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On iterative fixed point convergence in uniformly convex Banach space and Hilbert space

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Abstract

Some fixed point convergence properties are proved for compact and demicompact maps acting over closed, bounded and convex subsets of a real Hilbert space. We also show that for a generalized nonexpansive mapping in a uniformly convex Banach space the Ishikawa iterates converge to a fixed point. Finally, a convergence type result is established for multivalued contractive mappings acting on closed subsets of a complete metric space. These are extensions of results in Ciric, *et. al.* [7], Panyanak [2] and Agarwal, *et. al.* [9].

1 Introduction

Let H be a Hilbert space and K be a nonempty subset of H. A mapping $T : K \to H$ is said to be pseudo-contractive if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2 , \quad for\ all\ \ x, y \in K.$$

A mapping $T : K \to H$ is called hemicontractive if $F(T) = \{x \in K : Tx = x\} \neq \phi$ and

$$\|Tx - x^*\|^2 \leq \|x - x^*\|^2 + \|x - Tx\|^2 , \quad for\ all\ x^* \in F(T)\ and\ for\ all\ x \in K.$$

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It is easy to observe that each pseudo-contractive mapping with fixed points is hemicontractive. The reciprocal is not in general true; see [1],[4].

There are two well known methods of approximating a fixed point of a pseudo-contractive mapping, viz. Mann [11] iterative and Ishikawa [10] iterative processes. In 1991, Xu [3] introduced the following iteration process: For $T : K \to E$, let a sequence $\{x_n\}$ and $x_0 \in K$, where K is a nonempty subset of a Banach space E, defined iteratively as follows :

$$\begin{aligned} x_{n+1} &= a_n x_n + b_n T y_n + c_n u_n \\ y_n &= a_n{}' x_n + b_n{}' T x_n + c_n{}' v_n, \quad n \geq 0, \end{aligned} \tag{1}$$

where $\{u_n\}$ and $\{v_n\}$ are bounded sequences in K and $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}$, $\{b'_n\}$ and $\{c'_n\}$ are sequences in $[0, 1]$, such that $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$, for all $n \geq 1$. If, in (1), $b'_n = 0 = c'_n$, then we obtain the Mann iterative sequence in the sense of Xu. If $c_n = 0 = c'_n$ in (1), then we obtain the Ishikawa iterative sequence.

In [7], Ciric, *et al.* have introduced and investigated the following modified Mann implicit iterative process. Let K be a closed convex subset of a real normed space N and $T : K \to K$ be a mapping. Define $\{x_n\}$ in K as follows :

$$\begin{aligned} x_0 &\in K, \\ x_n &= a_n x_{n-1} + b_n T v_n + c_n u_n, \quad n \geq 1, \end{aligned} \tag{2}$$

where $\{a_n\}, \{b_n\}, \{c_n\}$ are real sequences in [0,1] such that $a_n + b_n + c_n = 1$, for each $n \in \mathbb{N}$ and $\{u_n\}$ and $\{v_n\}$ are sequences in K.

Let H be a Hilbert space and C a subset of H. A mapping $T : C \to H$ is called demicompact if it has the property that whenever $\{u_n\}$ is bounded sequence in H and $\{Tu_n - u_n\}$ is strongly convergent, there exists a strongly convergent subsequence $\{u_{n_k}\}$ of $\{u_n\}$.

In section two of the present paper, we have shown that if K is closed, bounded and convex subset of a real Hilbert space H, $T : K \to K$ a compact hemicontractive map with $x_0 \in T(K)$ and sequence $\{x_n\}$ in $T(K)$ be defined by (1) and $\{b_n\}$, $\{c_n\}$ and $\{v_n\}$ satisfy some appropriate conditions, then the sequence $\{x_n\}$ converges strongly to a fixed point of T. Also, we have investigated that if K is closed, bounded and convex subset of a real Hilbert space H and the mapping $T : K \to K$ is continuous demicompact hemicontractive map and $\{a_n\}, \{b_n\}, \{c_n\}$ are real sequences in [0,1] such that $a_n + b_n + c_n = 1$, for each $n \in \mathbb{N}$ and $\{b_n\}, \{c_n\}, \{v_n\}$ satisfy some appropriate conditions, then the sequence $\{x_n\}$, defined by (2), converges strongly to some fixed point of T.

Let E be a Banach space. A subset K of E is called proximinal if for each $x \in E$, there exists an element $k \in K$ such that

$$d(x,k) = dist(x,K) = \inf\{\|x-y\| : y \in K\}.$$

It is well known that every closed convex subset of a uniformly convex Banach space is proximinal. We shall denote by $P(K)$, the family of nonempty bounded proximinal subsets of K. We say that the mapping $T : E \to P(E)$ is generalized nonexpansive if

$$H(Tx,Ty) \leq a\|x-y\| + b\{d(x,Tx) + d(y,Ty)\} + c\{d(x,Ty) + d(y,Tx)\},$$

for all $x, y \in X$, where $a + 2b + 2c \leq 1$.

Bancha Panyanak proved the following Theorem in [2].

Theorem 1.1. *Let K be a nonempty compact convex subset of a uniformly convex Banach Space E. Suppose $T : K \to P(K)$ is a nonexpansive map with a fixed point p. Let $\{x_n\}$ be the sequence of Ishikawa iterates defined by $x_0 \in K$,*

$$y_n = (1-\beta_n)x_n + \beta_n z_n \quad \beta_n \in [0,1], \ n \geq 0,$$

where $z_n \in Tx_n$ is such that $\|z_n - p\| = dist(p, Tx_n)$, and

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n z_n{}', \quad \alpha_n \in [0,1],$$

where $z_n{}' \in Ty_n$ is such that $\|z_n{}' - p\| = dist(p, Ty_n)$. Assume that
(i) $0 \leq \alpha_n, \beta_n < 1$,
(ii) $\beta_n \to 0$ and
(iii) $\sum \alpha_n\beta_n = \infty$. Then the sequence $\{x_n\}$ converges to a fixed point of T.

In section three, we generalize the above theorem by taking generalized nonexpansive map in place of nonexpansive map in which the sequence of Ishikawa iterates converges to the fixed point of T.

Let X be a complete metric space and $C(X)$ is collection of all nonempty closed subsets of X, $CB(X)$ is the collection of all nonempty closed bounded subsets of X. Let H be a Hausdorff metric on $C(X)$, that is

$$H(A,B) = max\{sup_{x\in A} d(x,B), sup_{x\in B} d(x,A)\},$$

for any $A, B \in C(X)$, where $d(x,B) = \inf\{\|x-y\| : y \in B\}$.

A function $f : X \to \mathbb{R}$ is called lower semi-continuous, if for any sequence $\{x_n\}$ in X and $x \in X$,

$$x_n \to x \Longrightarrow f(x) \leq \underline{\lim}_{n\to\infty} f(x_n).$$

In section four, we generalize the following result (cf. Theorem 4.2.11 in [9]) by taking $C(X)$ in place of $CB(X)$.

Theorem 1.2. [9]. *Let X be a complete metric space and let $T_n : X \to CB(X)(n = 0, 1, 2, 3, ...)$ be contraction mappings each having Lipschitz constant $k < 1$, i.e.,*

$$H(T_nx, T_ny) \leq kd(x, y),$$

for all $x, y \in X$ and $n \in (0, 1, 2, 3, ...)$. If $\lim_{n\to\infty} H(T_n(x), T_0(x)) = 0$ uniformly for $x \in X$, then $\lim_{n\to\infty} H(F(T_n), F(T_0)) = 0$.

2 Fixed point theorems for hemicontractive map

We shall make use of the following Lemmas.

Lemma 2.1. [8]. *Let H be a Hilbert space, then for all $x, y, z \in H$,*

$$\|ax + by + cz\|^2 = a\|x\|^2 + b\|y\|^2 + c\|z\|^2 - ab\|x - y\|^2 - bc\|y - z\|^2 - ca\|z - x\|^2,$$

where $a, b, c \in [0, 1]$ and $a + b + c = 1$.

Lemma 2.2. [5]. *Suppose that $\{\rho_n\}, \{\sigma_n\}$ are two sequences of nonnegative numbers such that for some real number $N_0 \geq 1$,*

$$\rho_{n+1} \leq \rho_n + \sigma_n, \ \forall \ n \geq N_0.$$

(a) If $\sum_{n=1}^{\infty}\sigma_n < \infty$, then $\lim\{\rho_n\}$ exists.
(b) If $\sum_{n=1}^{\infty}\sigma_n < \infty$ and $\{\rho_n\}$ has a subsequence converging to zero, then $\lim_{n\to\infty} \rho_n = 0$.

Now we prove our main results in this section which is generalization of [[7],Theorem 4]

Theorem 2.3. *Let K be a closed bounded convex subset of a real Hilbert space H and $T : K \to K$ a compact, hemicontractive map. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be real sequences in [0,1] such that $a_n + b_n + c_n = 1$, for each $n \in \mathbb{N}$ and satisfying:*
(i) $\{b_n\} \subset [\delta, 1 - \delta]$ for some $\delta \in (0, 1/2]$,
(ii) $\sum_{n=1}^{\infty}c_n < \infty$.
For arbitrary $x_0 \in T(K)$, let a sequence $\{x_n\}$ in $T(K)$ be iteratively defined by

$$x_n = a_nx_{n-1} + b_nTv_n + c_nu_n, \quad n \geq 1, \tag{3}$$

where $v_n \in T(K)$ are chosen such that $\sum_{n=1}^{\infty} \|v_n - x_n\| < \infty$ and $\{u_n\}_{n=1}^{\infty}$ is arbitrary sequence in K. Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to some fixed point of T.

Proof. Let $T : K \to K$ be a continuous map, where K is a closed bounded convex subset of a real Hilbert space H. Then $T(K)$ is closed subset of K and $\overline{T(K)}$ is compact. Hence $T(K)$ is compact. Let $A = \overline{co}(T(K))$, convex closure of $T(K)$. Then $A \subset K$. Since $T(K)$ is a relatively compact subset of K, by Mazur's theorem $\overline{co}(T(K))$ is compact and convex. Furthermore, $T(A) \subset A$. Now we have to show that in restriction $T : A \to A$, $\{x_n\}_{n=1}^{\infty}$ converges strongly to some fixed point of T. Let $x^* \in T(K)$ be a fixed point of T and $M = diam(T(K))$, diameter of $T(K)$. Since T is hemicontractive,

$$\|Tv_n - x^*\|^2 \leq \|v_n - x^*\|^2 + \|v_n - Tv_n\|^2 , \tag{4}$$

for each $n \in \mathbb{N}$. By virtue of (3), Lemma 2.1 and (4), we obtain

$$\begin{aligned} \|x_n - x^*\|^2 &= \|a_n x_{n-1} + b_n Tv_n + c_n u_n - x^*\|^2 \\ &= \|a_n(x_{n-1} - x^*) + b_n(Tv_n - x^*) + c_n(u_n - x^*)\|^2 \\ &= a_n \|x_{n-1} - x^*\|^2 + b_n \|Tv_n - x^*\|^2 + c_n \|u_n - x^*\|^2 \\ &\quad - a_n b_n \|x_{n-1} - Tv_n\|^2 - b_n c_n \|Tv_n - u_n\|^2 \\ &\quad - a_n c_n \|x_{n-1} - u_n\|^2 \\ &\leq a_n \|x_{n-1} - x^*\|^2 + b_n \|Tv_n - x^*\|^2 \\ &\quad + c_n \|u_n - x^*\|^2 - a_n b_n \|x_{n-1} - Tv_n\|^2 \\ &\leq (1 - b_n) \|x_{n-1} - x^*\|^2 + b_n(\|v_n - x^*\|^2 \\ &\quad + \|v_n - Tv_n\|^2) + c_n M^2 - a_n b_n \|x_{n-1} - Tv_n\|^2 . \end{aligned} \tag{5}$$

Also, we have

$$\begin{aligned} \|v_n - x^*\|^2 &\leq \|v_n - x_n\|^2 + \|x_n - x^*\|^2 + 2 \|x_n - x^*\| \|v_n - x_n\| \\ &\leq \|v_n - x_n\|^2 + \|x_n - x^*\|^2 + 2M \|v_n - x_n\| , \end{aligned} \tag{6}$$

and

$$\begin{aligned} \|v_n - Tv_n\|^2 &\leq \|v_n - x_n\|^2 + \|x_n - Tv_n\|^2 + 2 \|x_n - Tv_n\| \|v_n - x_n\| \\ &\leq \|v_n - x_n\|^2 + \|x_n - Tv_n\|^2 + 2M \|v_n - x_n\| \end{aligned} \tag{7}$$

and

$$\begin{aligned} \|x_n - Tv_n\|^2 &= \|a_n x_{n-1} + b_n Tv_n + c_n u_n - Tv_n\|^2 \\ &= \|(1 - b_n - c_n)x_{n-1} + b_n Tv_n + c_n u_n - Tv_n\|^2 \\ &\leq [(1 - b_n) \|x_{n-1} - Tv_n\| + c_n \|u_n - x_{n-1}\|]^2 \\ &\leq [(1 - b_n) \|x_{n-1} - Tv_n\| + Mc_n]^2 \\ &\leq (1 - b_n)^2 \|x_{n-1} - Tv_n\|^2 + 3M^2 c_n. \end{aligned} \tag{8}$$

In view of (7) and (8), (5) takes the form

$$\begin{aligned}\|x_n - x^*\|^2 \quad &\leq \quad (1-b_n)^2 \|x_{n-1} - x^*\|^2 \\ &\quad + b_n \|x_n - x^*\|^2 + 2b_n \|v_n - x_n\|^2 + 4Mb_n \|v_n - x_n\| \\ &\quad + 4M^2 c_n - b_n[a_n - (1-b_n)^2] \|x_{n-1} - Tv_n\|^2 . \end{aligned} \tag{9}$$

Using $a_n + b_n + c_n = 1$ in condition (i), we have

$$\begin{aligned} a_n - (1-b_n)^2 \quad &= \quad 1 - b_n - c_n - (1-b_n)^2 \\ &= \quad b_n(1-b_n) - c_n \\ &\geq \quad \delta^2 - c_n. \end{aligned} \tag{10}$$

From condition (ii), it follows that there exists a positive integer $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $c_n \leq \delta^3$, i.e. $\delta^2 - c_n \geq \delta^2 - \delta^3 = \delta^2(1-\delta)$. Thus, from (10), we obtain

$$a_n - (1-b_n)^2 \quad \geq \quad \delta^2(1-\delta). \tag{11}$$

From (9) and (11), we have, for all $n \geq n_0$

$$\begin{aligned} (1-b_n)\|x_n - x^*\|^2 \quad &\leq \quad (1-b_n)\|x_{n-1} - x^*\|^2 + 2b_n \|v_n - x_n\|^2 \\ &\quad + 4Mb_n \|v_n - x_n\| + 4M^2 c_n \\ &\quad - b_n \delta^2(1-\delta) \|x_{n-1} - Tv_n\|^2 . \\ or \ \|x_n - x^*\|^2 \quad &\leq \quad \|x_{n_1} - x^*\|^2 + \frac{2b_n}{(1-b_n)} \|v_n - x_n\|^2 \\ &\quad + 4M\frac{b_n}{(1-b_n)} \|v_n - x_n\| + \frac{4M^2 c_n}{(1-b_n)} \\ &\quad + b_n \frac{\delta^2(1-\delta)}{(1-b_n)} \|x_{n-1} - Tv_n\|^2 . \end{aligned} \tag{12}$$

Since $\frac{1}{(1-b_n)} \leq \frac{1}{\delta}$ and $\frac{-1}{(1-b_n)} \leq \frac{-1}{1-\delta}$; $\delta \leq b_n \leq 1-\delta$, we have $\frac{b_n}{1-b_n} \leq \frac{1-\delta}{\delta} =$

$\frac{1}{\delta} - 1 < \frac{1}{\delta}$. Hence from (12), we have

$$\begin{aligned}
\|x_n - x^*\|^2 &\leq \|x_{n-1} - x^*\|^2 + \frac{2}{\delta}\|v_n - x_n\|^2 + \frac{4M}{\delta}\|v_n - x_n\| + \frac{4M^2 c_n}{\delta} \\
&\quad - \frac{\delta^3(1-\delta)}{(1-b_n)}\|x_{n-1} - Tv_n\|^2 \\
&\leq \|x_{n-1} - x^*\|^2 + \frac{2}{\delta}\|v_n - x_n\|^2 + \frac{4M}{\delta}\|v_n - x_n\| + \frac{4M^2 c_n}{\delta} \\
&\quad - \frac{\delta^3(1-\delta)}{(1-\delta}\|x_{n-1} - Tv_n\|^2 \\
&\leq \|x_{n-1} - x^*\|^2 + \frac{2}{\delta}\|v_n - x_n\|^2 + \frac{4M}{\delta}\|v_n - x_n\| + \frac{4M^2 c_n}{\delta} \\
&\quad - \delta^3\|x_{n-1} - Tv_n\|^2, \\
i.e.\ \|x_n - x^*\|^2 &\leq \|x_{n-1} - x^*\|^2 - \delta^3\|x_{n-1} - Tv_n\|^2 + \sigma_n,\ for\ all\ n \geq n_0,
\end{aligned} \tag{13}$$

where

$$\begin{aligned}
\sigma_n &= \left[\frac{2}{\delta}\|v_n - x_n\|^2 + \frac{4M}{\delta}\|v_n - x_n\| + \frac{4M^2}{\delta}c_n\right] \\
&= \frac{1}{\delta}[2\|v_n - x_n\|^2 + 4M\|v_n - x_n\| + 4M^2 c_n].
\end{aligned} \tag{14}$$

By the hypothesis of the theorem, we obtain

$$\sum_{j=n_0}^{\infty} \sigma_j \quad < \quad +\infty. \tag{15}$$

From (14), we get
$\delta^3\|x_{n-1} - Tv_n\|^2 \leq \|x_{n-1} - x^*\|^2 - \|x_n - x^*\|^2 + \sigma_n$,
and hence

$$\delta^3 \sum_{j=n_0}^{\infty} \|x_{j-1} - Tv_j\|^2 \leq \sum_{j=n_0}^{\infty} \sigma_j + \|x_{n_0-1} - x^*\|^2.$$

By (15) we get $\sum_{j=n_0}^{\infty} \|x_{j-1} - Tv_j\|^2 < +\infty$. This implies $\lim_{n\to\infty}\|x_{n-1} - Tv_n\| = 0$. From (8) and condition (ii), it further implies that $\lim_{n\to\infty}\|x_n - Tv_n\| = 0$. Also the condition $\sum_{j=n_0}^{\infty}\|v_n - x_n\| < \infty$ implies $\lim_{n\to\infty}\|v_n - x_n\| = 0$. Thus from (7), we have

$$\lim_{n\to\infty}\|v_n - Tv_n\| = 0. \tag{16}$$

By compactness of $\overline{T(K)}$, there is a convergent subsequence $\{v_{n_j}\}$ of $\{v_n\}$, such that it converges to some point $z \in \overline{T(K)} \subset \overline{co}(T(K)) = A$. By continuity of T, $\{Tv_{n_j}\}$ converges to Tz. Therefore, from (16), we conclude that $Tz = z$. Further, $\lim_{n\to\infty} \|v_n - x_n\| = 0$ implies

$$\lim_{j\to\infty} \left\|x_{n_j} - z\right\| = 0. \tag{17}$$

Since (13) holds for any fixed points of T, we have

$$\|x_n - z\|^2 \leq \|x_{n-1} - z\|^2 - \delta^3 \|x_{n-1} - Tv_n\|^2 + \sigma_n$$

and in view of (15), (17) and Lemma 2.2, we conclude that $\|x_n - z\| \to 0$ as $n \to \infty$ i.e $x_n \to z$ as $n \to \infty$. Thus, we have proved that a sequence $\{x_n\}$ converges strongly to some fixed point of T. This sequence in K automatically converges strongly to a fixed point of T. □

Theorem 2.4. *Let K be a closed bounded convex subset of a real Hilbert space H and $T : K \to K$ a continuous demicompact and hemicontractive map. Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be a real sequences in $[0, 1]$ such that $a_n + b_n + c_n = 1$ for each $n \in \mathbb{N}$ and satisfying:*
(i) $\{b_n\} \subset [\delta, 1 - \delta]$, for some $\delta \in (0, \frac{1}{2}]$,
(ii) $\sum_{n=1}^{\infty} c_n < \infty$.
For arbitrary $x_0 \in K$, let a sequence $x_n \in K$ be iteratively defined by

$$x_n = a_n x_{n-1} + b_n T v_n + c_n u_n, \quad n \geq 1, \tag{18}$$

where $v_n \in K$ are chosen such that $\sum_{n=1}^{\infty} \|v_n - x_n\| < \infty$. Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to some fixed point of T.

Proof. Let $x^* \in K$ be a fixed point of hemicontractive map T and $M = diam(K)$. Using inequality (4) as in the proof of Theorem 2.3 and proceeding in the similar manner we arrive at (16) which implies that the sequence $\{v_n - Tv_n\}_{n\in N}$ converges strongly to zero. As T is demicompact, it results that there exists a strongly convergent subsequence $\{v_{n_j}\}$ of $\{v_n\}$. such that $v_{n_j} \to z \in K$. By continuity of T, Tv_{n_j} converges to Tz. Therefore, from (16), we conclude that $Tz = z$. Further, $\lim_{n\to\infty} \|v_n - x_n\| = 0$ implies

$$\left\|x_{n_j} - z\right\| = 0. \tag{19}$$

Since (13) holds for any fixed points of T, we have

$$\|x_n - z\|^2 \leq \|x_{n-1} - z\|^2 - \delta^3 \|x_{n-1} - Tv_n\|^2 + \sigma_n. \tag{20}$$

In view of (15), (19) and Lemma 2.2, we conclude that $\|x_n - z\| \to 0$ as $n \to \infty$ i.e. $x_n \to z$ as $n \to \infty$. Thus, we have proved that $\{x_n\}$ converges strongly to some fixed point of T. □

3 Ishikawa iteration for multivalued generalized nonexpansive map

To prove the main theorem of this section, we need the following Lemmas:

Lemma 3.1. [3]. *Let E be a Banach space. Then E is uniformly convex if and only if for any given number $\rho > 0$, the square norm $\|.\|^2$ of E is uniformly convex on B_ρ, the closed ball centered at the origin with radius ρ; namely, there exists a continuous strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that*

$$\|\alpha x + (1-\alpha)y\|^2 \leq \alpha \|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\phi(\|x-y\|),$$

for all $x, y \in B_\rho, \alpha \in [0,1]$.

Lemma 3.2. [2]. *Let $\{\alpha_n\}, \{\beta_n\}$ be two real sequences such that*
(i) $0 \leq \alpha_n, \beta_n < 1$,
(ii) $\beta_n \to 0$ as $n \to \infty$ and
(iii) $\sum_{n=1}^{\infty}\alpha_n\beta_n = \infty$.
Let $\{\gamma_n\}$ be a nonnegative real sequence such that $\sum_{n=1}^{\infty}\alpha_n\beta_n(1-\beta_n)\gamma_n$ is bounded. Then $\{\gamma_n\}$ has a subsequence which converges to zero.

Theorem 3.3. *Let K be a nonempty compact convex subset of a uniformly convex Banach space E. Suppose $T : K \to P(K)$ is a generalized nonexpansive map with a fixed point p. Let $\{x_n\}$ be the sequence of Ishikawa iterates defined by $x_0 \in K$,*

$$y_n = (1-\beta_n)x_n + \beta_n z_n \quad \beta_n \in [0,1], \ \ n \geq 0,$$

where $z_n \in Tx_n$ is such that $\|z_n - p\| = dist(p, Tx_n)$, and

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n {z_n}', \quad \alpha_n \in [0,1]$$

where ${z_n}' \in Ty_n$ is such that $\|{z_n}' - p\| = dist(p, Ty_n)$. Assume that
(i) $0 \leq \alpha_n, \beta_n < 1$
(ii) $\beta_n \to 0$ and
(iii) $\sum_{n=1}^{\infty}\alpha_n\beta_n = \infty$. Then the sequence $\{x_n\}$ converges to a fixed point of T.

Proof. By using Lemma 3.1, we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|(1-\alpha_n)x_n + \alpha_n {z_n}' - p\|^2 \\ &\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n \|{z_n}' - p\|^2 - \alpha_n(1-\alpha_n)\phi(\|x_n - {z_n}'\|) \\ &\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n H^2(Ty_n, Tp) \\ &\quad - \alpha_n(1-\alpha_n)\phi(\|x_n - {z_n}'\|). \qquad (21)\end{aligned}$$

By generalized nonexpansive property of T, we have

$$\begin{aligned}
H(Tp, Ty_n) &\le a\|y_n - p\| + bd(y_n, Ty_n) + c\{d(p, Ty_n) + d(y_n, Tp)\} \\
&\le a\|y_n - p\| + b\{\|y_n - p\| + d(p, Ty_n)\} + c\{d(p, Ty_n) + d(y_n, Tp)\} \\
&\le (a+b+c)\|y_n - p\| + (b+c)d(p, Ty_n) \\
&\le (a+b+c)\|y_n - p\| + (b+c)H(Tp, Ty_n) \\
H(Tp, Ty_n) &\le \frac{a+b+c}{1-(b+c)}\|y_n - p\|. \qquad (22)
\end{aligned}$$

Since $\frac{a+b+c}{1-(b+c)} \le 1$, it follows that

$$H(Ty_n, Tp) \le \|y_n - p\| \qquad (23)$$

From (21) and (23), we get

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|y_n - p\|^2 \\
&\quad -\alpha_n(1-\alpha_n)\phi(\|x_n - z_n{}'\|). \qquad (24)
\end{aligned}$$

Now

$$\begin{aligned}
\|y_n - p\|^2 &= \|(1-\beta_n)x_n + \beta_n z_n - p\|^2 \\
&\le (1-\beta_n)\|x_n - p\|^2 + \beta_n\|z_n - p\|^2 - \beta_n(1-\beta_n)\phi(\|x_n - z_n\|) \\
&\le (1-\beta_n)\|x_n - p\|^2 + \beta_n H^2(Tx_n, Tp) - \beta_n(1-\beta_n)\phi(\|x_n - z_n\|) \\
&\le \|x_n - p\|^2 - \beta_n(1-\beta_n)\phi(\|x_n - z_n\|). \qquad (25)
\end{aligned}$$

From (24) and (25), we get

$$\|x_{n+1} - p\|^2 \le \|x_n - p\|^2 - \alpha_n\beta_n(1-\beta_n)\phi(\|x_n - z_n\|). \qquad (26)$$

Therefore

$$\alpha_n\beta_n(1-\beta_n)\phi(\|x_n - z_n\|) \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

This implies

$$\sum_{n=1}^{\infty}\alpha_n\beta_n(1-\beta_n)\phi(\|x_n - z_n\|) \le \|x_1 - p\|^2 < \infty.$$

By Lemma 3.2, there exists a subsequence $\{x_{n_k} - z_{n_k}\}$ of $\{x_n - z_n\}$ such that $\phi(\|x_{n_k} - z_{n_k}\|) \to 0$ as $k \to \infty$ and hence $\|x_{n_k} - z_{n_k}\| \to 0$, by continuity and strictly increasing nature of ϕ. By compactness of K, we may assume that $x_{n_k} \to q$, for some $q \in K$. Thus,

$$\begin{aligned} dist(q,Tq) &\leq \|q - x_{n_k}\| + dist(x_{n_k}, Tx_{n_k}) + H(Tx_{n_k}, Tq) \\ &\leq \|q - x_{n_k}\| + \|x_{n_k} - z_{n_k}\| + \|x_{n_k} - q\| \to 0 \ as \ k \to \infty \quad (27) \end{aligned}$$

Hence q is a fixed point of T. Now on taking q in place of p, we get $\|x_n - q\|$ as a decreasing sequence by (26). Since $\|x_{n_k} - q\| \to 0$ as $k \to \infty$, it follows that $\{\|x_n - q\|\}$ converges to zero, so that the conclusion of the theorem follows. □

4 Fixed point theorem for multivalued contractive mappings

The main result of this section is as follows:

Proposition 4.1. *Let X be a complete metric space and let $S, T : X \to C(X)$ be a multivalued mapping. If there exists a constant $c \in (0,1)$ such that for any $x \in X$ there is $y \in I_b^{(S)x}$ and $I_b^{(T)x}$ satisfying $d(y, S(y)) \leq cd(x,y)$ and $d(y,Ty) \leq cd(x,y)$ with $c < b$ and f is lower semi-continuous, then*

$$H(F(s), F(T)) \quad \leq \quad (b-c)^{-1} sup_{x\in X} H(Sx, Tx), \tag{28}$$

where the following have been taken from [12], for mapping $f : X \to R$, $f(x)$ is defined as $f(x) = d(x,Tx)$ and for mapping S, $f(x)$ is defined as $f(x) = d(x, Sx)$,

$$I_b^{(S)x} = \{y \in S(x) : bd(x,y) \leq d(x,Sx)\}$$

and

$$I_b^{(T)x} = \{y \in T(x) : bd(x,y) \leq d(x,Tx)\}.$$

Proof. Since $S(x), T(x) \in C(X)$ for any $x \in X$, $I_b^{(S)x}$and $I_b^{(T)x}$ are nonempty for any constant $b \in (0,1)$. Let $x_0 \in F(S)$ implies $x_0 \in S(x_0)$. Then there is another point $x_1 \in S(x_0)$ such that for any initial point $x_0 \in X$, there exists $x_1 \in I_b^{s(x_0)}$. For x_1, there exists Sx_1 such that

$$d(x_1, Sx_1) \leq cd(x_0, x_1),$$

and for any $x_0 \in X$, there exists $x_1 \in I_b^{(T)x_0}$ i.e. $\{x_1 \in T(x_0) : bd(x_0,x_1) \leq d(x_0, Tx_1)\}$ satisfying

$$d(x_1, Tx_1) \leq cd(x_0, x_1),$$

and for $x_1 \in X$, there is $x_2 \in I_b^{(T)x_1}$ satisfying

$$d(x_2, Tx_2) \leq cd(x_1, x_2).$$

Continuing this process, we can get an iterative sequence $\{x_n\}_{n=0}^{\infty}$, where $x_{n+1} \in I_b^{(T)x_n}$ and

$$d(x_{n+1}, Tx_{n+1}) \leq cd(x_n, x_{n+1}), \quad n = 0, 1, 2, ... \tag{29}$$

On the other hand $x_{n+1} \in I_b^{(T)x_n}$ implies

$$bd(x_n, x_{n+1}) \quad \leq \quad d(x_n, Tx_n), \quad n = 0, 1, 2, ... \tag{30}$$

From (30) and (31), we have

$$d(x_{n+1}, Tx_{n+1}) \leq \frac{c}{b} d(x_n, Tx_n), \quad n = 0, 1, 2, ...$$

and

$$d(x_{n+1}, x_{n+2}) \leq \frac{c}{b} d(x_n, x_{n+1}), \quad n = 0, 1, 2, ...$$

Observe that

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \frac{c}{b} d(x_{n-1}, x_n) \\ &\leq \frac{c}{b} \left[\frac{c}{b} d(x_{n-2}, x_{n-1}) \right] \\ &= \frac{c^2}{b^2} d(x_{n-2}, x_{n-1}) \\ &\quad .... \\ &\quad .... \\ &\quad .... \\ &= \frac{c^n}{b^n} d(x_0, x_1). \end{aligned} \tag{31}$$

Since $c < b$, $\frac{c}{b} < 1$, therefore $\lim_{n\to\infty} (\frac{c}{b})^n \to 0$, which means that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. By the completeness of X, there exists $v \in X$ such that $\{x_n\}_{n=0}^{\infty}$ converges to v.

Now we have to show that $v \in F(T)$. We have given $\{f(x_n)\}_{n=0}^{\infty} = \{d(x_n, Tx_n)\}_{n=0}^{\infty}$ to be a decreasing sequence and hence it converges to zero. Since f is lower semi-continuous, as $x_n \to v$, we have $0 \leq f(v) \leq \underline{\lim}_{n\to\infty} f(x_n) = 0$. Hence $f(v) = 0$. Finally the closeness of $T(v)$ implies $v \in T(v)$. Hence

$v \in F(T)$.
Now, we observe that

$$\begin{aligned} d(x_0, v) &\leq \sum_{n=0}^{\infty} d(x_n, x_{n+1}) \\ &\leq \sum_{n=0}^{\infty} \left(\frac{c}{b}\right)^n d(x_0, x_1) \\ &\leq \left(\frac{1}{1-\frac{c}{b}}\right) d(x_0, x_1) \\ &\leq \left(1-\frac{c}{b}\right)^{-1} \frac{1}{b} d(x_0, Tx_0). \end{aligned} \tag{32}$$

Now

$$\begin{aligned} d(x_0, Tx_0) &\leq sup_{x \in Sx_0} d(x, Tx_0) \\ &\leq max\{sup_{x \in Sx_0} d(x, Tx_0), sup_{x \in Tx_0} d(x, Sx_0)\} \\ &= H(Sx_0, Tx_0). \end{aligned} \tag{33}$$

Hence we get

$$\begin{aligned} d(x_0, v) &\leq b(b-c)^{-1} \frac{1}{b} d(x_0, Tx_0) \\ &\leq (b-c)^{-1} H(Sx_0, Tx_0). \end{aligned} \tag{34}$$

Interchanging the roles of S and T, for each $y_0 \in F(T)$ and $y_1 \in Sy_0$, for any $y_0 \in X$ and $u \in F(S)$, we have

$$d(y_0, u) \leq (b-c)^{-1} H(Sy_0, Ty_0).$$

Thus, we have

$$H(F(S), F(T)) \leq (b-c)^{-1} sup_{x \in X} H(Sx, Tx).$$

□

Example Let $X = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...\frac{1}{2^n}, ...\right\} \cup \{0, 1\}$, $d(x, y) = |x - y|$ for any $x, y \in X$, be a complete metric space. Define the mappings $S, T : X \to C(X)$ as and

$$S(x) = \begin{cases} \left\{\dfrac{1}{2^{n+2}}, 1\right\}, & \text{if } x = \dfrac{1}{2^n}, \quad n = 0, 1, 2, ... \\ \left\{0, \dfrac{1}{2}\right\}, & \text{if } x = 0. \end{cases}$$

Now

$$f(x) = d(x, Tx) = \begin{cases} \frac{1}{2^{n+1}}, & \text{if } x = \frac{1}{2^n}, n = 1, 2, ... \\ 0, & \text{if } x = 0, 1 \end{cases}$$

and

$$f(x) = d(x, Sx) = \begin{cases} \frac{3}{2^{n+2}}, & \text{if } x = \frac{1}{2^n}, n = 1, 2, ... \\ 0, & \text{if } x = 0, 1 \end{cases}$$

Hence f is continuous for both mappings S and T. Obviously, S and T are not contractive mappings. It is clear that

$$H\left(T\left(\frac{1}{2^n}\right), T(0)\right) = \frac{1}{2}.$$

Hence

$$H\left(T\left(\frac{1}{2^n}\right), T(0)\right) = \frac{1}{2} \geq \frac{1}{2^n} = \left|\frac{1}{2^n} - 0\right| = d\left(\frac{1}{2^n}, 0\right) \quad n = 1, 2, 3....$$

For mapping $S : X \to C(X)$

$$H\left(S\left(\frac{1}{2^n}\right), S(0)\right) = \frac{1}{2}.$$

Hence

$$H\left(S\left(\frac{1}{2^n}\right), S(0)\right) = \frac{1}{2} \geq \frac{1}{2^n} = \left|\frac{1}{2^n} - 0\right| = d\left(\frac{1}{2^n}, 0\right), \quad n = 1, 2, 3....$$

Furthermore, there exists $y \in I_{0.7}^x$, for any $x \in X$, such that $d(y, T(y)) = \frac{1}{2}d(x, y)$ and $d(y, S(y)) < \frac{1}{2}d(x, y)$, then

$$H(F(S), F(T)) = 0$$

and

$$Sup_{x\in X} H(Sx, Tx) = \frac{1}{4}.$$

Hence, we get $H(F(S), F(T)) \leq (b - c)^{-1} Sup_{x\in X} H(Sx, Tx)$.

Theorem 4.2. *Let X be a complete metric space and let $T_n : X \to C(X)$ $(n = 0, 1, 2, 3, ...)$ be multivalued mappings. If there exists a constant $c \in (0, 1)$ such that for any $x \in X$, there is $y \in I_b^{(n)x}$ satisfying*

$$d(y, T_n y) \leq cd(x, y), \quad for\ n = 1, 2, 3, 4....$$

If $\lim_{n\to\infty} H(T_n x, T_0 x) = 0$ *uniformly for* $x \in X$, *then* $\lim_{n\to\infty} H(F(T_n), F(T_0)) = 0$

Proof. Since

$$\lim_{n\to\infty} H(T_n(x), T_0(x)) = 0$$

uniformly for $x \in X$, it is possible to select $n_0 \in \mathbb{N}$, such that

$$sup_{x\in X} H(T_n x, T_0 x) \leq (b-c)\epsilon, \quad for \;\; all \;\; n \geq n_0.$$

By proposition 4.1, we have

$$H(F(T_n), F(T_0)) < \epsilon, \quad for \;\; all \;\; n \geq n_0.$$

Hence

$$\lim_{n\to\infty} H(F(T_n), F(T_0)) = 0.$$

□

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