v)$ of $(P_\epsilon)$ of the form:

$$\begin{cases} u(x) = U(x) + \theta_1(\zeta) + \rho_{1\epsilon}(x), & x \in [a, b], \\ v(x) = V(x) + \theta_2(\zeta) + \rho_{2\epsilon}(x), & x \in [b, c], \end{cases} \tag{2.1}$$

where $\zeta{:=}\epsilon^{-1}(x-a)$ is the fast variable; $(U,\ V)$ is the zero-th order term of the regular series; $\theta_1,\ \theta_2$ are boundary layer functions (correctors); $\rho_{1\epsilon},\ \rho_{2\epsilon}$ denote the remainder terms of zero-th order.

We substitute formally in $(E_\epsilon)$ $(u,\ v)$ given by (2.1) and then we identify the coefficients of $\epsilon^k$ $(k = -1, 0)$, separately those depending on $x$ from those depending on $\zeta$. So, we get

$$\begin{cases} \alpha(x)U'(x) + \beta(x)U(x) = f(x), \quad a < x < b, \\ -(\mu(x)V'(x))' + \alpha(x)V'(x) + \beta(x)V(x) = g(x), \quad b < x < c. \end{cases} \tag{$E_0$}$$

For $\theta_1{=}\theta_1(\zeta)$ we derive the equation

$$\theta_1''(\zeta) - \alpha(a)\theta_1'(\zeta) = 0. \tag{2.2}$$

Eq. (2.2) and the fact that $\theta_1$ is a boundary layer function (in particular, $\theta_1(\zeta) \longrightarrow 0,\ \ as\ \ \zeta \longrightarrow \infty$) implies that $\theta_1(\zeta) = ke^{\alpha(a)\zeta}$, where $k$ is a constant which will be determined from $(BC_\epsilon)$. Also, we can deduce that $\theta_2 = 0$.

For the remainder terms we have the equations

$$\begin{cases} -\varepsilon\ \rho_{1\epsilon}''(x) + \alpha(x)\rho_{1\epsilon}'(x) + \beta(x)\rho_{1\epsilon}(x) = \\ = \epsilon U''(x) + \epsilon^{-1}\big(\alpha(x) - \alpha(a)\big)\theta_1'\big(\zeta(x)\big) - \beta(x)\theta_1\big(\zeta(x)\big), \quad a < x < b, \\ -\big((\mu\rho_{2\epsilon}')(x)\big)' + \alpha(x)\rho_{2\epsilon}'(x) + \beta(x)\rho_{2\epsilon}(x) = 0, \quad b < x < c, \end{cases} \tag{$ER$}$$

where $\zeta(x) = (x-a)/\epsilon$.

From $(BC_\epsilon)$, we can derive

$$k = -U(a), \text{ hence } \theta_1(\xi) = -U(a)e^{\alpha(a)\zeta} \tag{2.3}$$

$$V(c) = 0, \tag{$BC_0$}$$

$$\begin{cases} \rho_{1\epsilon}(a) = 0, \\ \rho_{2\epsilon}(c) = 0. \end{cases} \tag{$BCR$}$$

By replacing (2.1) into $(TC)_\epsilon$ , we can see that $U$ and $V$ satisfy the transmission conditions

$$\begin{cases} U(b) = V(b), \\ V'(b) = 0, \end{cases} \tag{$TC_0$}$$

and for the remainder terms we have

$$\begin{cases} \rho_{1\epsilon}(b) = \rho_{2\epsilon}(b) + U(a)e^{\alpha(a)\theta_1(\zeta(b))}, \\ \epsilon\rho'_{1\epsilon}(b) = -\epsilon U'(b) + U(a)\alpha(a)e^{\alpha(a)\theta_1(\zeta(b))} + \left(\mu\rho'_{2\epsilon}\right)(b). \end{cases} \tag{$TCR$}$$

Summarizing, the reduced problem, $P_0$, is $(E_0)-(BC_0)-(TC_0)$, while the problem satisfied by the remainder terms is $(ER)-(BCR)-(TCR)$. As we shall see later on, the last problem is satisfied by $(\rho_{1\epsilon},\ \rho_{2\epsilon})$ in a generalized sense.

## 3. Existence and regularity for the problems $(P_\epsilon)$ and $P_0$.

For the problem $(P_\epsilon)$ we have the following result, whose proof is essentially known (see [1]):

**Proposition 3.1.** *Assume that $(A_1)$–$(A_4)$ are satisfied. Then, the problem $P_\epsilon$ admits a unique solution $(u,v) \in H^2(a,b) \times H^2(b,c)$ satisfying $(E_\epsilon)$ a.e. in $(a,b)$ and in $(b,c)$, respectively, as well as $(BC_\epsilon)$ and $(TC_\epsilon)$.*

In the following, we are going to investigate the reduced problem $(P_0)$. In fact, we can split it in two separate problems, with the unknowns $U$ and $V$, respectively. Clearly, $V$ is a solution of $(E_0)_2$, with the boundary conditions

$$V'(b) = 0, V(c) = 0. \tag{3.1}$$

By the Lax-Milgram lemma, there exists a unique solution $V \in H^2(b,c)$ of this problem. Obviously, Eq. $(E_0)_1$, with $U(b) = V(b)$, has a unique solution $U \in H^1(a,b)$. Therefore, we have the following result

**Proposition 3.2.** *Assume that $(A_1)-(A_4)$ are satisfied. Then, the problem $P_0$ has a unique solution $U\in H^1(a,b)$, $V\in H^2(a,c)$, which satisfies $(E_0)$ a.e. in $(a,b)$ and $(b,c)$, respectively, as well as $(BC_0)$ and $(TC_0)$.*

If we denote

$$\tilde{\rho}_{1\epsilon}(x) := \rho_{1\epsilon}(x) + A_\epsilon x + B_\epsilon,\ A_\epsilon := (b-a)\chi_\epsilon,\ B_\epsilon := -aA_\epsilon, \tag{3.2}$$

where $\chi_\epsilon = -U(a)e^{\alpha(a)\theta_1(\zeta(b)}$ then we have $\tilde{\rho}_{1\epsilon}(a)=0$, $\tilde{\rho}_{1\epsilon}(b) = \rho_{2\epsilon}(b)$ and taking into account $(P_\epsilon),(P_0)$, we can see that

$$\rho_\epsilon := \begin{cases} \tilde{\rho}_{1\epsilon} & \text{in } [a,b] \\ \rho_{2\epsilon} & \text{in } (b,c] \end{cases},$$

satisfies $\rho_\epsilon\in H_0^1(a,c)$ and

$$\epsilon\int_a^b \tilde{\rho}_{1\epsilon}'\varphi' dx + \int_b^c \mu{\rho_{2\epsilon}}'\varphi' dx + \int_a^b \alpha\tilde{\rho}_{1\epsilon}'\varphi dx + \int_b^c \alpha{\rho_{2\epsilon}}'\varphi dx$$

$$= -\epsilon\int_a^b U'\varphi' dx - \int_a^c h_\epsilon\varphi dx + \epsilon A_\epsilon\varphi(b) + \alpha(a)\varphi(b)\chi_\epsilon,\quad \forall\ \varphi\in H_0^1(a,c), \tag{3.3}$$

where

$$h_\epsilon(x) := \begin{cases} \beta(x)\theta_1\big((\zeta(x)\big) + \alpha(a)\big(\alpha(x)-\alpha(a)\big)\theta_1\big(\zeta(x)\big) + \\ +\Big[A_\epsilon\alpha(x) + \beta(x)(A_\epsilon x + B_\epsilon)\Big], \quad \text{in } (a,b), \\ 0, \quad \text{in } (b,c). \end{cases}$$

Indeed, by $(E_\epsilon)$ and $(E_0)$, we obtain

$$\int_a^b \epsilon u'\varphi' dx + \int_a^b \alpha u'\varphi dx + \int_a^b \beta u\varphi dx + \int_b^c (\mu v')\varphi' dx +$$

$$+\int_b^c \alpha v'\varphi dx + \int_b^c \beta v\varphi dx = \int_a^b f\varphi dx + \int_b^c g\varphi dx,\ \forall\ \varphi\in H_0^1(a,c),$$

$$\int_a^b \alpha U'\varphi dx + \int_a^b \beta U\varphi dx = \int_a^b f\varphi dx,$$

$$\int_b^c (\mu V')\varphi' dx + \int_b^c \alpha V'\varphi dx + \int_b^c \beta V\varphi dx - \mu V'\varphi|_b^c = \int_b^c \varphi g dx,\ \forall\ \varphi\in H_0^1(a,c).$$

Now, subtracting the last two equalities from the first one, we obtain that

$$\int_a^b \epsilon\Big(U' + \frac{d}{dx}\theta_1(\zeta(x)) + \varrho_{1\epsilon}'\Big)\varphi' dx + \int_a^b \alpha\Big(\frac{d}{dx}\theta_1(\zeta(x)) + \rho_{1\epsilon}'\Big)\varphi dx +$$

$$+\int_a^b \beta\Big(\theta_1(\zeta(x))+\rho_{1\epsilon}\Big)\varphi dx+$$

$$+\int_b^c (\mu\rho'_{2\epsilon})\varphi' dx+\int_b^c \alpha\rho'_{2\epsilon}\varphi dx+\int_b^c \beta\rho_{2\epsilon}\varphi dx-(\mu V')(b)\varphi(b)=0.$$

From

$$\epsilon\int_a^b \varphi'\frac{d}{dx}\theta_1(\zeta(x))dx=\ \epsilon\varphi(x)\frac{d}{dx}\theta_1(\zeta(x))|_a^b-\epsilon\int_a^b \varphi\frac{d^2}{dx^2}\theta_1(\zeta)dx=$$

$$=\varphi(b)\chi_\epsilon\alpha(a)-\epsilon^{-1}\alpha(a)^2\int_a^b \varphi\theta_1(\zeta(x))dx,$$

we can see that,

$$\int_a^b \epsilon\big(\ \tilde{\rho}_{1\epsilon}{}'-A_\epsilon\varphi'\big)dx+\int_a^b \alpha\big(\ \tilde{\rho}_{1\epsilon}{}'-A_\epsilon-\epsilon^{-1}\alpha(a)\theta_1(\zeta)\big)\varphi dx+$$

$$+\int_a^b \beta\big(\theta_1(\xi)+\tilde{\rho}_{1\epsilon}-A_\epsilon x-B_\epsilon)\big)\varphi dx+$$

$$+\int_b^c \mu\rho'_{2\epsilon}\varphi' dx+\int_b^c \alpha\rho'_{2\epsilon}\varphi dx+\int_b^c \beta\rho_{2\epsilon}\varphi dx-$$

$$-\epsilon^{-1}\alpha(a)^2\int_a^b \varphi\theta_1(\zeta)dx=-\epsilon\int_a^b U'\varphi' dx+\alpha(a)\varphi(b)\chi_\epsilon.$$

In conclusion, we obtain (3.3).

An elementary computation shows that, if $\rho_{1\epsilon}$, $\rho_{2\epsilon}$ are smooth functions, then they satisfy problem $(ER)-(BCR)-(TCR)$ in a classical sense.

## 4. Estimates for the remainder terms

The main result of this section is

**Theorem 4.1.** *Assume that $(A_1)-(A_4)$ are satisfied and $\alpha$ is Lipschitzian in $[a,b]$ (i.e., $\exists\ L>0$, such that $|\alpha(x)-\alpha(y)\ |\leq L|x-y|$, $\forall\ x,y\in[a,b]$). Then, for every $\epsilon>0$, the problem $(P_\epsilon)$ has a unique solution $(u,v)\in H^2(a,b)\times H^2(b,c)$ of the form (2.1), where $col(U,V)\in H^1(a,b)\times H^2(b,c)$ is the solution of $(P_0)$, $\theta_1$ is given by (2.3), $\theta_2=0$ and $(\rho_{1\epsilon},\rho_{2\epsilon})\in H^1(a,b)\times H^2(b,c)$.*

*In addition, we have the estimates $\|\rho_{1\epsilon}\|_{C[a,b]}=O(\sqrt{\epsilon})$, $\|\rho_{2\epsilon}\|_{C[b,c]}=O(\sqrt{\epsilon})$.*

**Proof.** By Propositions 3.1 and 3.2, $(\rho_{1\epsilon},\rho_{2\epsilon})\in H^1(a,b)\times H^2(b,c)$. For the sake of simplicity, we assume that $\beta-\alpha'/2\geq\gamma_0>0$ a.e. in $(a,c)$. If we choose in (3.3) $\varphi=\rho_\epsilon\in H^1_0(a,c)$,, we can see that

$$\epsilon\int_a^b\ ({\tilde{\rho}'}_{1\epsilon})^2dx+\int_b^c\mu\ (\rho'_{2\epsilon})^2dx+\int_a^b(\beta-\alpha'/2)\ \ {\tilde{\rho}_{1\epsilon}}^2dx+$$

$$+\int_b^c(\beta-\alpha'/2)\rho^2_{2\epsilon}dx=-\epsilon\int_a^b U'\ \ {\tilde{\rho}_{1\epsilon}}'dx-\int_a^b h_\epsilon\ \tilde{\rho}_{1\epsilon}dx+\gamma_\epsilon, \tag{4.1}$$

where $\gamma_\epsilon=\alpha(a)\varphi(b)\ \tilde{\rho}_{1\epsilon}(b)+\epsilon A_\epsilon\ \tilde{\rho}_{1\epsilon}(b)$. In the case $\beta-\alpha'/2\geq0$ a.e. in $(a,c)$, we choose in (3.3)

$$\varphi(x):=\begin{cases} e^{-x}\ \tilde{r}_{1\epsilon}(x) & \text{in } [a,b]\\ e^{-b}r_{2\epsilon}(x) & \text{in } (b,c]\end{cases}$$

and we can use a slight modification of our reasoning below. Denote by $\|\cdot\|_1$, $\|\cdot\|_2$ the norms of $L^2(a,b)$, $L^2(b,c)$, respectively. As $\beta-\ \alpha'/2\geq\gamma_0>0$ a.e. in $(a,c)$ and $\mu\geq\mu_0>0$ in $[b,c]$, it follows from (4.1) that

$$\epsilon\|\ {\tilde{\rho}_{1\epsilon}}'\|_1^2+\mu_0\|\rho'_{2\epsilon}\|_2^2+\gamma_0\big(\|\tilde{\rho}_{1\epsilon}\|_1^2+\|\rho_{2\epsilon}\|_2^2\big)\leq$$

$$\leq(1/2)\Big[\epsilon\|U'\|_1^2+\epsilon\|\tilde{\rho}'_{1\epsilon}\|_1^2+\gamma_0^{-1}\|h_\epsilon\|_1^2+\gamma_0\|\tilde{\rho}_{1\epsilon}\|_1^2\Big]+|\ \gamma_\epsilon|. \tag{4.2}$$

In the following, we can show that $\|h_\epsilon\|_1=O(\sqrt{\epsilon})$ and $\gamma(\epsilon)=O(\epsilon^k)$, $\forall k\geq0$. From equations $(E_\epsilon)$, we obtain

$$\epsilon\|u'\|_1^2+\mu_0\|v'\|_2^2+\gamma_0\big(\|u\|_1^2+\|v\|_2^2\big)\leq$$

$$\leq(1/2)\Big[\gamma_0^{-1}\big(\|f\|_1^2+\|g\|_2^2\big)+\gamma_0\big(\|u\|_1^2+\|v\|_2^2\big)\Big]. \tag{4.3}$$

This implies that $\|u\|_1 = O(1)$, $\|v\|_2 = O(1)$ and $\|v'\|_2 = = O(1)$, $\epsilon\ \|u'\|_2^2 = O(1)$. Since $v(c)=0$ and $H^1(b,c) \subset C[b,c]$, with a compact injection, we get $\|v\|_{C[b,c]} = O(1)$. For $\gamma_\epsilon$, we have

$$\lim_{\epsilon \to 0} \epsilon^{-k} \gamma_\epsilon = 0, \text{ for every } k \geq 0,$$

and therefore $\gamma_\epsilon = O(\epsilon^k)$. Now, as $\beta \in L^\infty(a,c)$, $\mid A_\epsilon \mid, \mid B_\epsilon \mid = O(\epsilon^k), k \geq 1$, $\alpha$ is Lipschitzian in $[a,b]$, one gets by an easy computation that $\|h_\epsilon\|_1^2 = O(\epsilon)$.

Now, from (4.2), it follows

$$\|\tilde{\rho}'_{1\epsilon}\|_1 = O(1), \quad \|\tilde{\rho}_{1\epsilon}\|_1 = O(\sqrt{\epsilon}), \quad \|\rho'_{2\epsilon}\|_2 = O(\sqrt{\epsilon}), \quad \|\rho_{2\epsilon}\|_2 = O(\sqrt{\epsilon}),$$

therefore

$$\|\rho_{1\epsilon}\|_1 \leq \ \|\tilde{\rho}_{1\epsilon}\|_1 + \ \|(A_\epsilon x + B_\epsilon)\omega(\epsilon)\|_1 = O(\sqrt{\epsilon}). \tag{4.4}$$

From

$$\rho_{2\epsilon}^2(x) = (-1/2) \int_x^c {\rho_{2\epsilon}}'(s) \rho_{2\epsilon}(s) ds \leq 2\ \|\rho_{2\epsilon}\|_2\ \ \|\rho_{2\epsilon}\|_2,$$

one gets

$$\|\rho_{2\epsilon}\|_{C[b,c]} = O(\sqrt{\epsilon}). \tag{4.5}$$

In that which follows, we shall prove that $\|\rho_{1\epsilon}\|_{C[b,c]} = O(\sqrt{\epsilon})$. To do that, we integrate $(E_\epsilon)_1$ on $[y,b]$, $y \in [a,b]$:

$$\epsilon\big(u'(y) - u'(b)\big) + \int_y^b \alpha(s) u'(s) ds + \int_y^b \beta(s) u(s) ds = \int_y^b f(s) ds. \tag{4.6}$$

By replacing

$$u_(x) = U(x) + \theta_1\big(\zeta(x)\big) + \rho_{1\epsilon}(x),$$

in (4.6), one obtains

$$\epsilon\ \rho'_{1\epsilon}(y) + \int_y^b \alpha(s) \rho'_{1\epsilon}(s) ds + \int_y^b \beta(s) \rho_{1\epsilon}(s) ds =$$

$$= \epsilon \Big[ U'(y) + \frac{d}{dy} \theta_1\big(\zeta(y)\big) \Big] -$$

$$- \int_y^b \Big[ \alpha(s) \Big( \frac{d}{ds} \theta_1\big(\zeta(s)\big) \Big) + \beta(s) \Big( \theta_1\big(\zeta(s)\big) \Big) \Big] ds + \epsilon u'(b) \text{a.e. in } (a,x). \tag{4.7}$$

Now we multiply the above inequality by $\rho'_{1\epsilon}(y)$ and then integrate on $[a,x]$.

Finally, we obtain that

$$\big(\alpha(x)/2\big)\ \rho_{1\epsilon}^2(x) + \tilde{b}(x)\rho_{1\epsilon}(x) + \tilde{c}(x) = 0, \tag{4.8}$$

where

$$\tilde{b}(x) := -\int_x^b \Big(\alpha'(y) - \beta(y)\Big)(\rho_{1\epsilon}(y)dy - \epsilon u'(b) - \alpha(b)\theta_1(\zeta(b) + \theta_1(\zeta(y)),$$

$$\tilde{c}(x) := -\int_a^x \big(\alpha'/2 - \beta\big)(y)\tilde{\rho}_{1\epsilon}^2(y)dy -$$

$$-\int_a^x \rho_{1\epsilon}'(y)\Big[\epsilon U'(y) - \big(\alpha(y) - \alpha(a)\big)\theta_1\big(\zeta(y)\big)\Big]dy -$$

$$-\int_a^x \Big[\Big(\beta(y) - \alpha'(y)\Big)\theta_1\big(\zeta(y)\big)\rho_{1\epsilon}(y)dy - \epsilon\int_a^x \big(\rho_{1\epsilon}'\big)^2(y)dy.$$

By (4.4), one obtains that $\tilde{c}(x){=}O(\epsilon)$. Integrating (4.6) on $[a, b]$, one obtains $\epsilon u'(b){=}O(\sqrt{\epsilon})$, hence $\tilde{b}(x){=}O(\sqrt{\epsilon})$.

Finally, from

$$(1/2)\alpha(x)\rho_{1\epsilon}^2(x) + O(\sqrt{\epsilon})\rho_{1\epsilon}(x) + O(\epsilon) = 0 \text{ a.e. in } (a, b),$$

$$\alpha(x)/2 \le \alpha_0/2, < 0 \ \text{ in } [a, b],$$

we can deduce that $\mid \rho_{1\epsilon}(x) \mid \le C\ \sqrt{\epsilon}$, $x \in [a, b]$, so we have $\|r_{1\epsilon}\|_{C[a,b]} = O(\sqrt{\epsilon})$.

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