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Kantrovich Type Generalization of Meyer-König and Zeller Operators via Generating Functions

Ali Olgun, H. Gül İnce and Fatma Taşdelen

Abstract

In the present paper, we study a Kantorovich type generalization of Meyer-König and Zeller type operators via generating functions. Using Korovkin type theorem we first give approximation properties of these operators defined on the space $C[0,A], 0<A<1$. Secondly, we compute the rate of convergence of these operators by means of the modulus of continuity and the elements of the modified Lipschitz class. Finally, we give an *r-th* order generalization of these operators in the sense of Kirov and Popova and we obtain approximation properties of them.

1 Introduction

For a function f on $[0,1)$, the Meyer -König and Zeller operators (MKZ) (see[11]) are given by

$$M_n(f;x) = (1-x)^{n+1}\sum_{k=0}^{\infty} f\left(\frac{k}{n+k+1}\right)\binom{n+k}{k} x^k \ , \ n\in\mathbb{N}, \tag{1}$$

Where $x\in[0,1)$. The approximation properties of the operators have been studied by Lupaş and Müller [12]. A slight modification of these operatos, called Bernstein power series, was introduced by Cheney and Sharma[3]

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and Khan[8] obtained the rate of convergence of Bernstein power series for functions of bounded variation. The Meyer-König and Zeller operators were also generalized in [4] by Doğru. A Stancu type generalization of the operators (1) have been studied by Agratini [2]. Doğru and Özalp [5] studied a Kantorovich type generalization of the operators. Using statistically convergence, a Kantorovich type generalization of Agratini's operators have been studied by Doğru, Duman and Orhan in [6]. Furthermore Altın, Doğru and Taşdelen [1] introduced a generalization of the operators (1) via linear generating functions as follows:

$$L_n(f;x) = \frac{1}{h_n(x,t)} \sum_{k=0}^{\infty} f\left(\frac{a_{k,n}}{a_{k,n}+b_n}\right) \Gamma_{k,n}(t)\ x^k, \tag{2}$$

where $0 < \frac{a_{k,n}}{a_{k,n}+b_n} < A$, $A \in (0,1)$, and $\{h_n(x,t)\}_{n\in\mathbb{N}}$ is the generating function for the sequence of function $\{\Gamma_{k,n}(t)\}_{n\in\mathbb{N}_0}$ in the form

$$h_n(x,t) = \sum_{k=0}^{\infty} \Gamma_{k,n}(t)\ x^k, \quad t \in I, \tag{3}$$

where I is any subinterval of $\mathbb{R}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The Authors studied approximation properties of the operators (2) under the following conditions (see [1]):

$i)$ $h_n(x,t) = (1-x)\, h_{n+1}(x,t)$
$ii)$ $b_n \Gamma_{k,n+1}(t) = a_{k+1,n} \Gamma_{k+1,n}(t)$
$iii)$ $b_n \to \infty, \frac{b_{n+1}}{b_n} \to 1,\ b_n \neq 0,\ for\ \ all\ n \in \mathbb{N}$
$iv)$ $\Gamma_{k,n}(t) \geq 0, \quad for\ all\ I \subset \mathbb{R}$
$v)$ $a_{k+1,n} = a_{k,n+1} + \varphi_n,\ \ |\varphi_n| \leq m < \infty,\ a_{0,n} = 0$

On the other hand, Korovkin type theorems (see [10]) on some general Lipschitz type maximal functions spaces were given by Gadjiev and Çakar [7] and Doğru [4], including the test functions $\left(\frac{x}{1-x}\right)^{\upsilon}$ and $\left(\frac{a_n x}{1+a_n x}\right)^{\upsilon}$ $(\upsilon = 0,1,2)$, respectively. The space of Lipschitz type maximal functions was defined by Lenze in [11]. Altın, Doğru and Taşdelen [1] obtained a Korovkin type theorem using the test functions $\left(\frac{x}{1-x}\right)^{\upsilon}$ $(\upsilon = 0,1,2)$, for the investigation of the approximation properties of the operators (2). They used the nodes $s = \frac{a_{k,n}}{a_{k,n}+b_n}$, with $\frac{s}{1-s} = \frac{a_{k,n}}{b_n}$.

We known from [1] that the operators L_n given by (2) satisfy the the following equalities:

$$L_n(1;x) = 1, \tag{4}$$

$$L_n(\theta(s);x) = \theta(x), \tag{5}$$

$$L_n\left(\theta^2(s);x\right)=(\theta(x))^2\,\frac{b_{n+1}}{b_n}+\frac{\varphi_n}{b_n}\theta(x)\,, \tag{6}$$

where $\theta(s)=\frac{s}{1-s}$, $x\in[0,A]\,,\ 0<A<1,\ t\in I,\ n\in\mathbb{N}$.

Throughout our present investigation, by $\|.\|$, we denote the usual supremum norm on the space $C[0,A]$ which is the space of all continuous functions on $[0,A]$.

Clearly by (4), (5) and (6) it follows that

$$\lim_{n\to\infty}\|L_n(\theta^{\upsilon})-\theta^{\upsilon}\|=0;\ \upsilon=0,1,2. \tag{7}$$

The objective of this paper is to investigate the Kantorovich type generalization of the operators L_n. In the next section, we present the integral type extension of L_n. In section 3, we study the rate of convergence of our operators by means of the modulus of continuity and the elements of the modified Lipschitz class. Finally, in section 4, we give an r-th order generalization for these operators and approximation properties of them.

2 Construction of the Kantorovich-type Operators

Now, we set

$$\theta(s):=\frac{s}{1-s}.$$

In this section, we construct a Kantorovich type generalization of MKZ type operators via generating functions and we investigate the approximation properties of these operators with the aid of the test functions $\theta^{\upsilon},\ \upsilon=0,1,2$. We should note here that in the sequel we shall use the letter θ for the test function $\theta(s)$.

Let ω be the modulus of continuity of f defined as

$$\omega(f,\delta):=\sup\{|f(s)-f(x)|;\ s,x\in[0,A]\,,|s-x|<\delta\}$$

for $f\in C[0,A]$.

Let $\omega=\omega(t)$ be arbitrary modulus of continuity defined on $[0,A]$, which satisfies the following conditions [14]

$a)$ ω is non- decreasing,

$b)$ $\omega(\delta_1+\delta_2)\le\omega(\delta_1)+\omega(\delta_2)$ for $\delta_1,\delta_2\in[0,A]$

$c)$ $\lim\limits_{\delta\to0^+}\omega(\delta)=0$.

Inspired of the well-known H_ω class (see for ex. p 108 of [16]), we now define the following W_ω class. We say that a function $f\in C[0,A]$ belongs to the class $W_\omega[0,A]$ if $|f(t_1)-f(t_2)|\le\omega\left(\left|\frac{t_1}{1-t_1}-\frac{t_2}{1-t_2}\right|\right)$ for all $t_1,t_2\in[0,A]$. Clearly, if $f\in W_\omega[0,A]$, then $\omega(f,\delta)\le\omega(\delta)$ for all $\delta\in[0,A]$. Now, fix

$\omega(t) = t$, an example of the function belonging to the W_ω class can be given by $f(x) = \frac{1}{1-x}$. Indeed, $|f(s) - f(x)| \leq \frac{|s-x|}{(1-x)(1-s)} = \omega\left(\left|\frac{s}{1-s} - \frac{x}{1-x}\right|\right)$.

Let W_ω be the space of real valued functions $f \in C[0, A]$ satisfying

$$|f(s) - f(x)| \leq \omega\left(\left|\frac{s}{1-s} - \frac{x}{1-x}\right|\right), \; for \; any \;\; x, s \in [0, A].$$

It can easily be obtained that ω satisfies

$$\omega(n\delta) \leq n\omega(\delta), \quad n \in \mathbb{N},$$

and from the property $b)$ of the modulus of continuity ω we have

$$\omega(\lambda\delta) \leq \omega((1 + \|\lambda\|)\delta) \leq (1+\lambda)\omega(\delta), \quad \lambda > 0,$$

where $\|\lambda\|$ is the greatest integer of λ.

Let us assume that the properties $(i) - (v)$ are satisfied. We also assume that $\{b_n\}$ is a positive increasing sequence.

Now we modify the operators L_n by replacing $f\left(\frac{a_{k,n}}{a_{k,n}+b_n}\right)$ in (2) by an integral mean of f over an interval $I_{k,n} = [a_{k,n}, a_{k,n} + c_{k,n}]$ as follows:

$$L_n^*(f(s); x) = L_n^*(f; x) = \frac{1}{h_n(x,t)} \sum_{k=0}^{\infty} \frac{\Gamma_{k,n}(t)}{c_{k,n}} \; x^k \int_{a_{k,n}}^{a_{k,n}+c_{k,n}} f\left(\frac{\xi}{\xi+b_n}\right) d\xi \qquad (8)$$

where f is an integrable function on the interval $(0, 1)$ and $\{c_{k,n}\}$ is a sequence such that

$$0 < c_{k,n} \leq 1, \;\; for \;\; every \;\; k \in \mathbb{N}. \qquad (9)$$

Clearly, L_n^* is a positive linear operator and also by (4)

$$L_n^*(1; x) = 1. \qquad (10)$$

Now, we give the following Korovkin type theorem for the test functions proved by Altın, Doğru and Taşdelen [1].

Theorem A. Let $\{A_n\}$ be a sequence of linear positive operators from W_ω into $C[0, A]$ and satisfies the three conditions (7). Then for all $f \in W_\omega$, we have

$$\lim_{n\to\infty} \|A_n(f) - f\| = 0 \qquad (11)$$

[1].

To prove the main result for the sequence of linear positive operators L_n^*, we need the following two lemmas.

Lemma 1. We have

$$\lim_{n\to\infty}\|L_n^*(\theta)-\theta\|=0.$$

Proof. From (iii) and (2) we get

$$\begin{aligned} L_n^*(\theta;x) &= \tfrac{1}{h_n(x,t)}\sum_{k=0}^{\infty}\tfrac{\Gamma_{k,n}(t)}{c_{k,n}}x^k\int_{a_{k,n}}^{a_{k,n}+c_{k,n}}\tfrac{\xi}{b_n}d\xi \\ &= \tfrac{1}{h_n(x,t)}\sum_{k=0}^{\infty}\tfrac{a_{k,n}}{b_n}\Gamma_{k,n}(t)\,x^k+\tfrac{1}{h_n(x,t)}\sum_{k=0}^{\infty}\tfrac{c_{k,n}}{2b_n}\Gamma_{k,n}(t)\,x^k \\ &= L_n(\theta;x)+\tfrac{1}{2b_n}\tfrac{1}{h_n(x,t)}\sum_{k=0}^{\infty}c_{k,n}\Gamma_{k,n}(t)\,x^k. \end{aligned}$$

In this way, we obtain from (5)

$$L_n^*(\theta;x)-\theta(x)=\frac{1}{2b_n}\frac{1}{h_n(x,t)}\sum_{k=0}^{\infty}c_{k,n}\Gamma_{k,n}(t)\,x^k. \tag{12}$$

Following (iii), (iv) and (9), each term of the right hand side is non- negative, we have

$$L_n^*(\theta;x)-\theta(x)\geq 0. \tag{13}$$

Hence from (9) and (4) we can write

$$0\leq L_n^*(\theta;x)-\theta(x)\leq\frac{1}{2b_n}.$$

So we have

$$\|L_n^*(\theta)-\theta\|\leq\frac{1}{2b_n}. \tag{14}$$

Taking limit for $n\to\infty$ in (14), (iii) yields that

$$\lim_{n\to\infty}\|L_n^*(\theta)-\theta\|=0.$$

Lemma 2. we have

$$\lim_{n\to\infty}\left\|L_n^*\left(\theta^2\right)-\theta^2\right\|=0.$$

Proof. From (8) and (2) we have

$$\begin{aligned} L_n^*\left(\theta^2;x\right) &= \tfrac{1}{h_n(x,t)}\sum_{k=0}^{\infty}\tfrac{\Gamma_{k,n}(t)}{c_{k,n}}x^k\int\limits_{a_{k,n}}^{a_{k,n}+c_{k,n}}\tfrac{\xi^2}{b_n^2}d\xi \\ &= L_n\left(\theta^2;x\right)+\tfrac{1}{b_n}\tfrac{1}{h_n(x,t)}\sum_{k=0}^{\infty}\tfrac{a_{k,n}}{b_n}c_{k,n}\Gamma_{k,n}\left(t\right)x^k \\ &+\tfrac{1}{3b_n^2}\tfrac{1}{h_n(x,t)}\sum_{k=0}^{\infty}c_{k,n}^2\Gamma_{k,n}\left(t\right)x^k. \end{aligned}$$

Hence by (9) we get

$$\begin{aligned} L_n^*\left(\theta^2;x\right)-\theta^2\left(x\right) &\leq L_n\left(\theta^2;x\right)-\theta^2\left(x\right) \\ &+\tfrac{1}{b_n}L_n\left(\theta;x\right)+\tfrac{1}{3b_n^2}L_n\left(1;x\right). \end{aligned}$$

So, using (4), (5) and (6) we have

$$\begin{aligned} L_n^*\left(\theta^2;x\right)-\theta^2\left(x\right) &\leq \theta^2\left(x\right)\left(\tfrac{b_{n+1}}{b_n}-1\right) \\ &+\left(\tfrac{\varphi_n}{b_n}+\tfrac{1}{b_n}\right)\theta\left(x\right)+\tfrac{1}{3b_n^2}. \end{aligned} \tag{15}$$

Since

$$\left(\frac{s}{1-s}\right)^2=\left(\frac{s}{1-s}-\frac{x}{1-x}\right)^2+\frac{2x}{1-x}\frac{s}{1-s}-\left(\frac{x}{1-x}\right)^2$$

we get

$$\begin{aligned} L_n^*\left(\theta^2;x\right)-\theta^2\left(x\right) &= L_n^*\left(\left(\theta\left(s\right)-\theta\left(x\right)\right)^2;x\right) \\ &+2\theta\left(x\right)L_n^*\left(\theta\left(s\right)-\theta\left(x\right);x\right)\geq 0. \end{aligned} \tag{16}$$

Here we have also used the positivity of L_n^* and (13). By taking the relations (15) and (16) into account we obtain

$$0\leq\left\|L_n^*\left(\theta^2\right)-\theta^2\right\|\leq B^*\left(\left|\tfrac{b_{n+1}}{b_n}-1\right|+\tfrac{\varphi_n+1}{b_n}+\tfrac{1}{3b_n^2}\right), \tag{17}$$

where $B^*=\max\left\{1,\tfrac{A}{1-A},\left(\tfrac{A}{1-A}\right)^2\right\}$.

Now taking limit for $n\to\infty$ in (17), (iii) and (v) yield

$$\lim_{n\to\infty}\left\|L_n^*\left(\theta^2\right)-\theta^2\right\|=0.$$

Then by using (10), Lemma 1, Lemma 2 and Theorem A we can state the following approximation theorem for the operators L_n^* at once.

Theorem 1. Let L_n^* be the operator given by (8). Then for all $f \in W_w$, we have

$$\lim_{n\to\infty} \|L_n^*(f) - f\| = 0.$$

3 Rates of Convergence Properties

In this section, we compute the rate of convergence of the sequence$\{L_n^*(f;.)\}$ to function f by means of the modulus of continuity and the elements of modified Lipschitz class.

Theorem 2. Let L_n^* be the operator given by (8). Then for all $f \in W_\omega$, we have

$$\|L_n^*(f) - f\| \leq \left(1 + \sqrt{B^*}\right)\omega(\delta_n),$$

where $\delta_n := \left\{\left|\frac{b_{n+1}}{b_n} - 1\right| + \frac{\varphi_n + 1}{b_n} + \frac{1}{3b_n^2}\right\}^{\frac{1}{2}}$ and B^* is given as in Lemma 2.

Proof. We suppose that $f \in W_\omega$; then, by (10),

$$\begin{aligned} |L_n^*(f;x) - f(x)| &\leq \omega(\delta_n)\, L_n^*\left(\tfrac{|s-x|}{\delta_n} + 1; x\right) \\ &\leq \omega(\delta_n)\, L_n^*\left(1 + \tfrac{1}{\delta_n}|\theta(s) - \theta(x)|; x\right) \end{aligned}$$

$$\begin{aligned} |L_n^*(f;x) - f(x)| &= \omega(\delta_n)\left[L_n^*(1;x) + \tfrac{1}{\delta_n} L_n^*(|\theta(s) - \theta(x)|; x)\right] \\ &= \omega(\delta_n)\left[1 + \tfrac{1}{\delta_n}\tfrac{1}{h_n(x,t)}\sum_{k=0}^{\infty}\tfrac{\Gamma_{k,n}(t)}{c_{k,n}}\, x^k \int\limits_{a_{k,n}}^{a_{k,n}+c_{k,n}}\left|\tfrac{\xi}{b_n} - \theta(x)\right| d\xi\right]. \end{aligned}$$

By applying the Cauchy-Schwarz inequality to the integral first and to the sum next, we obtain

$$\begin{aligned}
&\left|L_n^*(f;x)-f(x)\right| \\
&\leq \omega(\delta_n)\left[1+\tfrac{1}{\delta_n}\tfrac{1}{h_n(x,t)}\sum_{k=0}^{\infty}\tfrac{\Gamma_{k,n}(t)}{c_{k,n}}x^k\left(\int_{a_{k,n}}^{a_{k,n}+c_{k,n}}\left(\tfrac{\xi}{b_n}-\theta(x)\right)^2 d\xi\right)^{\frac{1}{2}}\left(\int_{a_{k,n}}^{a_{k,n}+c_{k,n}}d\xi\right)^{\frac{1}{2}}\right] \\
&\leq \omega(\delta_n)\left[1+\tfrac{1}{\delta_n}\left(\tfrac{1}{h_n(x,t)}\sum_{k=0}^{\infty}\tfrac{\Gamma_{k,n}(t)}{c_{k,n}}x^k\int_{a_{k,n}}^{a_{k,n}+c_{k,n}}\left(\tfrac{\xi}{b_n}-\theta(x)\right)^2 d\xi\right)^{\frac{1}{2}}\right. \\
&\left.\times\left(\tfrac{1}{h_n(x,t)}\sum_{k=0}^{\infty}\tfrac{\Gamma_{k,n}(t)}{c_{k,n}}x^k\right)^{\frac{1}{2}}\right] \\
&\leq \omega(\delta_n)\left[1+\tfrac{1}{\delta_n}\left(\tfrac{1}{h_n(x,t)}\sum_{k=0}^{\infty}\tfrac{\Gamma_{k,n}(t)}{c_{k,n}}x^k\int_{a_{k,n}}^{a_{k,n}+c_{k,n}}\left(\tfrac{\xi}{b_n}-\theta(x)\right)^2 d\xi\right)^{\frac{1}{2}}\right] \\
&\leq \omega(\delta_n)\left[1+\tfrac{1}{\delta_n}\left(L_n^*\left((\theta(s)-\theta(x))^2;x\right)\right)^{\frac{1}{2}}\right].
\end{aligned} \tag{18}$$

This implies that

$$\left\|L_n^*(f)-f\right\| \leq w(f,\delta_n)\left[1+\tfrac{1}{\delta_n}\left(\sup_{x\in[0,A]}L_n^*\left((\theta(s)-\theta(x))^2;x\right)\right)^{\frac{1}{2}}\right]. \tag{19}$$

By using the equality (16), from (14) and (17) we get that

$$\begin{aligned}
\left\|L_n^*\left((\theta(s)-\theta(x))^2\right)\right\| &\leq \left\|L_n^*\left(\theta^2\right)-\theta^2\right\| \\
&+\max_{x\in[0,A]}2\theta(x)\left\|\left(L_n^*(\theta)-\theta\right)\right\| \\
&\leq B^*\left[\left|\tfrac{b_{n+1}}{b_n}-1\right|+\tfrac{\varphi_n+1}{b_n}+\tfrac{1}{3b_n^2}\right].
\end{aligned} \tag{20}$$

So, combining (19) with (20) we can write

$$\left\|L_n^*(f)-f\right\| \leq \omega(\delta_n)\left\{1+\frac{1}{\delta_n}\left[B^*\left(\left|\frac{b_{n+1}}{b_n}-1\right|+\frac{\varphi_n+1}{b_n}+\frac{1}{3b_n^2}\right)\right]^{\frac{1}{2}}\right\}.$$

For $\delta := \delta_n = \left[\left|\frac{b_{n+1}}{b_n} - 1\right| + \frac{\varphi_n + 1}{b_n} + \frac{1}{3b_n^2}\right]^{\frac{1}{2}}$ the proof is completed.

Now we will study the rate of convergence of the positive linear operators L_n^* by means of the elements of the modified Lipschitz class $\widetilde{Lip}_M(\alpha)$.

Let us consider the class of functions f, denoted by $\widetilde{Lip}_M(\alpha)$, as follows:

$$|f(s) - f(x)| \le M\,|\theta(s) - \theta(x)|^{\alpha}\ ,\ s, x \in [0, A],\ 0 < A < 1,$$

where $M > 0,\ 0 < \alpha \le 1$ and $f \in C[0, A]$. We can call the class $\widetilde{Lip}_M(\alpha)$ as "the modified Lipschitz class".

Theorem 3. Let L_n^* be the operator given by (8). Then for all $f \in \widetilde{Lip}_M(\alpha)$, we have

$$\|L_n^*(f) - f\| \le M\ (B^*)^{\frac{\alpha}{2}}\ \delta_n^{\alpha},$$

where δ_n and B^* are the same as in Theorem 2.

Proof. Let $f \in \widetilde{Lip}_M(\alpha)$ and $0 < \alpha \le 1$. By linearity and monotonicity of L_n^*, we have

$$\begin{aligned}
|L_n^*(f;x) - f(x)| &\le L_n^*(|f(s) - f(x)|\,; x) \\
&\le M L_n^*(|\theta(s) - \theta(x)|^{\alpha}\,; x) \\
&= M\frac{1}{h_n(x,t)}\sum_{k=0}^{\infty}\frac{\Gamma_{k,n}(t)}{c_{k,n}}\ x^k \int_{a_{k,n}}^{a_{k,n}+c_{k,n}} \left|\frac{\xi}{b_n} - \theta(x)\right|^{\alpha} d\xi.
\end{aligned}$$

By applying the Hölder inequality with $p = \frac{2}{\alpha}, q = \frac{2}{2-\alpha}$, to the integral first and to the sum next , then last formula is reduced to

$$\begin{aligned}
|L_n^*(f;x) - f(x)| &\le M\left\{\frac{1}{h_n(x,t)}\sum_{k=0}^{\infty}\frac{\Gamma_{k,n}(t)}{c_{k,n}}\ x^k \int_{a_{k,n}}^{a_{k,n}+c_{k,n}}\left(\frac{\xi}{b_n} - \theta(x)\right)^2 d\xi\right\}^{\frac{\alpha}{2}} \\
&\quad\times\left\{\frac{1}{h_n(x,t)}\sum_{k=0}^{\infty}\Gamma_{k,n}(t)\ x^k\right\}^{\frac{2}{2-\alpha}} \\
&= M\left\{L_n^*\left((\theta(s) - \theta(x))^2\,; x\right)\right\}^{\frac{\alpha}{2}}.
\end{aligned}$$

By taking into account the inequality (20) in the last inequality we have

$$\begin{aligned}
\|L_n^*(f) - f\| &\le M\left\|L_n^*\left((\theta(s) - \theta(x))^2\right)\right\|^{\frac{\alpha}{2}} \\
&\le M\left\{B^*\left[\left|\frac{b_{n+1}}{b_n} - 1\right| + \frac{\varphi_n + 1}{b_n} + \frac{1}{3b_n^2}\right]\right\}^{\frac{\alpha}{2}}.
\end{aligned}$$

Thus we obtain the desired result.

4 An r-th order generalization of the operators L_n^*

By $C^r[0,A]$ $(0<A<1,\ r\in N_o)$ we denote the space of all functions of having continuous r-th order derivative $f^{(r)}$ on the segment $[0,A]$, $(0<A<1)$, where as usual, $f^{(0)}(x)=f(x)$.

We consider the following r-th order generalization of the positive linear operators L_n^* defined by (8):

$$L_{n,r}^*(f;x)=\frac{1}{h_n(x,t)}\sum_{k=0}^{\infty}\frac{\Gamma_{k,n}(t)}{c_{k,n}}\,x^k\int_{a_{k,n}}^{a_{k,n}+c_{k,n}}\sum_{j=0}^{r}f^{(j)}\left(\frac{\xi}{\xi+b_n}\right)\frac{\left(x-\frac{\xi}{\xi+b_n}\right)^j}{j!}d\xi, \tag{21}$$

where $f\in C^r[0,A]$ $(0<A<1,\ r\in\mathbb{N}_o)$, $n\in\mathbb{N}$.

The r-th order generalization of the positive linear operators was given in [9]. But we remark that the r-th order generalization for the Kantorovich-type operators are first introduced by Özarslan, Duman and Srivastava in [15]. Note that taking $r=0$, we obtain the operators $L_n^*(f;x)$ defined by (8).

We recall that a function $f\in[0,A]$ belongs to $Lip_M(\alpha)$ if the following inequality holds:

$$|f(y)-f(x)|\le M|y-x|^{\alpha},\quad (x,y\in[0,A]).$$

Theorem 4. For any $f\in C^r[0,A]$ such that $f^{(r)}\in Lip_M(\alpha)$ we have

$$\left\|L_{n,r}^*(f)-f\right\|_{C[0,A]}\le\frac{M}{(r-1)!}\frac{\alpha}{\alpha+r}B(\alpha,r)\left\|L_n\left(|s-x|^{\alpha+r};x\right)\right\|_{C[0,A]}, \tag{22}$$

where $B(\alpha,r)$ is the beta function and $r\in\mathbb{N}_o, n\in\mathbb{N}$.

Proof. We can write from (21) that

$$\begin{aligned}f(x)-L_{n,r}^*(f;x)&=\frac{1}{h_n(x,t)}\sum_{k=0}^{\infty}\frac{\Gamma_{k,n}(t)}{c_{k,n}}\,x^k\\&\times\int_{a_{k,n}}^{a_{k,n}+c_{k,n}}\left[f(x)-\sum_{j=0}^{r}f^{(j)}\left(\frac{\xi}{\xi+b_n}\right)\frac{\left(x-\frac{\xi}{\xi+b_n}\right)^j}{j!}\right]d\xi.\end{aligned} \tag{23}$$

It is also known from Taylor's formula that

$$\begin{aligned}&f(x)-\sum_{j=0}^{r}f^{(j)}\left(\frac{\xi}{\xi+b_n}\right)\frac{\left(x-\frac{\xi}{\xi+b_n}\right)^j}{j!}\\&=\frac{\left(x-\frac{\xi}{\xi+b_n}\right)^r}{(r-1)!}\int_0^1(1-s)^{r-1}\left[f^{(r)}\left(\frac{\xi}{\xi+b_n}+s\left(x-\frac{\xi}{\xi+b_n}\right)\right)-f^{(r)}\left(\frac{\xi}{\xi+b_n}\right)\right]ds.\end{aligned} \tag{24}$$

Because of $f^{(r)} \in Lip_M(\alpha)$, one can write

$$\left| f^{(r)}\left(\frac{\xi}{\xi+b_n}+s\left(x-\frac{\xi}{\xi+b_n}\right)\right)-f^{(r)}\left(\frac{\xi}{\xi+b_n}\right)\right| \leq Ms^{\alpha}\left|x-\frac{\xi}{\xi+b_n}\right|^{\alpha}. \tag{25}$$

From the well known beta function, we can write

$$\int_0^1 s^{\alpha}(1-s)^{r-1}\,ds = B(1+\alpha,r)=\frac{\alpha}{\alpha+r}B(\alpha,r). \tag{26}$$

Substituting (25) and (26) in (24), we conclude that

$$\left| f(x)-\sum_{j=0}^{r} f^{(j)}\left(\tfrac{\xi}{\xi+b_n}\right)\frac{\left(x-\frac{\xi}{\xi+b_n}\right)^j}{j!}\right| \leq \tfrac{M}{(r-1)!}\tfrac{\alpha}{\alpha+r}B(\alpha,r)\left|x-\tfrac{\xi}{\xi+b_n}\right|^{\alpha+r}. \tag{27}$$

By taking (23) and (27) into consideration, we arrive at (22).

Now, we consider the function $g \in C[0,A]$ defined by

$$g(s)=|s-x|^{r+\alpha}. \tag{28}$$

Since $g(x)=0$ we can write $\left\|L_n\left(|s-x|^{\alpha+r}\right)\right\|_{C[0,A]}=0$. Theorem 4 yields that for all $f \in C^r[0,A]$ such that $f^{(r)} \in Lip_M(\alpha)$, we have

$$\lim_{n\to\infty}\left\|L_n^{*[r]}(f)-f\right\|=0.$$

Finally, in view of Theorem 2, Theorem 3 and $g \in Lip_{A^r}(\alpha)$, one can deduce the following result from Theorem 4 immediately:

Corollary 1. For all $f \in C^r[0,A]$, such that $f^{(r)} \in Lip_M(\alpha)$, we have

$$\|L_n^*(f)-f\| \leq \frac{M}{(r-1)!}\frac{\alpha}{\alpha+r}B(\alpha,r)\left(1+\sqrt{B^*}\omega(g,\delta_n)\right).$$

where B^* is the same as in Lemma 2, δ_n is the same as in Theorem 2 and g is defined by (28).

Corollary 2. For all $f \in C^r[0,A]$ such that $f^{(r)} \in Lip_M(\alpha)$, we have

$$\left\|L_n^{[r]}(f)-f\right\| \leq \frac{MA^r}{(r-1)!}\frac{\alpha}{\alpha+r}B(\alpha,r)(B^*)^{\frac{\alpha}{2}}\delta_n^{\alpha}$$

where B^* is the same as in Lemma 2 and δ_n is the same as in Theorem 2.

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Ali Olgun
Kırıkkale University
Faculty of Sciences and Arts,
Department of Mathematics,
Kırıkkale-TURKEY
Email: aliolgun71@gmail.com

H. Gül İnce
Gazi University
Faculty of Sciences,
Department of Mathematics,
Ankara-TURKEY
Email: ince@gazi.edu.tr

Fatma Taşdelen
Ankara University
Faculty of Sciences,
Department of Mathematics
Ankara-TURKEY
Email: tasdelen@science.ankara.edu.tr