Fixed points of Kannan mappings in metric spaces endowed with a graph

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Abstract

Let (X, d) be a metric space endowed with a graph G such that the set $V(G)$ of vertices of G coincides with X. We define the notion of G-Kannan maps and obtain a fixed point theorem for such mappings.

1 Introduction

Let T be a selfmap of a metric space (X, d). Following Petruşel and Rus [10], we say that T is a Picard operator (abbr., PO) if T has a unique fixed point x^* and $\lim_{n\to\infty} T^n x = x^*$ for all $x \in X$ and is a weakly Picard operator (abbr. WPO) if the sequence $(T^n x)_{n\in\mathbb{N}}$ converges , for all $x \in X$ and the limit (which may depend on x) is a fixed point of T.

Let (X, d) be a metric space. Let Δ denote the diagonal of the Cartesian product $X \times X$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X, and the set $E(G)$ of its edges contains all loops, i.e., $E(G) \supseteq \Delta$. We assume G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$. Moreover, we may treat G as a weighted graph (see [[6], p. 309]) by assigning to each edge the distance between its vertices. By G^{-1} we denote the conversion of a graph G, i.e., the graph obtained from G by reversing the direction of edges. Thus we have

$$E\left(G^{-1}\right) = \{(x, y) \,|\, (y, x) \in G\}.$$

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The letter $\widetilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges. Actually, it will be more convenient for us to treat $\widetilde{G}$ as a directed graph for which the set of its edges is symmetric. Under this convention,

$$E\left(\widetilde{G}\right) = E\left(G\right) \cup E\left(G^{-1}\right) \tag{1.1}$$

We call (V', E') a subgraph of G if $V' \subseteq V\left(G\right)$, $E' \subseteq E\left(G\right)$ and for any edge $(x, y) \in E'$, $x, y \in V'$.

Now we recall a few basic notions concerning the connectivity of graphs. All of them can be found, e.g., in [6]. If x and y are vertices in a graph G, then a path in G from x to y of length N ($N \in \mathbb{N}$) is a sequence $(x_i)_{i=0}^{N}$ of $N+1$ vertices such that $x_0 = x$, $x_N = y$ and $(x_{n-1}, x_n) \in E\left(G\right)$ for $i = 1, ..., N$. A graph G is connected if there is a path between any two vertices. G is weakly connected if $\widetilde{G}$ is connected. If G is such that $E\left(G\right)$ is symmetric and x is a vertex in G, then the subgraph G_x consisting of all edges and vertices which are contained in some path beginning at x is called the component of G containing x. In this case $V\left(G_x\right) = [x]_G$, where $[x]_G$ is the equivalence class of the following relation R defined on $V\left(G\right)$ by the rule:

$$yRz \text{ if there is a path in } G \text{ from } y \text{ to } z.$$

Clearly, G_x is connected.

Recently, two results appeared giving sufficient conditions for f to be a PO if (X, d) is endowed with a graph. The first result in this direction was given by J. Jakhymski [5] who also presented its applications to the Kelisky-Rivlin theorem on iterates of the Bernstein operators on the space $C\left[0, 1\right]$.

Definition 1 ([5], Def. 2.1)**.** *We say that a mapping $f : X \to X$ is a Banach G-contraction or simply G-contraction if f preserves edges of G, i.e.,*

$$\forall x, y \in X \left((x, y) \in E\left(G\right) \Rightarrow \left(f\left(x\right), f\left(y\right)\right) \in E\left(G\right)\right) \tag{1.2}$$

and f decreases weights of edges of G in the following way:

$$\exists \alpha \in (0, 1), \forall x, y \in X \left((x, y) \in E\left(G\right) \Rightarrow d\left(f\left(x\right), f\left(y\right)\right) \leqslant \alpha d\left(x, y\right)\right) \tag{1.3}$$

Theorem 1 ([5], Th 3.2)**.** *Let (X, d) be complete, and let the triple (X, d, G) have the following property:*

for any $(x_n)_{n\in\mathbb{N}}$ in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E\left(G\right)$ for $n \in \mathbb{N}$ then there is a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ *with* $(x_{k_n}, x) \in E\left(G\right)$ *for* $n \in \mathbb{N}$.

Let $f : X \to X$ be a G-contraction, and $X_f = \{x \in X \,|(x, fx) \in E\left(G\right)\}$. Then the following statements hold.

1. $\mathrm{card} Fix\, f = \mathrm{card}\left\{[x]_{\widetilde{G}} \,|x \in X_f\right\}$.

2. $\mathrm{Fix}\, f \neq \emptyset$ iff $X_f \neq \emptyset$.

3. *f has a unique fixed point iff there exists $x_0 \in X_f$ such that $X_f \subseteq [x_0]_{\widetilde{G}}$.*

4. *For any $x \in X_f$, $f \Big|_{[x]_{\widetilde{G}}}$ is a PO.*

5. *If $X_f \neq \emptyset$ and G is weakly connected, then f is a PO.*

6. *If $X' := \cup \left\{ [x]_{\widetilde{G}} \, | x \in G \right\}$ then $f\,|_{X'}$ is a WPO.*

7. *If $f \subseteq E(G)$, then f is a WPO.*

Subsequently, Bega, Butt and Radojević extended Theorem 1 for set valued mappings.

Definition 2 ([1], Def. 2.6)**.** *Let $F : X \rightsquigarrow X$ be a set valued mapping with nonempty closed and bounded values. The mapping F is said to be a G-contraction if there exists a $\alpha \in (0,1)$ such that*

$$D(Fx, Fy) \leqslant \alpha d(x,y) \text{ for all } x, y \in E(G)$$

and if $u \in Fx$ and $v \in Fy$ are such that

$$d(u,v) \leqslant \alpha d(x,y) + k, \text{ for each } k > 0$$

then $(u,v) \in E(G)$.

Theorem 2 ([1], Th. 3.1)**.** *Let (X,d) be a complete metric space and suppose that the triple (X,d,G) has the property:*

for any $(x_n)_{n\in\mathbb{N}}$ in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$ then there is a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ *with* $(x_{k_n}, x) \in E(G)$ *for* $n \in \mathbb{N}$.

Let $F : X \rightsquigarrow X$ be a G-contraction and

$$X_f = \{x \in X : (x,u) \in E(G) \text{ for some } u \ \in Fx\}.$$

Then the following statements hold:

1. *For any $x \in X_F$, $F \Big|_{[x]_{\widetilde{G}}}$ has a fixed point.*

2. *If $X_F \neq \emptyset$ and G is weakly connected, then F has a fixed point in X.*

3. *If $X' := \cup \left\{ [x]_{\widetilde{G}} : x \in X_F \right\}$, then $F\,|_{X'}$ has a fixed point.*

4. *If $F \subseteq E(G)$ then F has a fixed point.*

5. *$Fix\, F \neq \emptyset$ if and only if $X_F \neq \emptyset$.*

Definition 3. *Let (X, d) be a metric space. $T : X \to X$ is called a Kannan operator if there exists $a \in \left[0, \frac{1}{2}\right)$ such that:*

$$d(Tx, Ty) \leqslant a[d(x, Tx) + d(y, Ty)]$$

for all $x, y \in X$.

Kannan [7] proved that if X is complete, then every Kannan mapping has a fixed point. A number of interesting results have been obtained by different authors for Kannan mappings, see for example [3, 4, 14, 15].

The aim of this paper is to study the existence of fixed points for Kannan mappings in metric spaces endowed with a graph G by introducing the concept of G-Kannan mappings.

2 Main Results

Throughout this section we assume that (X, d) is a metric space, and G is a directed graph such that $V(G) = X$, $E(G) \supseteq \Delta$ and the graph G has no parallel edges. The set of all fixed points of a mapping T is denoted by $FixT$.

In this section, by using the idea of Jakhymski [5], we will consider the following concept:

Definition 4. *Let (X, d) be a metric space and G a graph. The mapping $T : X \to X$ is said to be a G-Kannan mapping if:*

1. $\forall x, y \in X\,(\text{If } (x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G))$.
2. *there exists $a \in \left[0, \frac{1}{2}\right)$ such that:*

$$d(Tx, Ty) \leqslant a[d(x, Tx) + d(y, Ty)]$$

for all $(x, y) \in E(G)$.

Remark 1. *If T is a G-Kannan mapping, then T is both a G^{-1}-Kannan mapping and a $\widetilde{G}$-Kannan mapping.*

Example 1. *Any Kannan mapping is a G_0-Kannan mapping, where the graph G_0 is defined by $E(G_0) = X \times X$.*

Example 2. *Let $X = \{0, 1, 3\}$ and the euclidean metric $d(x, y) = |x - y|$, $\forall\, x, y \in X$. The mapping $T : X \to X$, $Tx = 0$, for $x \in \{0, 1\}$ and $Tx = 1$, for $x = 3$ is a G-Kannan mapping with constant $a = \frac{1}{3}$, where $G = \{(0, 1); (1, 3); (0, 0); (1, 1)); (3, 3)\}$, but is not a Kannan mapping because $d(T0, T3) = 1$ and $d(0, T0) + d(3, T3) = 2$.*

Definition 5. *Let (X,d) be a metric space endowed with a graph G and $T: X \to X$ be a mapping. We say that the graph G is $T-$connected if for all vertices x,y of G with $(x,y) \notin E(G)$, there exists a path in G,$(x_i)_{i=0}^{N}$ from x to y such that $x_0 = x, x_N = y$ and $(x_i, Tx_i) \in E(G)$ for all $i = 1,...,N-1$. A graph G is weakly $T-$connected if $\widetilde{G}$ is $T-$connected.*

Lemma 1. *Let (X,d) be a metric space endowed with a graph G and $T: X \to X$ be a G-Kannan mapping with constant a. If the graph G is weakly $T-$connected then, given $x,y \in X$, there is $r(x,y) \geqslant 0$ such that*

$$d(T^n x, T^n y) \leqslant ad\left(T^{n-1}x, T^n x\right) + \left(\frac{a}{1-a}\right)^n r(x,y) + ad\left(T^{n-1}y, T^n y\right) \quad (2.1)$$

for all $n \in \mathbb{N}^*$.

Proof. Let $x,y \in X$. If $(x,y) \in E\left(\widetilde{G}\right)$ then by induction $(T^n x, T^n y) \in E\left(\widetilde{G}\right)$ so (2.1) is true, with $r(x,y) = 0$ for all $n \in \mathbb{N}$. If $(x,y) \notin E\left(\widetilde{G}\right)$ then there is a path $(x_i)_{i=0}^{N}$ in $\widetilde{G}$ from x to y, i.e., $x_0 = x, x_N = y$ with $(x_{i-1}, x_i) \in E\left(\widetilde{G}\right)$ for $i = 1,...,N$ and $(x_i, Tx_i) \in E\left(\widetilde{G}\right)$ for $i = 1,...,N-1$. By Remark 1, T is a $\widetilde{G}$-Kannan mapping. An easy induction shows $(T^n x_{i-1}, T^n x_i) \in E\left(\widetilde{G}\right)$ for $i = 1,...,N$, $\left(T^{n-1}x_i, T^n x_i\right) \in E\left(\widetilde{G}\right)$ and

$$d\left(T^{n-1}x_i, T^n x_i\right) \leqslant \left(\frac{a}{1-a}\right)^{n-1} d(x_i, Tx_i)$$

for all $n \in \mathbb{N}^*$ and $i = 1,...,N-1$. Hence by the triangle inequality, we get

$$\begin{aligned} d(T^n x, T^n y) &\leqslant \sum_{i=1}^{N} d(T^n x_{i-1}, T^n x_i) \\ &\leqslant a\left[d\left(T^{n-1}x, T^n x\right) + 2\sum_{i=1}^{N-1} d\left(T^{n-1}x_i, T^n x_i\right) + d\left(T^{n-1}y, T^n y\right)\right] \\ &\leqslant a\left[d\left(T^{n-1}x, T^n x\right) + 2\left(\frac{a}{1-a}\right)^{n-1} \sum_{i=1}^{N-1} d(x_i, Tx_i) + d\left(T^{n-1}y, T^n y\right)\right], \end{aligned}$$

so it suffices to set

$$r(x,y) = 2(1-a)\sum_{i=1}^{N-1} d(x_i, Tx_i).$$

□

The main result of this paper is given by the following theorem.

Theorem 3. *Let (X, d) be a complete metric space endowed with a graph G and $T : X \to X$ be a G-Kannan mapping. We suppose that:*

(i.) G is weakly $T-$connected;

(ii.) for any $(x_n)_{n\in\mathbb{N}}$ in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$ then there is a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ with $(x_{k_n}, x) \in E(G)$ for $n \in \mathbb{N}$.

Then T is a PO.

Proof. Let $x \in X$. By Lemma 1, there exists $r(x, Tx)$ such that:

$$d\left(T^n x, T^{n+1}x\right) \leqslant ad\left(T^{n-1}x, T^n x\right) + \left(\frac{a}{1-a}\right)^n r(x, Tx) + ad\left(T^n x, T^{n+1}x\right).$$

for all $n \in \mathbb{N}^*$. Hence

$$d\left(T^n x, T^{n+1}x\right) \leqslant \frac{a}{1-a} d\left(T^{n-1}x, T^n x\right) + \frac{a^n}{(1-a)^{n+1}} r(x, Tx). \tag{2.2}$$

Using the relation (2.2) and an elementary calculus we get

$$d\left(T^n x, T^{n+1}x\right) \leqslant \left(\frac{a}{1-a}\right)^n d(x, Tx) + n\frac{a^n}{(1-a)^{n+1}} r(x, Tx) \tag{2.3}$$

for all $n \in \mathbb{N}$.

Let $b := \frac{a}{1-a}$, because $a \in \left[0, \frac{1}{2}\right)$ then $b \in [0, 1)$. Using the relation (2.3) we have

$$\begin{aligned} \sum_{i=0}^{n} d\left(T^k x, T^{k+1}x\right) &\leqslant d(x, Tx) \sum_{i=0}^{n} b^k + \frac{r(x, Tx)}{1-a} \sum_{i=0}^{n} k \cdot b^k \\ &= d(x, Tx) \frac{1-b^{n+1}}{1-b} + \frac{r(x, Tx)}{1-a} \cdot \frac{nb^{n+2} - (n+1)b^{n+1} + b}{(1-b)^2} \end{aligned}$$

Hence $\sum\limits_{n=0}^{\infty} d\left(T^n x, T^{n+1}x\right) < \infty$ and because

$$d\left(T^n x, T^{n+p}x\right) \leqslant \sum_{i=0}^{n+p} d\left(T^i x, T^{i+1}x\right) - \sum_{i=0}^{n-1} d\left(T^i x, T^{i+1}x\right) \to 0$$

as $n \to \infty$ and for all $p \in \mathbb{N}$, we get $(T^n x)_{n\geqslant 0}$ is a Cauchy sequence. But (X, d) is a complete metric space, therefore $(T^n x)_{n\geqslant 0}$ converges to some $x^* \in X$.

Let $x, y \in X$ then $(T^n x)_{n\geqslant 0} \to x^*$ and $(T^n y)_{n\geqslant 0} \to y^*$, as $n \to \infty$.
By Lemma 1, for all $n \in \mathbb{N}^*$, we get

$$d(T^n x, T^n y) \leqslant ad\left(T^{n-1}x, T^n x\right) + \left(\frac{a}{1-a}\right)^n r(x,y) + ad\left(T^{n-1}y, T^n y\right)$$

Letting $n \to \infty$ obtain that $d(x^*, y^*) \leqslant 0$, hence $x^* = y^*$ and for all $x \in X$ there exists a unique x^* such that

$$\lim_{n\to\infty} T^n x = x^*.$$

Now we will prove that $x^* \in FixT$. Because the graph G is weakly $T-$connected, there is at least $x_0 \in X$ such that $(x_0, Tx_0) \in E\left(\widetilde{G}\right)$ so $\left(T^n x_0, T^{n+1}x_0\right) \in E\left(\widetilde{G}\right)$ for all $n \in \mathbb{N}$. But $\lim_{n\to\infty} T^n x_0 = x^*$, then by (ii.) there is a subsequence $\left(T^{k_n}x_0\right)_{n\in\mathbb{N}}$ with $\left(T^{k_n}x_0, x^*\right) \in E(G)$ for all $n \in \mathbb{N}$. Then, for all $n \in \mathbb{N}$, we get

$$\begin{aligned} d(x^*, Tx^*) &\leqslant d\left(x^*, T^{k_n+1}x\right) + d\left(T^{k_n+1}x_0, Tx^*\right) \\ &\leqslant d\left(x^*, T^{k_n+1}x\right) + a\left[d\left(T^{k_n}x_0, T^{k_n+1}x_0\right) + d(x^*, Tx^*)\right]. \end{aligned}$$

Hence

$$d(x^*, Tx^*) \leqslant \frac{1}{1-a} d\left(x^*, T^{k_n+1}x\right) + \frac{a}{1-a} d\left(T^{k_n}x_0, T^{k_n+1}x_0\right)$$

Now, letting $n \to \infty$, we obtain

$$d(x^*, Tx^*) = 0 \Leftrightarrow x^* = Tx^*, \text{ that is, } x^* \in \mathrm{Fix}T.$$

If we have $Ty = y$ for some $y \in X$, then from above, we must have $T^n y \to x^*$, so $y = x^*$ and therefore, T is a PO.

□

The next example shows that the condition (ii.) is a necessary condition for G-Kannan mapping to be a PO.

Example 3. *Let $X := [0,1]$ be endowed with the Euclidean metric d_E. Define the graph G by*

$$E(G) = \{(x,y) \in (0,1] \times (0,1] \,|\, x \geqslant y\} \cup \{(0,0), (0,1)\}$$

Set

$$Tx = \frac{x}{4} \text{ for } x \in (0,1], \text{ and } T0 = 1$$

It is easy to verify (X,d) is a complete metric space, G is weakly $T-$connected and T is a G-Kannan mapping with $a = \frac{3}{7}$. Clearly, $T^n x \to 0$ for all $x \in X$, but T has no fixed points.

The next example shows that the graph G must be $T-$connected so the G-Kannan mapping T to be a PO.

Example 4. *Let $X = \mathbb{N}\backslash\{0,1\}$ be endowed with the Euclidean metric and define $T : X \to X$, $\quad Tx = 2x$. Consider the graph G define by:*

$$V(G) = X \text{ and } E(G) = \left\{\left(2^k n, 2^k (n+1)\right) : n \in X, k \in \mathbb{N}\right\} \cup \Delta$$

Then T is G-Kannan operator with $a = \frac{2}{5}$ because

$$\begin{aligned} d_E\left(T2^k n, T2^k(n+1)\right) = 2^{k+1} &\leqslant \frac{2}{5}2^k(2n+1) \\ &= \frac{2}{5}\left[d_E\left(2^k n, T2^k n\right) + d_E\left(2^k(n+1), T2^k(n+1)\right)\right] \end{aligned}$$

for all $n \in X$ and for all $k \in \mathbb{N}$.

Then (X,d) is a complete metric space, G is weakly connected but not weakly $T-$connected because $(2,4) \notin E\left(\widetilde{G}\right)$ and the only path in $\widetilde{G}$ from 2 to 4 is
$y_0 = 2, y_1 = 3, y_2 = 4$ *and* $(3,T3) = (3,6) \notin E\left(\widetilde{G}\right)$.

Clearly, $T^n x$ not converge for all $x \in X$ and T has no fixed points.

From Theorem 3, we obtain the following corollary concerning the fixed point of Kannan operator in partially ordered metric spaces.

Corollary 1. *Let $(X,\leqslant)$ be a partially ordered set and d be a metric on X such that the metric space (X,d) is complete. Let $T : X \to X$ be an increasing operator such that the following three assertions hold:*

(i.) There exist $a \in \left[0, \frac{1}{2}\right)$ such that $d(Tx,Ty) \leqslant a\left[d(x,Tx) + d(y,Ty)\right]$ for each $x, y \in X$ with $x \leqslant y$;

(ii.) For each $x, y \in X$, incomparable elements of $(X,\leqslant)$, there exists $z \in X$ such that $x \leqslant z$, $y \leqslant z$ and $z \leqslant Tz$;

(iii.) If an increasing sequence (x_n) converges to x in X, then $x_n \leqslant x$ for all $n \in \mathbb{N}$.

Then T is a PO.

Proof. Consider the graph G with $V(G) = X$, and

$$E(G) = \{(x,y) \in X \times X \mid x \leqslant y\}.$$

Because the mapping T is increasing and (i.) holds we get the mapping T is a $G-$Kannan mapping. By (ii.) the graph G is weakly $T-$connected and the condition (iii.) implies the condition (ii.) from Theorem 3. The conclusion follows now from Theorem 3. □

In the next we show the fixed point theorem for cyclic Kannan mapping proved in [9] by Petric is a consequence of the Theorem 3.

Let $p \leqslant 2$ and $\{A_i\}_{i=1}^{p}$ be nonempty closed subsets of a complete metric space X. A mapping $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ is called a *cyclical operator* if

$$T(A_i) \subseteq A_{i+1}, \quad \text{for all } i \in \{1, 2, ..., p\} \tag{2.4}$$

where $A_{p+1} := A_1$.

Theorem 4. *Let $A_1, A_2, ..., A_p, A_{p+1} = A_1$ be nonempty closed subsets of a complete metric space (X, d) and suppose $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ is a cyclical operator, and there exists $a \in \left[0, \frac{1}{2}\right)$ such that for each pair $(x, y) \in A_i \times A_{i+1}$, for $i \in \{1, 2, ..., p\}$, we have*

$$d(Tx, Ty) \leqslant a\left[d(x, Tx) + d(y, Ty)\right].$$

Then T is a PO.

Proof. Let $Y = \cup_{i=1}^{p} A_i$ then (Y, d) is a complete metric space.

Consider the graph G with $V(G) = Y$, and

$$E(G) = \{(x, y) \in Y \times Y \,|\, \exists\, i \in \{1, 2, ..., p\} \text{ such that } x \in A_i \text{ and } y \in A_{i+1}\}$$

Because T is a cyclic operator we get

$$(Tx, Ty) \in E(G), \text{ for all } (x, y) \in E(G)$$

and via hypothesis the operator T is a G-Kannan operator and the graph G is weakly $T-$connected. Now let $(x_n)_{n\in\mathbb{N}}$ in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$. Then there is $i \in \{1, 2, ..., n\}$ such that $x \in A_i$. However in view of (2.4) the sequence $\{x_n\}$ has an infinite number of terms in each A_i, for all $i \in \{1, 2, ..., p\}$. The subsequence of the sequence $\{x_n\}$ formed by the terms which is in A_{i-1} satisfies the condition (ii.) from Theorem 3. In conclusion the operator T is PO.

□

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