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Commutativity of near-rings with (σ, τ)-derivations

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Abstract

In this paper we study some conditions under which a near-ring R admitting a (multiplicative) (σ, τ)-derivation d must be a commutative ring with constrained-suitable conditions on d, σ and τ. Consequently, we obtain some results which generalize some recent theorems in the literature.

1 Introduction

Let R be a left near-ring, $Z(R)$ its multiplicative center and σ, τ two maps from R to R. We say that R is 3-prime if, for all $x, y \in R$, $xRy = \{0\}$ implies $x = 0$ or $y = 0$. For all $x, y \in R$, we write $[x, y] = xy - yx$ for the multiplicative commutator, $[x, y]_{\sigma,\tau} = \sigma(x)y - y\tau(x)$,$x \circ y = xy + yx$ for the anti-commutator, $(x \circ y)_{\sigma,\tau} = \sigma(x)y + y\tau(x)$ and $(x, y) = x + y - x - y$ for the additive commutator. A map $d : R \to R$ is called a multiplicative (σ, τ)-derivation if $d(xy) = \sigma(x)d(y)+d(x)\tau(y)$ for all $x, y \in R$. If d is also an additive mapping, then d is called a (σ, τ)-derivation (see [1] and [6]). If $\tau = 1_R$, then d is called a (multiplicative) σ-derivation (see [8]). If $\sigma = \tau = 1_R$, then d is the usual (multiplicative) derivation. We say that $x \in R$ is constant if $d(x) = 0$. d will be called (σ, τ)-commuting ((σ, τ)-semicommuting) if $[x, d(x)]_{\sigma,\tau} = 0$ (if $[x, d(x)]_{\sigma,\tau} = 0$ or $(x \circ d(x))_{\sigma,\tau} = 0$) for all $x \in R$. An element $x \in R$ is called a left (right) zero divisor in R if there exists a non-zero element $y \in R$

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such that $xy = 0$ ($yx = 0$). A zero divisor is either a left or a right zero divisor. A near-ring R is called a constant near-ring, if $xy = y$ for all $x, y \in R$ and is called a zero-symmetric near-ring, if $0x = 0$ for all $x \in R$. A trivial zero-symmetric near-ring R is a zero-symmetric near-ring such that $xy = y$ for all $x \in R - \{0\}, y \in R$ [11]. We refer the reader to the books of Meldrum [11] and Pilz [12] for basic results of near-ring theory and its applications.

The study of commutativity of 3-prime near-rings by using derivations was initiated by H. E. Bell and G. Mason in 1987 [4]. In [8] A. A. M. Kamal generalizes some results of Bell and Mason by studying the commutativity of 3-prime near-rings using a σ-derivation instead of the usual derivation, where σ is an automorphism on the near-ring. M. Ashraf, A. Ali and Shakir Ali in [1] and N. Aydin and O. Golbasi in [6] generalize Kamal's work by using a (σ, τ)-derivation instead of a σ-derivation, where σ and τ are automorphisms. In this paper, we generalize many results on near-rings with (σ, τ)-derivations, where σ and τ are just two maps from the near-ring to itself which satisfy some other conditions.

In Section 2 we give some well-known results and we add some new auxiliary results on a near-ring R admitting a non-zero (σ, τ)-derivation d, which will be useful in the sequel. Proposition 2.7 determines the relation between zero-symmetric near-rings and (σ, τ)-derivations.

In Section 3 we give some examples of non-zero (σ, τ)-derivations on near-rings. Theorem 3.3 shows that under some conditions any zero-symmetric near-ring without non-zero zero divisors admitting a non-zero (σ, τ)-semicommuting (σ, τ)-derivation is an abelian near-ring. In Theorem 3.5 we show the whole cases for a trivial zero-symmetric near-ring to have a non-zero multiplicative (σ, τ)-derivation.

Section 4 is devoted to study the commutativity of a near-ring R admitting a non-zero (multiplicative) (σ, τ)-derivation d such that $d(R) \subseteq Z(R)$. As a consequence, we generalized Theorem 2 of [6], Theorem 3.1 of [1], Theorem 2.5 of [8] and Theorem 2 of [4].

Section 5 is focused on studying the commutativity of a near-ring R admitting a non-zero (multiplicative) (σ, τ)-derivation d such that $d(xy) = d(yx)$ for all $x, y \in R$. As a consequence of the results obtained in this section, we generalized Theorem 2.6 of [7] and Theorem 4.1 of [3]. The rest of Section 5 is devoted to study the commutativity under the condition $d(xy) = -d(yx)$ for all $x, y \in R$ to obtain that R is a commutative ring of characteristic 2. As a consequence, we generalized Theorem 4.2 of [3].

2 Preliminaries

In this section we give some well-known results and we add some new lemmas which will be used throughout the next sections of the paper. Throughout this section, R will be a near-ring.

Lemma 2.1 [6, Lemma 1] Let d and τ be additive mappings on a near-ring R and σ be any map from R to R. Then $d(xy) = d(x)\tau(y) + \sigma(x)d(y)$, for all $x, y \in R$ if and only if d is a (σ,τ)-derivation on R.

Lemma 2.2 [6, Lemma 2] For all $x, y, z \in R$ and σ and τ are multiplicative endomorphisms, we have that R satisfies the partial distributive law on a multiplicative (σ,τ)-derivation d, that means $(\sigma(x)d(y) + d(x)\tau(y))\tau(z) = \sigma(x)d(y)\tau(z) + d(x)\tau(y)\tau(z)$. Moreover, if τ is onto, then for all $x, y, c \in R$ we have $(\sigma(x)d(y) + d(x)\tau(y))c = \sigma(x)d(y)c + d(x)\tau(y)c$.

Lemma 2.3 Let $x \in Z(R)$ be not zero divisor. If either yx or xy is in $Z(R)$, then $y \in Z(R)$.

Proof. Suppose $xy \in Z(R)$. For all $r \in R$, we have $xyr = rxy = xry$. Thus, $x(yr - ry) = 0$. Since x is not a zero divisor in R, we get $y \in Z(R)$. The proof for $yx \in Z(R)$ is similar.

Lemma 2.4 [4, Lemma 3(ii)] If $x \in Z(R)$ is not a zero divisor in R and $x + x \in Z(R)$, then $(R, +)$ is abelian.

Lemma 2.5 [4, Lemma 3(i)] Let R be a 3-prime near-ring and $x \in Z(R) - \{0\}$. Then x is not a zero divisor in R.

Lemma 2.6 Let d be a non-zero (σ,τ)-derivation on R such that τ is an additive mapping on R and suppose $\sigma(u) \neq 0$ is not a left zero divisor in R for some $u \in R$. If $[u, d(u)]_{\sigma,\tau} = 0$ or $(u \circ d(u))_{\sigma,\tau} = 0$, then (x, u) is a constant for every $x \in R$.

Proof. We prove the lemma in the case $[u, d(u)]_{\sigma,\tau} = 0$. From $u(u + x) = u^2 + ux$ we obtain

$$d(u(u+x)) = \sigma(u)d(u+x)+d(u)\tau(u+x) = \sigma(u)d(u)+\sigma(u)d(x)+d(u)\tau(u)+d(u)\tau(x)$$

and

$$d(u^2 + ux) = d(u^2) + d(ux) = \sigma(u)d(u) + d(u)\tau(u) + \sigma(u)d(x) + d(u)\tau(x).$$

Comparing the previous two equations, we get $\sigma(u)d(x)+d(u)\tau(u) = d(u)\tau(u)+\sigma(u)d(x)$. Since $[u, d(u)]_{\sigma,\tau} = 0$, we have $\sigma(u)d(u) = d(u)\tau(u)$. So $\sigma(u)d(x) +$

$\sigma(u)d(u) = \sigma(u)d(u) + \sigma(u)d(x)$ and then $\sigma(u)d(x) + \sigma(u)d(u) - \sigma(u)d(x) - \sigma(u)d(u) = 0$. Therefore, $\sigma(u)d(x)+\sigma(u)d(u)+\sigma(u)(-d(x))+\sigma(u)(-d(u)) = 0$ and $\sigma(u)(d(x)+d(u)-d(x)-d(u)) = \sigma(u)d(x+u-x-u) = \sigma(u)d((x,u)) = 0$. Since $\sigma(u) \neq 0$ is not a left zero divisor in R, we get $d((x,u)) = 0$ and (x,u) is a constant. The proof is similar for the case $(u \circ d(u))_{\sigma,\tau} = 0$.

Proposition 2.7 A near-ring R is admitting a multiplicative (σ, τ)-derivation d such that σ and τ are multiplicative endomorphisms and $\tau(0) = 0$ where τ is either one-to-one or onto if and only if R is zero-symmetric.

Proof. By [11, Theorem 1.15] any near-ring can be expressed as the sum of $R_o = \{x \in R : 0x = 0\}$ the unique maximal zero-symmetric subnear-ring of R and $R_c = 0R = \{0r : r \in R\}$ the unique maximal constant subnear-ring of R.

1) Suppose that R admitting a multiplicative (σ, τ)-derivation d such that σ and τ are multiplicative endomorphisms and $\tau(0) = 0$ where τ is either one-to-one or onto. Suppose also that R is not zero-symmetric, so $\{0\} \subsetneqq 0R$. If $z \in 0R$, then $z = 0y$ for some $y \in R$. For all $x \in R$, we have $xz = x0y = 0y = z$ and $zx = 0yx \in 0R$. Observe that $\tau(z) = \tau(0y) = \tau(0)\tau(y) = 0\tau(y) \in 0R$. Thus, $z \in 0R$ implies $\tau(z) \in 0R$. Since τ is either one-to-one or onto, we have $\tau(0R) \neq \{0\}$. So there exists $z \in 0R$ such that $\tau(z) \neq 0$. Hence, $d(z) = d(z^2) = \sigma(z)d(z) + d(z)\tau(z) = \sigma(z)d(z) + \tau(z)$. Multiplying both sides by $\sigma(z)$, we have $\sigma(z)d(z) = \sigma(z)\sigma(z)d(z)+\sigma(z)\tau(z) = \sigma(z)d(z)+\tau(z)$. Thus, $\tau(z) = 0$, which is a contradiction. Therefore, R must be zero-symmetric.

2) Suppose R is zero-symmetric. It is easy to show that the zero map is a derivation on R which is called the zero derivation on R. Trivially this zero derivation on R is a $(1_R, 1_R)$-derivation on R where 1_R is the identity automorphism on R.

For the usual derivation, there are some classes of near-rings which has only the zero derivation. The most important one is the subclass of the class of simple near-rings with identity $\{M_o(G) : G \text{ is any group}\}$, where the near-ring $M_o(G)$ is the set of all zero preserving maps from G to itself with addition and composition of maps [5, Theorem 1.1]. For the (σ, τ)-derivation, we have a better result in the proof of Proposition 2.9 than the zero derivation.

Corollary 2.8 A near-ring R is admitting a multiplicative σ-derivation such that σ is a multiplicative endomorphism if and only if R is zero-symmetric.

Proposition 2.9 If R is a non-zero near-ring, then it has a non-zero (multiplicative) (σ, τ)-derivation d.

Proof. Take d to be any non-zero additive map (any non-zero map) from R to R such that $d(xy) = f(x)d(y)$ for all $x, y \in R$, where f is a map from R to itself (e. g. take $d = f$ as the identity map). Let $\sigma = f$ and $\tau = 0$. Then for all $x, y \in R$ we have $d(xy) = f(x)d(y) = f(x)d(y) + d(x)0 = \sigma(x)d(y) + d(x)\tau(y)$. Hence, d is a non-zero (σ, τ)-derivation.

Note that the (σ, τ)-derivation mentiond in the proof of Proposition 2.9 includes all endomorphisms (multiplicative endomorphisms) on R by putting $f = d$. Observe that also if d.is a right multiplcative map (i. e. there exists $c \in R$ such that $d(x) = xc$ for all $x \in R$), then $d(xy) = xd(y)$ for all $x, y \in R$. So the multiplicative (σ, τ)-derivation mentiond in the proof of Proposition 2.9 includes all right multiplcative maps by putting f equal to the identity map.

The following example shows that the condition "τ is either one-to-one or onto" in Proposition 2.7 is essential.

Example 2.1 Let R be any non-zero constant near-ring. Then R is not zero-symmetric. Suppose $\tau = 0$ and σ is any endomorphism on R. So for any additive mapping d of R and for all $x, y \in R$ we have $d(xy) = d(y) = \sigma(x)d(y) = \sigma(x)d(y) + d(x)\tau(y)$. Therefore, any additive mapping on R is a (σ, τ)-derivation on R which illustrates that Proposition 2.7 is not true if τ is neither one-to-one nor onto.

Lemma 2.10 Let R be a distributive near-ring such that there exists $a \in R$ which is not a left zero divisor for (x, y) for all $x, y \in R$. Then R is a ring.

Proof. Since R is distributive, we have $(r+r)(x+y) = (r+r)x+(r+r)y = rx+rx+ry+ry$ and $(r+r)(x+y) = r(x+y)+r(x+y) = rx+ry+rx+ry$ for all $r, x, y \in R$. Comparing the previous two expressions, we get $rx+ry = ry+rx$ and hence $r(x + y - x - y) = 0$ for all $r, x, y \in R$. Choosing $r = a$, we have $x + y - x - y = 0$ and $(R, +)$ is abelian. Hence, R is a ring.

Definition 2.1 [10] A near-ring R is called n-distributive, where n is a positive integer, if for all $a, b, c, d, r, a_i, b_i \in R$,

(i) $ab + cd = cd + ab$

(ii) $(\sum a_i b_i)r = \sum a_i b_i r$, where $i = 1, 2, \ldots, n$.

Lemma 2.11 Let R be a 2-distributive near-ring. Then

(i) R is zero-symmetric.

(ii) For all $x, y, r \in R$, we have $-xyr = (-xy)r$.

Proof. (i) For all $r \in R$, we get $0r + 0r = 00r + 00r = (00 + 00)r = 0r$. So $0r = 0$ and R is zero-symmetric.

(ii) For all $x, y, r \in R$, we have $xyr + (-xy)r = (xy + (-xy))r = 0r = 0$. Thus, $(-xy)r = -xyr$ for all $x, y, r \in R$.

Lemma 2.12 Let R be a 2-distributive near-ring with identity. Then R is a ring.

Proof. Let 1 be the identity of R. Using Definition 2.1, we have $r + s = r1 + s1 = s1 + r1 = s + r$ for all $r, s \in R$ and $(R, +)$ is an abelian group. Now, $(x + y)r = (x1 + y1)r = x1r + y1r = xr + yr$ for all $x, y, r \in R$, so R is distributive. Hence, R is a ring.

3 Examples and commutativity of $(R, +)$

We start this section by giving three examples of (σ, τ)-derivations on a near-ring.

Example 3.1 Let R be a 2-distributive near-ring with a distributive element a in R (see [9, Example 2.4] for an example of a 2-distributive near-ring with some distributive elements which is not a distributive near-ring). We will now prove that for any endomorphisms σ, τ on R, $d(x) = \sigma(x)a - a\tau(x)$ is a (σ, τ)-derivation on R. Using (i) and (ii) of Lemma 2.11 and Definition 2.1(i), observe that

$$\begin{aligned} d(x+y) &= \sigma(x+y)a - a\tau(x+y) = (\sigma(x) + \sigma(y))a - a(\tau(x) + \tau(y)) \\ &= \sigma(x)a + \sigma(y)a - a\tau(y) - a\tau(x) = \sigma(x)a - a\tau(x) + \sigma(y)a - a\tau(y) \\ &= d(x) + d(y) \end{aligned}$$

and d is an additive mapping. Also, from Definition 2.1(ii) we have

$$\begin{aligned} d(xy) &= \sigma(xy)a - a\tau(xy) = \sigma(x)\sigma(y)a - a\tau(x)\tau(y) \\ &= \sigma(x)\sigma(y)a - \sigma(x)a\tau(y) + \sigma(x)a\tau(y) - a\tau(x)\tau(y) \\ &= \sigma(x)[\sigma(y)a - a\tau(y)] + [\sigma(x)a - a\tau(x)]\tau(y) = \sigma(x)d(y) + d(x)\tau(y). \end{aligned}$$

In particular, If R has an identity, then R is a ring by Lemma 2.12. If we take a to be the identity, then for any endomorphisms σ, τ on R, $d(x) = \sigma(x) - \tau(x)$ is a (σ, τ)-derivation on R.

Example 3.2 Let R be an abelian near-ring with identity $1 \in R$ and without non-zero zero divisors which is not a ring (for example take R to be any near-field which is not a division ring). Take σ to be any non-zero multiplicative endomorphism on R such that $\sigma \neq \tau$ where τ is defined by $\tau(0) = 0$ and $\tau(x) = 1$ for all $x \in R - \{0\}$. Observe that τ is a multiplicative endomorphism on R. Define $d : R \to R$ by $d(x) = \sigma(x)a - a\tau(x)$ where

$a \in R - \{0\}$. So d is a non-zero multiplicative (σ,τ)-derivation on R. indeed, for all $x \in R, y \in R$, we have

$$\begin{aligned} d(xy) &= \sigma(xy)a - a\tau(xy) = \sigma(x)\sigma(y)a - a\tau(x)\tau(y) \\ &= \sigma(x)\sigma(y)a - \sigma(x)a\tau(y) + \sigma(x)a\tau(y) - a\tau(x)\tau(y) \\ &= \sigma(x)[\sigma(y)a - a\tau(y)] + [\sigma(x)a - a\tau(x)]\tau(y) = \sigma(x)d(y) + d(x)\tau(y). \end{aligned}$$

Also, for all $c \in R$ such that $d(c) \neq 0$, we obtain that $d(c)$ is not a left zero divisor in R.

Example 3.3 Let N be a zero-symmetric abelian near-ring which has a non-zero ideal I contained in $Z(N)$. Let $a \in I$ and define $d : N \to N$ by $d(x) = \sigma(x)a - \tau(x)a$ for all $x \in N$, where σ and τ are endomorphisms of N. Then $d(N) \subseteq I \subseteq Z(N)$ and d is a (σ,τ)-derivation on N. Indeed,

$$\begin{aligned} d(x+y) &= \sigma(x+y)a - \tau(x+y)a = \sigma(x)a + \sigma(y)a - \tau(y)a - \tau(x)a \\ &= \sigma(x)a - \tau(x)a + \sigma(y)a - \tau(y)a = d(x) + d(y) \end{aligned}$$

which means that d is an additive mapping.

$$\begin{aligned} d(xy) &= \sigma(xy)a - \tau(xy)a = \sigma(x)\sigma(y)a - \sigma(x)\tau(y)a + \sigma(x)\tau(y)a - \tau(x)\tau(y)a \\ &= \sigma(x)[\sigma(y)a - \tau(y)a] + \tau(y)a\sigma(x) - \tau(y)a\tau(x) \\ &= \sigma(x)[\sigma(y)a - \tau(y)a] + \tau(y)[\sigma(x)a - \tau(x)a] \\ &= \sigma(x)d(y) + \tau(y)d(x) = \sigma(x)d(y) + d(x)\tau(y). \end{aligned}$$

For example, take N to be the direct sum of M and R, where M is a zero-symmetric abelian near-ring and R a commutative ring, which generalizes an example due to Samman in 2009 [13].

Remark 3.1 We know from [14, Lemma 2] that for a derivation d on a near-ring R that if $x \in R$ is central, then so is $d(x)$. This is not true in a (σ,τ)-derivation on R, even if we take R to be a ring and σ,τ are automorphisms on R or $\sigma = \tau$ is an endomorphism on R which is not onto. The next example illustrates that.

Example 3.4 Let $R = M_2(\mathbb{Z}) \times M_2(\mathbb{Z})$ where $\mathbb{Z}$ is the ring of integers. Then R is a non-commutative ring which has a non-zero center $Z(R)$, where

$$Z(R) = \left\{ \left(\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} \right) : a, b \in \mathbb{Z} \right\}.$$

Define $d : R \to R$ by $d(x) = \sigma(x)A - A\tau(x)$ for all $x \in R$, where A is a non-zero element of R, σ is the identity map on R and $\tau(x,y) = (y,x)$ for all $x, y \in R$. Clearly that σ, τ are automorphisms on R. So d is a (σ, τ)-derivation on R by Example 3.1. Let $A = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right)$. Thus, for all $a, b, c, d, e, f, g, h \in \mathbb{Z}$

$$d\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} e & f \\ g & h \end{bmatrix}\right) = \left(\begin{bmatrix} a-e & -f \\ c & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right).$$

Now, we have $z = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) \in Z(R)$ and $d\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right)$ which means $d(z) \notin Z(R)$, since

$$\begin{aligned} \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right)\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) \\ &\neq \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right)\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) \\ &= \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right). \end{aligned}$$

Now take $R = \left\{\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in \mathbb{Z}\right\}$. Define $d : R \to R$ by $d(x) = \sigma(x)A - A\sigma(x)$ for all $x \in R$, where A is a non-zero element of R and σ is an endomorphism on R defined by $\sigma\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$. Clearly σ is not onto. Choosing $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, we have $d\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$ for all $a, b, c \in \mathbb{Z}$. Now $e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in Z(R)$, but $d(e) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \notin Z(R)$, since

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

For Remark 3.1, we have the following result:

Proposition 3.1 [2, Proposition 2.1] Let R be a near-ring with a (σ, σ)-derivation d such that σ is an epimorphism on R. If $x \in Z(R)$, then $d(x) \in Z(R)$.

Remark 3.2 In the usual derivation we have that for a derivation d on a near-ring R, $d(R) \subseteq Z(R)$ implies $d(xy) = d(yx)$ for all $x, y \in R$, but the converse is not true. For (σ, τ)-derivations, $d(R) \subseteq Z(R)$ does not imply $d(xy) = d(yx)$ for all $x, y \in R$ even for rings, as Example 3.5 shows.

Example 3.5 Let $R = \left\{ \begin{bmatrix} a & 3b \\ 3c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z}_9 \right\}$. Then R is a subring of $M_2(\mathbb{Z}_9)$. So $d : R \to R$ defined by $d(x) = \sigma(x) \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \tau(x)$ for all $x \in R$ where σ, τ are endomorphisms on R, is a (σ, τ)-derivation by Example 3.1. Take $\tau = 0$ and σ is the identity. Thus, for all $a, b, c, d \in \mathbb{Z}_9$

$$d\left(\begin{bmatrix} a & 3b \\ 3c & d \end{bmatrix}\right) = \sigma(\begin{bmatrix} a & 3b \\ 3c & d \end{bmatrix}) \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 3a & 0 \\ 0 & 3d \end{bmatrix} \in Z(R)$$

and then $d(R) \subseteq Z(R)$. Observe that $d\left(\begin{bmatrix} 1 & 3 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}\right) = d\left(\begin{bmatrix} 2 & 6 \\ 0 & 3 \end{bmatrix}\right) = \begin{bmatrix} 6 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} = d\left(\begin{bmatrix} 1 & 3 \\ 3 & 3 \end{bmatrix}\right) = d\left(\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 3 \end{bmatrix}\right)$.

The following result shows that when $d(R) \subseteq Z(R)$ implies $d(xy) = d(yx)$ for all $x, y \in R$.

Proposition 3.2 Let R be a near-ring with a (σ, τ)-derivation d such that $d(R) \subseteq Z(R)$ and τ is an additive mapping on R. Then d is a (τ, σ)-derivation on R if and only if $d(xy) = d(yx)$ for all $x, y \in R$.

Proof. Using $d(R) \subseteq Z(R)$ and Lemma 2.1, we have $d(xy) = d(x)\tau(y) + \sigma(x)d(y) = \tau(y)d(x) + d(y)\sigma(x)$ for all $x, y \in R$. Now suppose d is a (τ, σ)-derivation. Thus, $d(xy) = \tau(y)d(x) + d(y)\sigma(x) = d(yx)$. Conversely, suppose $d(xy) = d(yx)$ for all $x, y \in R$. Therefore, $d(yx) = d(xy) = \tau(y)d(x) + d(y)\sigma(x)$ for all $x, y \in R$ which means d is a (τ, σ)-derivation on R.

Theorem 3.3 Let R be a zero-symmetric near-ring without non-zero zero divisors. If R admits a non-zero (σ, τ)-semicommuting (σ, τ)-derivation d on R such that τ is a monomorphism on R. Then $(R, +)$ is abelian.

Proof. For any additive commutator (x, y), if $\sigma(y) \neq 0$ for some $y \in R$, then (x, y) is constant by Lemma 2.6. If $\sigma(y) = 0$, then for both cases $[y, d(y)]_{\sigma,\tau} = 0$ or $(y \circ d(y))_{\sigma,\tau} = 0$ we have $\sigma(y)d(y) = 0$ and hence $d(y)\tau(y) = 0$. Since R does not have non-zero zero divisors, we obtain that either $d(y) = 0$ or $\tau(y) = 0$. If $d(y) = 0$, then $d(x + y - x - y) = 0$ and (x, y) is constant. If $\tau(y) = 0$, then $y = 0$ as τ is a monomorphism. So $d(x + y - x - y) = 0$ and (x, y) is constant. Hence, in all cases (x, y) is constant. Since y is an arbitrary, we have (x, y) is constant for all additive commutators. Observe that $(zx, zy) = zx + zy - zx - zy = z(x + y - x - y) = z(x, y)$ for all $x, y, z \in R$. It follows that $d(z(x, y)) = 0$ and $\sigma(z)d(x, y) + d(z)\tau(x, y) = d(z)\tau(x, y) = 0$ for all $x, y, z \in R$. As d is non-zero, choose $z = t \in R$ such that $d(t) \neq 0$. Since $d(t)$ is not a zero divisor in R, we have $\tau(x, y) = 0$ and then $(x, y) = 0$ for all $x, y \in R$. Hence, $(R, +)$ is abelian.

In [9, Example 2.14], we mentioned an example of a class of 3-prime abelian near-rings which are not rings admitting a non-zero (σ, σ)-derivation and a non-zero $(1, \sigma)$-derivation, where $1 = i_R$ the identity map on R. Also, in Example 3.2 above, we have an example of a non-zero multiplicative (σ, τ)-derivation on a near-field (which is an abelian near-ring without non-zero zero divisors).

Corollary 3.4 Let R be a near-ring without non-zero zero divisors. If R admits a non-zero σ-semicommuting σ-derivation d on R, then $(R, +)$ is abelian.

The class of trivial zero-symmetric near-rings is very useful as a tool in some proofs of results in near-rings, for example, to prove the simplicity of $M(G)$ and $M_o(G)$ (see Lemma 1.34, Theorem1.37 and Theorem 1.42 of [11]). Observe that for any near-ring $R \neq \{0\}$, the identity i_R is a non-zero (σ, τ)-derivation on R with ($\sigma = 0$ and $\tau = i_R$) or ($\sigma = i_R$ and $\tau = 0$). In the following result we will show that if d is a non-zero multiplicative (σ, τ)-derivation on a trivial zero symmetric near-ring R, what are the possible cases.

Theorem 3.5 Let R be a trivial zero symmetric near-ring with a non-zero multiplicative (σ, τ)-derivation d. Then we have one of the following cases:

(i) $\sigma = 0$ and $d = \tau$.

(ii) $\tau = 0$, $\sigma(x) \neq 0$ for all $x \in R - \{0\}$ and $\sigma(0) = 0$ if and only if $d(0) = 0$. If $\sigma(0) \neq 0$, then d is a constant function.

(iii) $d = \tau$ and $\sigma \neq 0$ such that $\sigma(x)d(x) = 0 = \sigma(0) = d(0)$ and if $\sigma(x) = 0$ then $d(x) \neq 0$ for all $x \in R - \{0\}$.

(iv) $d(0) = \tau(x) \neq 0, \sigma(y) \neq 0$ and $d(x) = \tau(0) = 0$ for all $x \in R - \{0\}, y \in R$.

(v) $\tau(y) = d(0) \neq 0, \sigma(x) \neq 0$ and $d(x) = \sigma(0) = 0$ for all $x \in R - \{0\}, y \in R$.

Proof. Suppose $\sigma = 0$. Then for all $x \in R - \{0\}, y \in R$, we have $d(y) = d(xy) = \sigma(x)d(y) + d(x)\tau(y) = d(x)\tau(y)$. As $d \neq 0$, we have $d(x) \neq 0$ for all $x \in R - \{0\}$. That means $d(y) = \tau(y)$ for all $y \in R$ and $d = \tau$. Hence, we get (i).

Now suppose $\tau = 0$. Then for all $x \in R-\{0\}, y \in R$, we have $d(y) = d(xy) = \sigma(x)d(y)$. For all $x \in R-\{0\}$, we get that $d(a) = d(xa) = \sigma(x)d(a)$ which implies that $\sigma(x) \neq 0$ for all $x \in R-\{0\}$. If $d(0) = 0$, then $0 = d(0) = d(0a) = \sigma(0)d(a)$. Thus, $\sigma(0) = 0$. Now, if $\sigma(0) = 0$, then $d(0) = d(00) = \sigma(0)d(0) = 0$. Now, if $\sigma(0) \neq 0$, then $d(0) = d(0x) = \sigma(0)d(x) = d(x)$ for all $x \in R$. Thus, d is a constant function. Hence, we get (ii).

After that, suppose $\sigma \neq 0$ and $\tau \neq 0$. There exist $a, b, c \in R$ such that $d(a) \neq 0$, $\sigma(b) \neq 0$ and $\tau(c) \neq 0$. For all $x \in R-\{0\}, y \in R$, we have $d(y) = d(xy) = \sigma(x)d(y) + d(x)\tau(y)$. If there exists $x \in R-\{0\}$ such that $\sigma(x) = 0$ then for all $y \in R$, we have $d(y) = d(xy) = d(x)\tau(y)$. If $d(x) = 0$, then $d(y) = d(xy) = d(x)\tau(y) = 0$ for all $y \in R$ and hence $d = 0$, a contradiction. So $d(x) \neq 0$ and $d(y) = d(x)\tau(y) = \tau(y)$ for all $y \in R$. Thus, $d = \tau$. Therefore, $d(x) = d(xx) = \sigma(x)d(x)+d(x)d(x) = \sigma(x)d(x)+d(x)$ for all $x \in R$. That implies $\sigma(x)d(x) = 0$ for all $x \in R$. So $\sigma(a) = \sigma(c) = d(b) = 0$. Then $d(0) = d(0b) = \sigma(0)d(b) + d(0)d(b) = 0$. Also, $0 = d(0) = d(0a) = \sigma(0)d(a) + d(0)d(a) = \sigma(0)d(a)$. That means $\sigma(0) = 0$. So $a \neq 0$, $b \neq 0$ and $c \neq 0$. Hence, we get (iii).

Now, suppose that $\sigma(x) \neq 0$ for all $x \in R-\{0\}$. Then for all $x \in R-\{0\}, y \in R$, we have $d(y) = d(xy) = \sigma(x)d(y) + d(x)\tau(y) = d(y) + d(x)\tau(y)$. So $d(x)\tau(y) = 0$ for all $x \in R-\{0\}, y \in R$. As $\tau \neq 0$, we deduce that $d(x) = 0$ for all $x \in R-\{0\}$. That means $a = 0$ as $d \neq 0$. Therefore, $0 \neq d(0) = d(0x) = \sigma(0)d(x)+d(0)\tau(x) = d(0)\tau(x) = \tau(x)$ for all $x \in R-\{0\}$. If $\sigma(0) \neq 0$, then $d(0) = d(00) = \sigma(0)d(0) + d(0)\tau(0) = d(0) + \tau(0)$ and $0 = \tau(0)$. Hence, we get (iv).

If $\sigma(0) = 0$, then $d(0) = d(00) = \sigma(0)d(0) + d(0)\tau(0) = \tau(0)$. Hence, we get (v).

In the following example, we will give an example for each case of the five cases mentioned in Theorem 3.5.

Example 3.6 Let R be a non-zero trivial zero symmetric near-ring. For case (i), take $\sigma = 0$ and $d = \tau = i_R$ the identity map. For case (ii), if $\sigma(0) = 0$, then take $\sigma = d = i_R$ and $\tau = 0$. If $\sigma(0) \neq 0$, then take $\tau = 0$ and $\sigma = d$ as a constant map defined by $d(x) = c \neq 0$ for all $x \in R$. For case (iii), let $R-\{0\} = S \cup T$ such that $S \cap T = \phi$ and $S \neq \phi \neq T$. Let $d = \tau, \sigma$ be any maps defined as the following, $0 = \sigma(0) = d(0)$ and $d(x) = x, \sigma(x) = 0$ if $x \in S$ and $d(x) = 0, \sigma(x) = x$ if $x \in T$. For case (iv), take σ as a constant map defined by $\sigma(x) = c \neq 0$ for all $x \in R$ and define d and τ as the following $d(x) = \tau(0) = 0$ and $d(0) = \tau(x) = c$ for all $x \in R-\{0\}$. For case (v), take τ as a constant map defined by $\tau(x) = c \neq 0$ for all $x \in R$ and define d and σ as the following $d(x) = \sigma(0) = 0$ and $d(0) = \sigma(x) = c$ for all $x \in R-\{0\}$.

4 The condition $d(R) \subseteq Z(R)$

We shall prove some theorems in this section on commutativity of near-rings which generalize known results due to [4], [8], [1] and [6].

Theorem 4.1 Let R be a near-ring with a non-zero multiplicative (σ, τ)-derivation d such that σ and τ are multiplicative endomorphisms and τ is either one-to-one or onto. If $d(R) \subseteq Z(R)$ and there exists $a \in R$ such that $d(a)$ is not a left zero divisor in R, then R is a commutative ring.

Proof. For all $x, y \in R$, we have $d(xy) = \sigma(x)d(y) + d(x)\tau(y) \in Z(R)$. Multiplying $d(xy)$ by $\tau(y)$ in the right and the left respectively, we get

$$\begin{aligned} d(xy)\tau(y) &= (\sigma(x)d(y) + d(x)\tau(y))\tau(y) = \sigma(x)d(y)\tau(y) + d(x)\tau(y)\tau(y) \\ &= d(y)\sigma(x)\tau(y) + d(x)\tau(y)\tau(y) \end{aligned}$$

by using Lemma 2.2 and $\tau(y)d(xy) = \tau(y)\sigma(x)d(y)+\tau(y)d(x)\tau(y) = d(y)\tau(y)\sigma(x)+ d(x)\tau(y)\tau(y)$ for all $x, y \in R$. So $d(y)\sigma(x)\tau(y) = d(y)\tau(y)\sigma(x)$ which means that $d(y)[\sigma(x)\tau(y) - \tau(y)\sigma(x)] = 0$ for all $x, y \in R$. Since $d(a)$ is not a left zero divisor in R, we have $\sigma(x)\tau(a) = \tau(a)\sigma(x)$ for all $x \in R$. Multiplying $d(xy)$ by $\tau(a)$ in the right and the left respectively, we have $d(xy)\tau(a) = \sigma(x)d(y)\tau(a) + d(x)\tau(y)\tau(a) = d(y)\sigma(x)\tau(a) + d(x)\tau(y)\tau(a)$ and $\tau(a)d(xy) = d(y)\tau(a)\sigma(x) + d(x)\tau(a)\tau(y)$ for all $x, y \in R$. Using that $\sigma(x)\tau(a) = \tau(a)\sigma(x)$ for all $x \in R$, we have $d(x)\tau(a)\tau(y) = d(x)\tau(y)\tau(a)$. So $d(x)[\tau(a)\tau(y) - \tau(y)\tau(a)] = 0$ for all $x, y \in R$. Using $d(a)$ is not a left zero divisor in R, we get $\tau(a)\tau(y) = \tau(y)\tau(a)$ for all $y \in R$. Now, multiply $d(xa)$ by $\tau(z)$ in the right and the left respectively. It follows that $d(xa)\tau(z) = d(a)\sigma(x)\tau(z) + d(x)\tau(a)\tau(z)$ and $\tau(z)d(xa) = d(a)\tau(z)\sigma(x) + d(x)\tau(z)\tau(a)$ for all $x, z \in R$. Using that $\tau(a)\tau(y) = \tau(y)\tau(a)$ for all $y \in R$, we get $d(a)\sigma(x)\tau(z) = d(a)\tau(z)\sigma(x)$. So $d(a)[\sigma(x)\tau(z) - \tau(z)\sigma(x)] = 0$ and then

$$\sigma(x)\tau(z) = \tau(z)\sigma(x) \qquad \text{for all } x, z \in R. \tag{4.1}$$

Multiplying $d(ay)$ by $\tau(z)$ in the right and the left respectively, we have $d(ay)\tau(z) = d(y)\sigma(a)\tau(z) + d(a)\tau(y)\tau(z)$ and $\tau(z)d(ay) = d(y)\sigma(a)\tau(z) + d(a)\tau(z)\tau(y)$ for all $y, z \in R$. Using (4.1), we get $d(a)\tau(z)\tau(y) = d(a)\tau(y)\tau(z)$. So $d(a)[\tau(z)\tau(y) - \tau(y)\tau(z)] = 0$ and

$$\tau(z)\tau(y) = \tau(y)\tau(z) \qquad \text{for all } y, z \in R. \tag{4.2}$$

If τ is either one-to-one or onto, then R is a commutative near-ring. Using, $0 \neq d(a) \in Z(R)$ is not a left zero divisor in R and Lemma 2.10, we have that R is a commutative ring.

The condition "τ is either one-to-one or onto" in Theorem 4.1 is essential even for rings.

Example 4.1 Let $R = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ c & 0 & a \end{bmatrix} : a \in Z(S), b, c \in S \right\}$ where S is any non-commutative division ring which has non-zero center. Take for example

$$S = \left\{ \begin{bmatrix} z & w \\ -\overline{w} & \overline{z} \end{bmatrix}, z \text{ and } w \text{ are complex numbers} \right\}$$

where $\overline{z}$ is the complex conjugate of z. Then S is a non-commutative division ring which has a non-zero center as if r is a real number, then for every complex numbers z, w we have

$$\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} z & w \\ -\overline{w} & \overline{z} \end{bmatrix} = \begin{bmatrix} rz & rw \\ -r\overline{w} & r\overline{z} \end{bmatrix} = \begin{bmatrix} zr & wr \\ -\overline{w}r & \overline{z}r \end{bmatrix} = \begin{bmatrix} z & w \\ -\overline{w} & \overline{z} \end{bmatrix} \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}.$$

Then R is a non-commutative ring. Define $d : R \to R$ by $d\left(\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ c & 0 & a \end{bmatrix} \right) = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$. So d is an additive mapping. Taking $\sigma = d$, then σ is an endomorphism on R. Taking $\tau = 0$, then τ is neither one-to-one nor onto. Also, d is a non-zero (σ,τ)-derivation and $d(R) \subseteq Z(R)$. If there exists $\begin{bmatrix} e & 0 & 0 \\ 0 & f & 0 \\ g & 0 & e \end{bmatrix} \in R$ such that $d\left(\begin{bmatrix} e & 0 & 0 \\ 0 & f & 0 \\ g & 0 & e \end{bmatrix} \right) = \begin{bmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e \end{bmatrix} \neq 0$ and $\begin{bmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ c & 0 & a \end{bmatrix} = 0$ for some $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ c & 0 & a \end{bmatrix} \in R$, then $e \neq 0$ and $\begin{bmatrix} ea & 0 & 0 \\ 0 & eb & 0 \\ ec & 0 & ea \end{bmatrix} = 0$. Since S has no non-zero divisors of zero, we have $a = b = c = 0$ and hence $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ c & 0 & a \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. That means if $d(A) \neq 0$ for some $A \in R$, then it is not a zero divisor in R. Using the example above with $\sigma = 0$ and $\tau = d$, we get another counter example.

The next corollary generalizes Theorem 2 of O. Golbasi and N. Aydin [6] and Theorem 3.1 of M. Ashraf, A. Ali and Shakir Ali [1].

Corollary 4.2 Let R be a 3-prime near-ring with a non-zero multiplicative (σ, τ)-derivation d such that σ and τ are multiplicative endomorphisms and τ is either one-to-one or onto. If $d(R) \subseteq Z(R)$, then R is a commutative ring.

Proof. Since d is a non-zero multiplicative (σ, τ)-derivation, there exists $a \in R$ such that $0 \neq d(a)$ and by Lemma 2.5 $d(a)$ is not a left zero divisor in R. So R is a commutative ring by Theorem 4.1.

Corollary 4.3 Let R be a near-ring with a non-zero multiplicative σ-derivation d such that σ is a multiplicative endomorphism on R. If $d(R) \subseteq Z(R)$ and there exists $a \in R$ such that $d(a)$ is not a left zero divisor in R, then R is a commutative ring.

Proof. Since τ here is the identity isomorphism, we get the result from Theorem 4.1.

The following corollary generalizes Theorem 2.5 of Kamal [8] and Theorem 2 of Bell and Mason [4].

Corollary 4.4 Let R be a 3-prime near-ring with a non-zero multiplicative σ-derivation d such that σ is a multiplicative endomorphism on R and $d(R) \subseteq Z(R)$. Then R is a commutative ring.

Proof. Since τ here is the identity isomorphism, we get the result by Corollary 4.2.

Theorem 4.5 Let R be a 3-prime near-ring with a non-zero multiplicative (σ, τ)-derivation d that satisfies $d(R) \subseteq Z(R)$ such that σ and τ are endomorphisms on R and either $\ker\tau \cap \ker\sigma = \{0\}$ or $\tau(R) \cup \sigma(R) = R$. Then R is a commutative ring.

Proof. Since d is a non-zero multiplicative (σ, τ)-derivation, there exists $a \in R$ such that $0 \neq d(a)$ and by Lemma 2.5 $d(a)$ is not a left zero divisor in R. So the first Part of this proof is similar to the proof of 4.1 up to equation (4.2). Now, we have two possible cases:

Case 1: $d(b) = 0$ for all $b \in \ker\tau$.

From (4.2), we obtain that $0\tau(x) = \tau(x)0 = 0$ for all $x \in R$. Thus, $d(bx) = \sigma(b)d(x) + d(b)\tau(x) = \sigma(b)d(x)$ for all $x \in R$. Multiplying $d(bx)$ by $\sigma(y)$ in the left and the right respectively, we have $\sigma(y)d(bx) = \sigma(y)\sigma(b)d(x) = d(x)\sigma(y)\sigma(b)$ for all $x, y \in R$ and $d(bx)\sigma(y) = \sigma(b)d(x)\sigma(y) = d(x)\sigma(b)\sigma(y)$. Choosing $x = a$, we have $d(a)[\sigma(y)\sigma(b) - \sigma(b)\sigma(y)] = 0$ and then

$$\sigma(y)\sigma(b) - \sigma(b)\sigma(y) = 0 \qquad \text{for all } y \in R \text{ and for all } b \in \ker\tau. \qquad (4.3)$$

Suppose first that $\ker\tau \cap \ker\sigma = \{0\}$. So from (4.2) and (4.3) we conclude that $yb - by \in \ker\tau \cap \ker\sigma = \{0\}$ for all $y \in R$ and for all $b \in \ker\tau$. Thus,

$\ker\tau \subseteq Z(R)$. If τ is a monomorphism, then by (4.2) R is a commutative ring. If there exists $0 \neq b \in \ker\tau$, then $\tau(\sigma(x)b) = \tau(\sigma(x))\tau(b) = \tau(\sigma(x))0 = 0$ for all $x \in R$ which means $\sigma(x)b \in \ker\tau$. Thus, $\sigma(x)b \in Z(R)$ for all $x \in R$. By Lemma 2.3 and Lemma 2.5 we conclude that $\sigma(x) \in Z(R)$ for all $x \in R$. So

$$\sigma(x)\sigma(z) - \sigma(z)\sigma(x) = 0 \qquad \text{for all } x, z \in R. \tag{4.4}$$

Equations (4.2) and (4.4) imply that $xy - yx \in \ker\tau \cap \ker\sigma = \{0\}$ for all $x, y \in R$ and hence R is a commutative near-ring. Now, Suppose $\tau(R)\cup\sigma(R) = R$. From (4.1) and (4.3), we conclude that $\sigma(b) \in Z(R)$ for all $b \in \ker\tau$. Since $\tau(xb) = \tau(x)\tau(b) = 0$ for all $x \in R$ and for all $b \in \ker\tau$, we have $xb \in \ker\tau$ and hence $\sigma(xb) \in Z(R)$ for all $x \in R$ and for all $b \in \ker\tau$. If there exists $b \in \ker\tau$ such that $\sigma(b) \neq 0$, then we have $\sigma(x)\sigma(b) \in Z(R)$ for all $x \in R$. By Lemma 2.3 and Lemma 2.5 we conclude that $\sigma(x) \in Z(R)$ for all $x \in R$ and by the same way above we conclude equation (4.4). Now, suppose $r, s \in R$, then ($r = \sigma(a)$ or $r = \tau(b)$) and ($s = \sigma(c)$ or $s = \tau(d)$) for some $a, b, c, d \in R$ since $\tau(R) \cup \sigma(R) = R$. Using (4.1), (4.2) and (4.4) we conclude that $rs = sr$ and R is a commutative near-ring. If $\sigma(b) = 0$ for all $b \in \ker\tau$, then $\ker\tau \subseteq \ker\sigma$. Since $(\tau(R), +)$ and $(\sigma(R), +)$ are subgroups of $(R, +)$ whose union is R, we have either $\tau(R) \subseteq \sigma(R)$ or $\sigma(R) \subseteq \tau(R)$. Since $\ker\tau \subseteq \ker\sigma$, we get from isomorphism theorems that $(R/\ker\tau)/(\ker\sigma/\ker\tau)$ is isomorphic as near-rings to $R/\ker\sigma$. But $R/\ker\tau$ is isomorphic to $\tau(R)$ and $R/\ker\sigma$ is isomorphic to $\sigma(R)$, so $\tau(R)/(\ker\sigma/\ker\tau)$ is isomorphic to $\sigma(R)$. Thus, the cardinal number of $\tau(R)$ is greater than or equal to the cardinal number of $\sigma(R)$. Therefore $\sigma(R) \subseteq \tau(R)$ and $R = \tau(R) \cup \sigma(R) = \tau(R)$. So τ is an epimorphism and hence R is a commutative near-ring from (4.2).

Case 2: $d(b) \neq 0$ for some $b \in \ker\tau$.

So $d(b)$ is not a zero divisor in R by Lemma 2.5 and $d(xb) = \sigma(x)d(b) + d(x)\tau(b) = \sigma(x)d(b)$ for all $x \in R$. Multiplying $d(xb)$ by $\sigma(y)$ in the left and the right respectively, we have $\sigma(y)d(xb) = \sigma(y)\sigma(x)d(b) = d(b)\sigma(y)\sigma(x)$ and $d(xb)\sigma(y) = \sigma(x)d(b)\sigma(y) = d(b)\sigma(x)\sigma(y)$ for all $x, y \in R$. So $d(b)[\sigma(y)\sigma(x) - \sigma(x)\sigma(y)] = 0$ for all $x, y \in R$ and then we get (4.4). Suppose $\ker\tau \cap \ker\sigma = \{0\}$, then (4.2) and (4.4) imply that $xy - yx \in \ker\tau \cap \ker\sigma$ for all $x, y \in R$. So R is a commutative near-ring. Now, suppose $\tau(R) \cup \sigma(R) = R$. Then (4.1), (4.2) and (4.4) imply that R is a commutative near-ring by the same way above in case 1.

So from the above two cases, R is a commutative near-ring. Using $d(a)$ is not a left zero divisor in R and Lemma 2.11, we have that R is a commutative ring.

The next corollary is another generalization of Theorem 2 of O. Golbasi and N. Aydin [6] and Theorem 3.1 of M. Ashraf, A. Ali and Shakir Ali [1].

Corollary 4.6 Let R be a 3-prime near-ring with a non-zero multiplicative (σ, τ)-derivation d such that σ and τ are endomorphisms on R, σ or τ is a monomorphism or an epimorphism and $d(R) \subseteq Z(R)$. Then R is a commutative ring.

Proof. If σ or τ is a monomorphism, then $\ker \tau \cap \ker \sigma = \{0\}$. If σ or τ is an epimorphism, then $\tau(R) \cup \sigma(R) = R$. Therefore, we get the result by Theorem 4.5.

5 The condition $d(xy) = d(yx)$

In this section we study the commutativity of a near-ring R admitting a non-zero derivation d satisfying the condition $d(xy) = d(yx)$ $(d(xy) = -d(yx))$ for all $x, y \in R$. As a consequence of results obtained, we generalized some results due to Golbasi, Ashraf and S. Ali.

Proposition 5.1 Let R be a near-ring admitting a non-zero multiplicative (σ, τ)-derivation d such that τ is one-to-one. Then the following are equivalent:

(1) $d(xy) = d(yx)$ for all $x, y \in R$ and there exists $a \in R$ such that $d(a)$ is not a left zero divisor for $\tau(xy) - \tau(yx)$ for all $x, y \in R$.

(2) R is a commutative near-ring.

Proof. Suppose $d(xy) = d(yx)$ for all $x, y \in R$ and there exists $a \in R$ such that $d(a)$ is not a left zero divisor for $\tau(xy) - \tau(yx)$ for all $x, y \in R$. Replacing x by yx in $d(xy) = d(yx)$ we get $d(yxy) = d(yyx)$ and hence $\sigma(y)d(xy) + d(y)\tau(xy) = \sigma(y)d(yx) + d(y)\tau(yx)$. Then we have $d(y)\tau(xy) = d(y)\tau(yx)$. It follows that

$$d(y)(\tau(xy) - \tau(yx)) = 0 \text{ for all } x, y \in R. \tag{5.1}$$

But $d(a)$ is not a left zero divisor for $\tau(xy) - \tau(yx)$, so $d(a)(\tau(xa) - \tau(ax)) = 0$ implies $\tau(xa) = \tau(ax)$ for all $x \in R$. As τ is one-to-one, we obtain $xa = ax$ for all $x \in R$ which means $a \in Z(R)$. From $d(xy) = d(yx)$ for all $x, y \in R$, we have $d(a(xy)) = d((ax)y) = d(y(ax)) = d((ya)x) = d((ay)x) = d(a(yx))$ and then $\sigma(a)d(xy) + d(a)\tau(xy) = \sigma(a)d(yx) + d(a)\tau(yx)$. It follows that $d(a)\tau(xy) = d(a)\tau(yx)$ for all $x, y \in R$. So

$$d(a)(\tau(xy) - \tau(yx)) = 0 \text{ for all } x, y \in R. \tag{5.2}$$

Again, $d(a)$ is not a left zero divisor for $\tau(xy) - \tau(yx)$ implies that $\tau(xy) = \tau(yx)$ and hence $xy = yx$ for all $x, y \in R$. Therefore, R is a commutative near-ring.

Conversely, Suppose R is a commutative near-ring. Thus, $d(xy) = d(yx)$ and $\tau(xy) - \tau(yx) = 0$ for all $x, y \in R$. So for all $z \in R - \{0\}$, we get that z is not a left zero divisor for $\tau(xy) - \tau(yx)$ for all $x, y \in R$.

Theorem 5.2 Let R be a near-ring admitting a non-zero multiplicative (σ, τ)-derivation d such that τ is one-to-one. Then the following are equivalent:

(1) $d(xy) = d(yx)$ for all $x, y \in R$ and there exist $a, b \in R$ such that $d(a)$ is not a left zero divisor for $\tau(xy) - \tau(yx)$ and b is not a left zero divisor for $x + y - x - y = (x, y)$ for all $x, y \in R$.

(2) R is a commutative ring.

Proof. Suppose $d(xy) = d(yx)$ for all $x, y \in R$ and there exist $a, b \in R$ such that $d(a)$ is not a left zero divisor for $\tau(xy) - \tau(yx)$ for all $x, y \in R$ and b is not a left zero divisor for (x, y). By proposition 5.1 we deduce that R is a commutative near-ring. Since R is commutative, it is distributive. So by Lemma 2.10, R is a ring. Conversely, suppose R is a commutative ring. By proposition 5.1 $d(xy) = d(yx)$ for all $x, y \in R$ and there exists $a \in R$ such that $d(a)$ is not a left zero divisor for $\tau(xy) - \tau(yx)$ for all $x, y \in R$. Since $(R, +)$ is abelian, we obtain $(x, y) = 0$ for all $x, y \in R$. So for all $z \in R - \{0\}$, we get that z is not a left zero divisor for (x, y) for all $x, y \in R$.

Corollary 5.3 Let R be a near-ring with a non-zero multiplicative σ-derivation d such that $d(xy) = d(yx)$ for all $x, y \in R$ and there exists $a \in R$ such that $d(a)$ is not a left zero divisor in R. Then R is a commutative ring.

We generalize Theorem 2.6 of [7] and Theorem 4.1 of [3] in the following theorem.

Theorem 5.4 Let R be a 3-prime near-ring with a non-zero multiplicative (σ, τ)-derivation d such that τ is a multiplicative automorphism and $d(xy) = d(yx)$ for all $x, y \in R$. Then R is a commutative ring.

Proof. Using the proof of Proposition 5.1, we get (5.1) and then $d(y)\tau(x)\tau(y) = d(y)\tau(y)\tau(x)$ for all $x, y \in R$. Putting xz instead of x, we have $d(y)\tau(x)\tau(z)\tau(y) = d(y)\tau(y)\tau(x)\tau(z) = d(y)\tau(x)\tau(y)\tau(z)$ for all $x, y, z \in R$. Thus, $d(y)\tau(x)[\tau(z)\tau(y) - \tau(y)\tau(z)] = 0$. Since τ is onto, we obtain $d(y)R[\tau(z)\tau(y) - \tau(y)\tau(z)] = \{0\}$. Using primeness of R, for all $y \in R$ either $d(y) = 0$ or $\tau(zy) = \tau(yz)$. As d is a non-zero multiplicative (σ, τ)-derivation, there exists $a \in R$ such that $d(a) \neq 0$. So $a \in Z(R)$ since τ is a monomorphism. By the same way again in the proof of Proposition 5.1, we have (5.2) and then $d(a)\tau(x)\tau(y) = d(a)\tau(y)\tau(x)$. Putting xz instead of x, we get $d(a)\tau(x)\tau(z)\tau(y) = d(a)\tau(y)\tau(x)\tau(z) = d(a)\tau(x)\tau(y)\tau(z)$ for all $x, y, z \in R$. Therefore, $d(a)\tau(x)[\tau(z)\tau(y) - \tau(y)\tau(z)] = 0$ for all $x, y, z \in R$ and then $d(a)R[\tau(z)\tau(y) - \tau(y)\tau(z)] = \{0\}$. Using the primeness of R and $d(a) \neq 0$, we have $\tau(z)\tau(y) = \tau(y)\tau(z)$ for all $y, z \in R$ and R is a commutative near-ring. Since R is commutative and $d(a) \neq 0$ is not a left zero divisor in R by Lemma 2.5, then R is a commutative ring by Lemma 2.10.

Remark 5.1 Since a near-ring R which satisfies the hypothesis of Theorem 5.4 will be commutative, we have $d(a) \neq 0$ is not a zero divisor in R for some $a \in R$ by Lemma 2.5. So the condition "R is a 3-prime near-ring with a non-zero multiplicative (σ, τ)-derivation d such that τ is a multiplicative automorphism and $d(xy) = d(yx)$ for all $x, y \in R$" implies the condition "R is a near-ring with a non-zero multiplicative (σ, τ)-derivation d such that τ is one-to-one, $d(xy) = d(yx)$ for all $x, y \in R$ and there exists $a \in R$ such that $d(a)$ is not a left zero divisor in R". The converse is not true as the following example shows, let R be the polynomial ring $\mathbb{Z}_4[x]$ and d the usual derivative. Then R is commutative and $d(xy) = d(yx)$ for all $x, y \in R$. Moreover, $d(x^5) = 5(x^4) = x^4$ is not a zero divisor in R. But R is not prime since $2xR2x = R(2x)(2x) = R(4x^2) = \{0\}$ and $2x \neq 0$. So the second condition is weaker than the first one.

Corollary 5.5 Let R be a 3-prime near-ring with a non-zero (σ, τ)-derivation d such that $[x, d(y)]_{\sigma,\tau} = 0$ for all $x, y \in R$. If τ is an automorphism on R, then R is commutative ring.

Proof. Using $[x, d(y)]_{\sigma,\tau} = 0$ and Lemma 2.1, we have $d(xy) = \sigma(x)d(y) + d(x)\tau(y) = d(y)\tau(x) + \sigma(y)d(x) = \sigma(y)d(x) + d(y)\tau(x) = d(yx)$. Hence, we get the result by Theorem 5.4.

Corollary 5.6 Let R be a 3-prime near-ring with a non-zero multiplicative σ-derivation d such that $d(xy) = d(yx)$ for all $x, y \in R$. Then R is a commutative ring.

Theorem 5.7 Let R be a near-ring with a (σ, τ)-derivation d such that $d(xy) = -d(yx)$ for all $x, y \in R$ and there exists $a \in R$ such that $d(a)$ is not a left zero divisor in R. If τ is a monomorphism on R, then R is a commutative ring of characteristic 2.

Proof. Replacing x by yx in $d(xy) = -d(yx)$, we get $d(yxy) = -d(yyx)$ and hence $d(y(xy + yx)) = 0$. Then $\sigma(y)d(xy + yx) + d(y)\tau(xy + yx) = 0$ for all $x, y \in R$. Since $d(xy) = -d(yx)$, we have $d(y)\tau(xy + yx) = 0$ for all $x, y \in R$. As $d(a)$ is not a left zero divisor in R, then $\tau(xa+ax) = 0$ and hence $xa = -ax$ for all $x \in R$. For all $x, y \in R$, we have $d(a(xy)) = -d((xy)a) = -d(x(ya)) = -d(x(-ay)) = -d(-xay) = d(x(ay)) = -d((ay)x) = -d(a(yx))$ for all $x, y \in R$. It follows that $d(a(xy+yx)) = 0$. So $d(a)\tau(xy+yx) = 0$. and then $xy = -yx$ for all $x, y \in R$. Observe that $(x+y)z = -[z(x+y)] = -[zx+zy] = -zy - zx = yz + xz$ for all $x, y, z \in R$. Since $0x = (0 + 0)x = 0x + 0x$ for all $x \in R$, we have $0x = 0$ and R is zero-symmetric. Now, $0 = 0x = (y + (-y))x = (-y)x + yx$ which means $(-y)x = -yx$ for all $x, y \in R$. Therefore, $(x+y)z = -(z(x+y)) = (-z)(x+y) = (-z)x+(-z)y = -zx+(-zy) = xz+yz$ for all $x, y, z \in R$ and R is distributive. Since $xy = -yx$ for all $x, y \in R$, we have $x^2 = -x^2$ for all $x \in R$ and then $0 = x^2 + x^2 = x(x+x) = x(2x)$. Choosing $x = d(a)$, we have $d(a)(2d(a)) = 0$ and hence $2d(a) = 0$. Using distributivity of R, observe that $d(a)(2y) = d(a)(y+y) = d(a)y + d(a)y = (d(a)+d(a))y = (2d(a))y = 0y = 0$ which means $2y = 0$ for all $y \in R$. Thus, $2R = \{0\}$ and R is of characteristic 2. Therefore, R is an abelian near-ring and $xy = -yx = yx$ for all $x, y \in R$. Therefore, R is a commutative ring.

Corollary 5.8 Let R be a near-ring with a σ-derivation d such that $d(xy) = -d(yx)$ for all $x, y \in R$ and there exists $a \in R$ such that $d(a)$ is not a left zero divisor in R. Then R is a commutative ring of characteristic 2.

We generalize Theorem 4.2 of [3] in the next result.

Theorem 5.9 Let R be a 3-prime near-ring with a non-zero (σ,τ)-derivation d such that $d(xy) = -d(yx)$ for all $x, y \in R$. If τ is an automorphism on R, then R is a commutative ring of characteristic 2.

Proof. Replacing x by yx in $d(xy) = -d(yx)$, we get $d(y)\tau(xy+yx) = 0$ and then $d(y)\tau(x)\tau(y) = -d(y)\tau(y)\tau(x)$ for all $x, y \in R$. Replacing x by xz, we get

$$\begin{aligned} d(y)\tau(x)\tau(z)\tau(y) &= -d(y)\tau(y)\tau(x)\tau(z) = -(-d(y)\tau(x)\tau(y))\tau(z) \\ &= -[d(y)\tau(x)\tau(-y)\tau(z)] \end{aligned}$$

and hence $d(y)\tau(x)[\tau(z)\tau(y) + \tau(-y)\tau(z)] = 0$ for all $x, y, z \in R$. So we have $d(y)R[\tau(z)\tau(y)+\tau(-y)\tau(z)] = \{0\}$ and then for each $y \in R$ either $d(y) = 0$ or $\tau(zy+(-y)z) = 0$. As d is non-zero, there exists $a \in R$ such that $d(a) \neq 0$. So $\tau(za+(-a)z) = 0$ and then $za = -(-a)z = (-a)(-z)$ for all $z \in R$. Observe that $z(-a) = -za = (-a)z$ and $-a \in Z(R)$. So $d((-a)(xy)) = d(((-a)x)y) = -d(y((-a)x)) = -d((y(-a))x) = -d(((-a)y)x) = -d((-a)(yx))$. Thus, $d((-a)(xy+yx)) = 0$ and then $d(-a)\tau(xy+yx) = 0$ for all $x, y \in R$. So $d(-a)\tau(x)\tau(y) = -d(-a)\tau(y)\tau(x)$ for all $x, y \in R$. Replacing x by xz, we get $d(-a)\tau(x)\tau(z)\tau(y) = -d(-a)\tau(x)\tau(-y)\tau(z)$ by the same way above. Hence $d(-a)\tau(x)[\tau(z)\tau(y) + \tau(-y)\tau(z)] = 0$ for all $x, y, z \in R$. Since $d(-a) \neq 0$, we have $\tau(zy + (-y)z) = 0$ which means $zy = (-y)(-z) = -(-y)z$ for all $y, z \in R$. It follows that $z(-y) = -zy = (-y)z$ for all $y, z \in R$ and R is a commutative near-ring. Since R is commutative and $a \neq 0$ is not a zero divisor in R, we have that R is a commutative ring by Lemma 2.10. Since $d \neq 0$ and R is commutative, there exists $a \in R$ such that $d(a) \neq 0$ is not a left zero divisor in R by Lemma 2.5 and hence R is a ring of characteristic 2 by Theorem 5.7.

Corollary 5.10 Let R be a 3-prime near-ring with a non-zero σ-derivation d such that $d(xy) = -d(yx)$ for all $x, y \in R$. Then R is a commutative ring of characteristic 2.

Example 5.1 Let $R = \mathbb{Z}_2[x]$ with $d = \tau$ is the identity map and $\sigma = 0$. Then d is a non-zero (σ, τ)-derivation on R and R is a commutative prime ring of characteristic 2. Clearly $d(xy) = d(yx) = -d(yx)$ and $d(x) = x$ is not a left zero divisor in R for all $x \in R - \{0\}$.

The following example shows that the condition "$d(xy) = -d(yx)$ for all $x, y \in R$" is not redundant in Theorem 5.9.

Example 5.2 Let $R = M_2(\mathbb{Z}_2)$ with $d = \tau$ is the identity map and $\sigma = 0$. Then R is a non-commutative 2-torsion prime ring and d is a non-zero (σ, τ)-derivation on R. Observe that

$$\begin{aligned} d\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) &= d\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \\ &\neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = d\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) \\ &= d\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) \end{aligned}$$

and hence $d(xy) \neq d(yx) = -d(yx)$. Also, $d\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in Z(R) - \{0\}$ is not a left zero divisor in R by Lemma 2.5.

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References

[1] M. Ashraf, A. Ali and S. Ali, (σ, τ)- *derivations on prime near-rings*, Arch. Math. (Brno) **40** (2004), no. 3, 281–286.

[2] M. Ashraf and S. Ali, *On (σ, τ)-derivations of prime near-rings*, Trends in Theory of Rings and Modules. (S. Tariq Rizvi & S. M. A. Zaidi, (Eds.)), Anamaya Publishers, New Delhi, (2005), 5-10.

[3] M. Ashraf and S. Ali, *On (σ, τ)-derivations of prime near-rings-II*, Sarajevo J. Math. **4(16)** (2008), no. 1, 23–30.

[4] H. E. Bell and G. Mason, *On derivations in near-rings*, Near-rings and near-fields (Tübingen, 1985), 31–35, North-Holland Math. Stud., 137, *North-Holland, Amsterdam,* 1987.

[5] Y. Fong, W.-F. Ke, and C.-S. Wang, *Nonexistence of derivations on transformation near-rings*, Comm. Algebra **28** (2000), 1423–1428.

[6] O. Golbasi, and N. Aydin, *Results on prime near-rings with (σ, τ)-derivation*, Math. J. Okayama Univ. **46** (2004), 1–7.

[7] O. Golbasi, *On a theorem of Posner for 3-prime near-rings with (σ, τ)-derivation*, Hacet. J. Math. Stat. **36** (2007), no. 1, 43–47.

[8] A. A. M. Kamal, *σ-derivations on prime near-rings*, Tamkang J. Math. **32**(2001), no.2, 89–93.

[9] A. A. M. Kamal and K. H. Al-shaalan, *Existence of derivations on near-rings*, to appear in *Math. Slovaca.*

[10] S. Ligh, *A note on matrix near rings*, J. London Math. Soc. (2) **11** (1975), no. 3, 383–384.

[11] J. D. P. Meldrum, "*Near-rings and their links with groups*", Research Notes in Mathematics, 134. Pitman (Advanced Publishing Program), Boston, MA, 1985.

[12] G. Pilz, "*Near-rings. The theory and its applications*", Second edition. North-Holland Mathematics Studies, 23. North-Holland Publishing Co., Amsterdam, 1983.

[13] M. S. Samman, *Existence and Posner's theorem for α-derivations in prime near-rings*, Acta Math. Univ. Comenian., (N.S.) **78** (2009), no.2, 37-42.

[14] X. K. Wang, *Derivations in prime near-rings*, Proc. Amer. Math. Soc. **121** (1994), no. 2, 361–366.

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