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Noor Iterative Approximation for Solutions to Variational Inclusions with k-Subaccretive Type Mappings in Reflexive Banach Spaces

Shuyi Zhang, Xinqi Guo and Jun Wang

Abstract

In this paper, we introduce and study a new class of nonlinear variational inclusion problems with Lipschitz k-subaccretive type mappings in real reflexive Banach spaces. The existence and uniqueness of such solutions are proved and the convergence and stability of Noor iterative sequences with errors are also discussed. Furthermore, general convergence rate estimates are given in our results, which essentially improve and extend the corresponding results in Chang[1, 2], Ding[3], Gu[5, 6, 7], Hassouni and Moudafi[8], Kazmi[9], Noor[11, 12], Siddiqi and Ansari[13], Siddiqi, Ansari and Kazmi[14] and Zeng[16].

1 Introduction and Preliminaries

Let X be an arbitrary real Banach space with norm $\|\cdot\|$ and dual X^*, and J denote by the normalized duality mapping from X into 2^{X^*} given by $J(x) = \left\{ f \in X^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2 \right\}, \forall x \in X$, where $\langle \cdot, \cdot \rangle$ is the generalized duality pairing. In the sequel, I denotes the identity operator on X, $D(T)$ denote the domain of the mapping T.

Let $T, A, B : X \to X, N(\cdot,\cdot,\cdot) : X \times X \times X \to X, g : X \to X^*$, $\eta : X^* \times X^* \to X^*$ be mappings and $\varphi : X^* \times X \to R \cup \{+\infty\}$ be such that for

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each fixed $y \in X, \varphi(\cdot, y)$ is a proper convex lower semicontinuous function, we consider the following problem. For any given $f \in X$, to find $u \in X$ such that

$$\begin{cases} g(u) \in D(\partial_\eta \varphi(\cdot, u)), \\ \langle N(Tu, Au, Bu) - f, \eta(v, g(u)) \rangle \geq \varphi(g(u), u) - \varphi(v, u), \forall v \in X^*, \end{cases} \tag{1.1}$$

where $\partial_\eta \varphi(\cdot, u)$ denotes the η-subdifferential of $\varphi(\cdot, u)$.

Now we consider some special cases of (1.1).

(1) If $N(x, y, z) = N(x, y), \varphi(x_1, y) = \varphi(x_1), \forall x, y, z \in X, \forall x_1 \in X^*$, then the problem (1.1) reduces to problem (1.2). For any given $f \in X$, to find $u \in X$ such that

$$\begin{cases} g(u) \in D(\partial_\eta \varphi), \\ \langle N(Tu, Au) - f, \eta(v, g(u)) \rangle \geq \varphi(g(u)) - \varphi(v), \forall v \in X^*, \end{cases} \tag{1.2}$$

where $\partial_\eta \varphi$ denotes the η-subdifferential of φ which has been studied in Gu[5, 6].

(2) If $N(x, y, z) = N(x, y), \varphi(x_1, y) = \varphi(x_1)$ and $\eta(x_1, y_1) = x_1 - y_1$ for all $x, y, z \in X$ and $x_1, y_1 \in X^*$, then the problem (1.1) reduces to problem (1.3). For any given $f \in X$, to find $u \in X$ such that

$$\begin{cases} g(u) \in D(\partial \varphi), \\ \langle N(Tu, Au) - f, v - g(u) \rangle \geq \varphi(g(u)) - \varphi(v), \ \forall v \in X^*, \end{cases} \tag{1.3}$$

where $\partial \varphi$ denotes the subdifferential of φ which has been studied in Zeng[16].

(3) $N(x, y, z) = x - y, \varphi(x_1, y) = \varphi(x_1)$ and $\eta(x_1, y_1) = x_1 - y_1$ for all $x, y, z \in X$ and $x_1, y_1 \in X^*$, then the problem (1.1) reduces to problem (1.4). For any given $f \in X$, to find $u \in X$ such that

$$\begin{cases} g(u) \in D(\partial \varphi), \\ \langle Tu - Au - f, v - g(u) \rangle \geq \varphi(g(u)) - \varphi(v), \ \forall v \in X^*, \end{cases} \tag{1.4}$$

where $\partial \varphi$ denotes the subdifferential of φ which has been studied in Chang[1, 2] and Gu[7].

(4) If X is a Hilbert space H, $\varphi(x_1, y) = \varphi(x_1), N(x, y, z) = x - y$ and $\eta(x_1, y_1) = x_1 - y_1$ for all $x, y, z \in X$ and $x_1, y_1 \in X^*$, then the problem (1.1) reduces to problem (1.5). For given $f \in H$, to find $u \in H$ such that

$$\begin{cases} g(u) \in D(\partial \varphi), \\ \langle Tu - Au - f, v - g(u) \rangle \geq \varphi(g(u)) - \varphi(v), \ \forall v \in H, \end{cases} \tag{1.5}$$

(1.5) is said to be a variational inclusion problem in a Hilbert space which has been studied in Ding[3], Hassouni and Moudafi[8] and Kazmi[9]. If $\varphi = \delta_K$, where K is a nonempty closed convex subset of H and δ_K is the indicator function of K, i.e.,

$$\delta_K(x) = \begin{cases} 0, & x \in K, \\ +\infty, & x \notin K. \end{cases}$$

Then the variational inclusion problem (1.5) is equivalent to to find $u \in K$ for given f such that

$$\begin{cases} g(u) \in K, \\ \langle Tu - Au - f, v - g(u) \rangle \geq 0, \ \forall v \in K, \end{cases} \tag{1.6}$$

(1.6) is said to be the strongly nonlinear quasi-variational inequality problem which has been studied in Noor[11,12], Siddiqi and Ansari[13] and Siddiqi, Ansari and Kazmi[14].

The following definitions and lemmas will be needed in the sequel.

Definition 1.1 ([15]). An operator $T : D(T) \subset X \to X$ is called k-subaccretive, if for all $x, y \in D(T)$, there exist $j(x-y) \in J(x-y)$ and a constant $k \in (-\infty, +\infty)$ such that

$$\langle Tx - Ty, j(x-y) \rangle \geq k \|x - y\|^2 .$$

It is clear that k-subaccretive operator must be accretive and strongly accretive operator, respectively. However, the converse doesn't hold in general(see[15]). It is well known that T is accretive if and only if

$$\|x - y\| \leq \|x - y + r(Tx - Ty)\| \tag{1.7}$$

for all $x, y \in D(T)$ and $r > 0$.

Definition 1.2 ([4]). Let X be a real Banach space, and $\eta : X \times X \to X$ be a mapping. A proper convex function $\varphi : X^* \to R \cup \{+\infty\}$ is said to be η-subdifferentiable at $x_0 \in X$ if there exist $f \in X^*$ such that

$$\varphi(y) - \varphi(x_0) \geq \langle f, \eta(y, x_0) \rangle, \ \forall y \in X,$$

where f is called a η-subgradient of φ at x_0. The set of all η-subgradients of φ at x_0 is denoted by $\partial_\eta \varphi(x_0)$.

Suppose that T is an operator on X . Let x_0 be a point in X and let $x_{n+1} = f(T, x_n)$ $(n \geq 0)$ denote an iteration procedure which yields a sequence of points $\{x_n\}$ in X. Assume that $F(T) = \{x \in X : Tx = x\} \neq \emptyset$ and that $\{x_n\}$ converges strongly to $q \in F(T)$. Let $\{y_n\}$ be an arbitrary sequence in X and set $\varepsilon_n = \|y_{n+1} - f(T, y_n)\|$. If $\lim_{n\to\infty} \varepsilon_n = 0$ implies that $\lim_{n\to\infty} y_n = q$, then the iteration procedure defined by $x_{n+1} = f(T, x_n)$ is said to be T-stable.

Lemma 1.1. Let X be a real reflexive Banach space, then the following conclusions are equivalent
(i) $x^* \in X$ is a solution of the variational inclusion problem (1.1).
(ii) $x^* \in X$ is a fixed point of the mapping $S : X \to 2^X$, where

$$S(x) = f - (N(Tx, Ax, Bx) + \partial_\eta \varphi(g(x), x)) + x, \ \forall x \in X.$$

(iii) $x^* \in X$ is a solution of equation $f \in N(Tx, Ax, Bx) + \partial_\eta \varphi(g(x), x)$.

Proof. (i)$\Rightarrow$(iii). If x^* is a solution of the variational inclusion problem (1.1), then $g(x^*) \in D(\partial_\eta \varphi(\cdot, x^*))$ and

$$\langle N(Tx^*, Ax^*, Bx^*) - f, \eta(v, g(x^*))\rangle \geq \varphi(g(x^*), x^*) - \varphi(v, x^*), \ \forall v \in X^*.$$

By the definition of η-subdifferential $\partial_\eta \varphi(\cdot, x^*)$, we have

$$f - N(Tx^*, Ax^*, Bx^*) \in \partial_\eta \varphi(g(x^*), x^*).$$

therefore, $x^* \in X$ is a solution of equation $f \in N(Tx, Ax, Bx) + \partial_\eta \varphi(g(x), x)$.

(iii)$\Rightarrow$(ii). If (iii) is holds, then it is easy to see that $x^* \in f - (N(Tx^*, Ax^*, Bx^*) + \partial_\eta(g(x^*), x^*)) + x^* = Sx^*$. This implies that x^* is a fixed point of S in X.

(ii)$\Rightarrow$(i). If (ii) is holds, then $f - N(Tx^*, Ax^*, Bx^*) \in \partial_\eta \varphi(g(x^*), x^*)$, hence from the definition of $\partial_\eta \varphi(\cdot, x^*)$, it follows that

$$\varphi(v, x^*) - \varphi(g(x^*), x^*) \geq \langle f - N(Tx^*, Ax^*, Bx^*), \eta(v, g(x^*))\rangle, \ \forall v \in X^*,$$

i.e.,

$$\langle N(Tx^*, Ax^*, Bx^*) - f, \eta(v, g(x^*))\rangle \geq \varphi(g(x^*), x^*) - \varphi(v, x^*), \ \forall v \in X^*.$$

This implies that x^* is a solution of the variational inclusion problem (1.1). This completes the proof.

Lemma 1.2 ([5,6]). Let X be an arbitrary real Banach space and $T : X \to X$ be continuous k-subaccretive operator. If $k > -1$, then the equation $x + Tx = f$ has a unique solution for any $f \in X$.

Lemma 1.3 ([10]). Suppose that $\{a_n\}$ and $\{b_n\}$are nonnegative sequence satisfying the following inequality

$$a_{n+1} \leq \lambda a_n + b_n, \quad n \geq 0,$$

where $\lambda \in (0, 1)$ and $\lim_{n\to\infty} b_n = 0$, Then $\lim_{n\to\infty} a_n = 0$.

2. Main Results

Theorem 2.1. Suppose that X is a real reflexive Banach space. Let $T, A, B : X \to X, N(\cdot,\cdot,\cdot) : X \times X \times X \to X, g : X \to X^*, \eta : X^* \times X^* \to X^*$ be mappings and $\varphi : X^* \times X \to R \cup \{+\infty\}$ be such that for each fixed $y \in X, \varphi(\cdot, y)$ is a proper convex lower semicontinuous function with η-subdifferential $\partial_\eta \varphi$. Assume that $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are sequences in X, and that $\{\alpha_n\}$ is sequence in $(0, \frac{1}{2}) \subset (0,1)$ and $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1)$, satisfying the following conditions

(i)$N(T(\cdot), A(\cdot), B(\cdot)) + \partial_\eta \varphi(g(\cdot), \cdot) - I : X \to X$ is a Lipschitz continuous k-subaccretive with $k \in (-1,1)$ and Lipschitz constant $L \geq 1$;

(ii) $\beta_n L(1 + L(1 + \gamma_n L)) + \alpha_n L\left(1 + L + \beta_n L\left(L - 1 + \gamma_n L^2\right)\right) \leq 1 + k - \mu$, where $\mu \in (0, 1+k)$;

(iii) $0 < a \leq \alpha_n$, $n \geq 0$, where a is a constant;

(iv) $\lim\limits_{n\to\infty} \|u_n\| = \lim\limits_{n\to\infty} \|v_n\| = \lim\limits_{n\to\infty} \|w_n\| = 0$.

For arbitrary $f \in X$ define the operator $S : X \to X$ by

$$S(x) = f - (N(Tx, Ax, Bx) + \partial_\eta \varphi(g(x), x)) + x, \ \forall x \in X.$$

For arbitrary $x_0, u_0, v_0, w_0 \in X$ define Noor iterative sequence with errors $\{x_n\}$ by

$$\begin{cases} z_n = (1-\gamma_n) x_n + \gamma_n S x_n + w_n \\ y_n = (1-\beta_n) x_n + \beta_n S z_n + v_n \\ x_{n+1} = (1-\alpha_n) x_n + \alpha_n S y_n + u_n. \end{cases} \tag{2.1}$$

Let $\{g_n\}$ be any sequence in X and define $\{\varepsilon_n\}$ by

$$\begin{cases} \varepsilon_n = \|g_{n+1} - (1-\alpha_n) g_n - \alpha_n S\xi_n - u_n\| \\ \xi_n = (1-\beta_n) g_n + \beta_n S\eta_n + v_n \\ \eta_n = (1-\gamma_n) g_n + \gamma_n S g_n + w_n. \end{cases} \tag{2.2}$$

Then the following conclusions hold:

(I) The variational inclusion problem (1.1) has a unique solution $x^* \in X$, and Noor iterative sequence with errors $\{x_n\}$ defined by (2.1) converges strongly to the unique solution x^* of the variational inclusion (1.1); moreover,

$$\|x_n - x^*\| \leq \left(1 - \frac{\mu a}{2}\right)^n \|x_0 - x^*\| + \begin{cases} \frac{2(1-(1-(1/2)\mu a)^n)M}{\mu a(1+k)}, & \text{if } -1 < k < 0, \\ \frac{2(1-(1-(1/2)\mu a)^n)M}{\mu a}, & \text{if } 0 \leq k < 1, \end{cases}$$

where $n \geq 0$ and $M = \sup\limits_{n\geq 0} \left\{L(L+1)\|v_n\| + (L+1)\|u_n\| + L^2(1+L)\|w_n\|\right\}$.

(*II*)

$$\|g_{n+1} - x^*\| \leq \|g_n - x^*\| + \varepsilon_n$$
$$+ \begin{cases} \frac{1}{1+k} \left(L(L+1) \|v_n\| + (L+1) \|u_n\| + L^2 (1+L) \|w_n\|\right), \text{if } -1 < k < 0, \\ \\ L(L+1) \|v_n\| + (L+1) \|u_n\| + L^2 (1+L) \|w_n\|, \quad \text{if } \ 0 \leq k < 1, \end{cases}$$

for all $n \geq 0$.
(*III*) $\lim_{n\to\infty} g_n = x^*$ if and only if $\lim_{n\to\infty} \varepsilon_n = 0$.

Proof. Since $N(T(\cdot), A(\cdot), B(\cdot)) + \partial_\eta \varphi(g(\cdot), \cdot) - I$ is continuous and k-subaccret- ive, by Lemma 1.2, for any given $f \in X$, the equation $x + (N(Tx, Ax, Bx) + \partial_\eta \varphi(g(x), x)) - x) = f$ has a unique solution $x^* \in X$, i.e., the equation $N(Tx, Ax, Bx) + \partial_\eta \varphi(g(x), x)) = f$ has a unique solution $x^* \in X$. From X is a reflexive Banach space, hence, using Lemma 1.1, we know that x^* is a unique solution of the variational inclusion problem (1.1), and so x^* is also the unique fixed point of S in X, i.e., $Sx^* = x^*$.

Now we prove Noor iterative convergence and stability and give convergence rate estimate for solutions to the variational inclusion problems (1.1) with k-subaccretive operator. From conditions (i), the mapping $N(T(\cdot), A(\cdot), B(\cdot)) + \partial_\eta \varphi(g(\cdot), \cdot) - I$ is k-subaccretive with $k \in (-1, 1)$. Hence, by Definition 1.1 for all $x, y \in X$, there exists $j(x-y) \in J(x-y)$ such that

$$\begin{aligned} &\langle Sx - Sy, j(x-y)\rangle = \langle f - (N(T(x), A(x), B(x)) + \partial_\eta \varphi(g(x), x)) + x \\ &- (f - (N(T(y), A(y), B(y)) + \partial_\eta \varphi(g(y), y)) + y), j(x-y)\rangle \\ &= -\langle (N(T(x), A(x), B(x)) + \partial_\eta \varphi(g(x), x) - x) \\ &- (N(T(y), A(y), B(y)) + \partial_\eta \varphi(g(y), y) - y), j(x-y)\rangle \leq -k \|x-y\|^2. \end{aligned} \quad (2.3)$$

From (2.3), we have

$$\langle (-S - kI)x - (-S - kI)y, j(x-y)\rangle \geq 0.$$

Hence $-S - kI$ is an accretive operator, and it follows from (1.7) that

$$\|x - y\| \leq \|x - y - r[(S + kI)x - (S + kI)y]\| \quad (2.4)$$

for all $x, y \in X$ and $r > 0$. Using (2.1), we easily conclude that for all $n \geq 0$,

$$(1 - \alpha_n) x_n = (1 + k\alpha_n) x_{n+1} - \alpha_n (S + kI) x_{n+1} + \alpha_n S x_{n+1} - \alpha_n S y_n - u_n. \quad (2.5)$$

Note that

$$(1 - \alpha_n) x^* = (1 + k\alpha_n) x^* - \alpha_n (S + kI) x^*, \quad (2.6)$$

for all $n \geq 0$. It follows from (2.4), (2.5) and (2.6) that

$$\begin{aligned}
&(1-\alpha_n)\|x_n - x^*\| \\
&\geq (1+k\alpha_n)\left\|x_{n+1} - x^* - \frac{\alpha_n}{1+k\alpha_n}\left[(S+kI)x_{n+1} - (S+kI)x^*\right]\right\| \\
&- \alpha_n \|Sx_{n+1} - Sy_n\| - \|u_n\| \\
&\geq (1+k\alpha_n)\|x_{n+1} - x^*\| - \alpha_n\|Sx_{n+1} - Sy_n\| - \|u_n\|,
\end{aligned}$$

which means that

$$\begin{aligned}
&\|x_{n+1} - x^*\| \\
&\leq \frac{1-\alpha_n}{1+k\alpha_n}\|x_n - x^*\| + \frac{\alpha_n}{1+k\alpha_n}\|Sx_{n+1} - Sy_n\| + \frac{\|u_n\|}{1+k\alpha_n} \\
&= \left(1 - \frac{\alpha_n(1+k)}{1+k\alpha_n}\right)\|x_n - x^*\| + \frac{\alpha_n}{1+k\alpha_n}\|Sx_{n+1} - Sy_n\| + \frac{\|u_n\|}{1+k\alpha_n} \qquad (2.7)
\end{aligned}$$

for all $n \geq 0$.

Because $N(T(\cdot), A(\cdot), B(\cdot)) + \partial_\eta \varphi(g(\cdot), \cdot) - I$ is a Lipschitz mapping with the Lipschitz constant L, then it is easy to know by definition of S that S is also a Lipschitz mapping with the Lipschitz constant L. Hence from (2.1) we have the following estimates

$$\begin{aligned}
\|z_n - x^*\| &\leq (1+\gamma_n L)\|x_n - x^*\| + \|w_n\|, \\
\|y_n - x^*\| &\leq (1-\beta_n)\|x_n - x^*\| + \beta_n\|Sz_n - x^*\| + \|v_n\| \\
&\leq (1-\beta_n)\|x_n - x^*\| + \beta_n L\|z_n - x^*\| + \|v_n\| \\
&\leq \left(1+\beta_n\left(L-1+\gamma_n L^2\right)\right)\|x_n - x^*\| + \beta_n L\|w_n\| + \|v_n\|, \\
\|x_n - Sy_n\| &\leq \|x_n - x^*\| + \|Sy_n - x^*\| \\
&\leq \|x_n - x^*\| + L\left(1+\beta_n\left(L-1+\gamma_n L^2\right)\right)\|x_n - x^*\| \\
&+ \beta_n L^2\|w_n\| + L\|v_n\| \\
&= \left(1+L+\beta_n L\left(L-1+\gamma_n L^2\right)\right)\|x_n - x^*\| \\
&+ \beta_n L^2\|w_n\| + L\|v_n\|, \\
\|x_n - y_n\| &\leq \beta_n\|Sz_n - x_n\| + \|v_n\| \\
&\leq \beta_n\left(L\|z_n - x^*\| + \|x_n - x^*\|\right) + \|v_n\| \\
&\leq \beta_n\left(1+L(1+\gamma_n L)\right)\|x_n - x^*\| + \beta_n L\|w_n\| + \|v_n\|,
\end{aligned}$$

and

$$\begin{aligned}
&\|Sx_{n+1} - Sy_n\| \leq L\,\|(1-\alpha_n)\,x_n + \alpha_n Sy_n + u_n - y_n\| \\
&\leq L\,\|x_n - y_n\| + \alpha_n L\,\|Sy_n - x_n\| + L\,\|u_n\| \\
&\leq \left\{\beta_n L(1 + L\,(1+\gamma_n L)) + \alpha_n L\left(1 + L + \beta_n L\left(L - 1 + \gamma_n L^2\right)\right)\right\}\|x_n - x^*\| \\
&+ L^2\,(1+\alpha_n L)\,\beta_n\,\|w_n\| + L\,(1 + L\alpha_n)\,\|v_n\| + L\,\|u_n\|\,. \qquad (2.8)
\end{aligned}$$

for all $n \geq 0$.
If $-1 < k < 0$, then $1 + k\alpha_n > 1 + k > 0$ because $\alpha_n < 1$. If $0 \leq k < 1$, then $1 + k\alpha_n \geq 1$, so we have

$$0 < \frac{\mu a}{2} \leq \frac{\mu\alpha_n}{1 + k\alpha_n} \leq \begin{cases} \dfrac{\mu\alpha_n}{1+k} < \dfrac{(1+k)\alpha_n}{1+k} = \alpha_n < 1, & \text{if } -1 < k < 0, \\ \mu\alpha_n < (1+k)\alpha_n < (1+1)\frac{1}{2} = 1\,, & \text{if } 0 \leq k < 1, \end{cases} \qquad (2.9)$$

for all $n \geq 0$. Substituting (2.8) into (2.7), and by condition (ii) and (2.9), we infer that

$$\begin{aligned}
&\|x_{n+1} - x^*\| \leq \Bigg(1 - \frac{\alpha_n\left[1 + k - \beta_n L\,(1 + L\,(1+\gamma_n L))\right.}{1 + k\alpha_n} \\
&- \frac{\left.\alpha_n L\left(1 + L + \beta_n L\left(L - 1 + \gamma_n L^2\right)\right)\right]}{1 + k\alpha_n}\Bigg)\|x_n - x^*\| \\
&+ \frac{L^2\,(1+\alpha_n L)\,\alpha_n\beta_n\,\|w_n\| + L\,(1 + L\alpha_n)\,\alpha_n\,\|v_n\| + (L\alpha_n + 1)\,\|u_n\|}{1 + k\alpha_n} \\
&\leq \left(1 - \frac{\mu\alpha_n}{1 + k\alpha_n}\right)\|x_n - x^*\| + B_n \leq \left(1 - \frac{\mu a}{2}\right)\|x_n - x^*\| + B_n, \qquad (2.10)
\end{aligned}$$

for all $n \geq 0$, where

$$\begin{aligned}
B_n &= \frac{L^2\,(1+\alpha_n L)\,\alpha_n\beta_n\,\|w_n\| + L\,(1 + L\alpha_n)\,\alpha_n\,\|v_n\| + (L\alpha_n + 1)\,\|u_n\|}{1 + k\alpha_n} \\
&\leq \begin{cases} \frac{1}{1+k}\left(L(L+1)\,\|v_n\| + (L+1)\,\|u_n\| + L^2\,(1+L)\,\|w_n\|\right), & \text{if } -1 < k < 0, \\ L(L+1)\,\|v_n\| + (L+1)\,\|u_n\| + L^2\,(1+L)\,\|w_n\|\,, & \text{if } 0 \leq k < 1, \end{cases} \qquad (2.11)
\end{aligned}$$

for all $n \geq 0$. Note that $B_n \to 0$ as $n \to \infty$ in (2.11). Set $\lambda = 1 - \frac{\mu a}{2}, a_n = \|x_n - x^*\|, b_n = B_n, n \geq 0$. By (2.10) and Lemma 1.3 ensures that $x_n \to x^*$ as $n \to \infty$, that is, $\{x_n\}$ converges strongly to the unique solution x^* of the variational inclusion problem (1.1). Furthermore, using (2.10) and (2.11), we have

$$\|x_n - x^*\| \le \left(1 - \frac{\mu a}{2}\right)\|x_{n-1} - x^*\| + \begin{cases} \dfrac{M}{1+k}, & \text{if } -1 < k < 0, \\ M, & \text{if } 0 \le k < 1 \end{cases}$$

$$\le \left(1 - \frac{\mu a}{2}\right)^n \|x_0 - x^*\| + \begin{cases} \dfrac{2\left(1 - \left(1 - (1/2)\,\mu a\right)^n\right) M}{\mu a\,(1+k)}, & \text{if } -1 < k < 0, \\ \dfrac{2\left(1 - \left(1 - (1/2)\,\mu a\right)^n\right) M}{\mu a}, & \text{if } 0 \le k < 1 \end{cases}$$

for all $n \ge 0$, completing the proof of (I).

Put $p_n = (1 - \alpha_n)\, g_n + \alpha_n S\xi_n + u_n$ for all $n \ge 0$. Note that

$$\|g_{n+1} - x^*\| \le \|p_n - x^*\| + \varepsilon_n \tag{2.12}$$

for all $n \ge 0$. Similar to (2.7) and (2.8), by (2.2), we have also the following

$$\begin{aligned} \|p_n - x^*\| &\le \left(1 - \frac{\alpha_n(1+k)}{1 + k\alpha_n}\right)\|g_n - x^*\| \\ &\quad + \frac{L\alpha_n}{1 + k\alpha_n}\|p_n - \xi_n\| + \frac{\|u_n\|}{1 + k\alpha_n}, \qquad (2.13) \\ \|\eta_n - x^*\| &\le (1 + \gamma_n L)\,\|g_n - x^*\| + \|w_n\|, \\ \|\xi_n - x^*\| &\le (1 - \beta_n)\,\|g_n - x^*\| + \beta_n\,\|S\eta_n - x^*\| + \|v_n\| \\ &\le \left(1 + \beta_n\left(L - 1 + \gamma_n L^2\right)\right)\|g_n - x^*\| + \beta_n L\,\|w_n\| + \|v_n\|, \\ \|g_n - S\xi_n\| &\le \|g_n - x^*\| + \|S\xi_n - x^*\| \\ &\le \left(1 + L + \beta_n L\left(L - 1 + \gamma_n L^2\right)\right)\|g_n - x^*\| \\ &\quad + \beta_n L^2\,\|w_n\| + L\,\|v_n\|, \\ \|g_n - \xi_n\| &\le \beta_n\,\|S\eta_n - g_n\| + \|v_n\| \\ &\le \beta_n\,(1 + L\,(1 + \gamma_n L))\,\|g_n - x^*\| + \beta_n L\,\|w_n\| + \|v_n\|, \end{aligned}$$

and

$$\begin{aligned} &\|p_n - \xi_n\| = \|(1 - \alpha_n)\, g_n + \alpha_n S\xi_n + u_n - \xi_n\| \\ &\le \|g_n - \xi_n\| + \alpha_n\,\|S\xi_n - g_n\| + \|u_n\| \\ &\le \left\{\beta_n\left[1 + L\,(1 + \gamma_n L)\right] + \alpha_n\left(1 + L + \beta_n L\left(L - 1 + \gamma_n L^2\right)\right)\right\}\|g_n - x^*\| \\ &+ L\,(1 + \alpha_n L)\,\beta_n\,\|w_n\| + (1 + L\alpha_n)\,\|v_n\| + \|u_n\|, \qquad (2.14) \end{aligned}$$

for all $n \ge 0$.

Substituting (2.14) into (2.13), and by condition (ii) and (2.9), we obtain that

$$\begin{aligned}\|p_n - x^*\| &\leq \left(1 - \frac{\mu\alpha_n}{1 + k\alpha_n}\right)\|g_n - x^*\| + B_n \\ &\leq \left(1 - \frac{\mu a}{2}\right)\|g_n - x^*\| + B_n. \qquad (2.15)\end{aligned}$$

for all $n \geq 0$, where $B_n = \frac{L^2(1+\alpha_n L)\alpha_n\beta_n\|w_n\|+L(1+L\alpha_n)\alpha_n\|v_n\|+(L\alpha_n+1)\|u_n\|}{1+k\alpha_n}$. Therefore, (II) follows immediately from (2.11),(2.12) and (2.15).

Suppose that $\lim_{n\to\infty} \varepsilon_n = 0$. Note that $\frac{\mu a}{2} \in (0,1)$, $B_n \to 0$ as $n \to \infty$. It follows from (2.12), (2.15) and Lemma 1.3 that $g_n \to x^*$ as $n \to \infty$. Suppose that $\lim_{n\to\infty} g_n = x^*$. Using (2.2) and (2.15), we immediately conclude that

$$\begin{aligned}\varepsilon_n &= \|g_{n+1} - (1-\alpha_n)\, g_n - \alpha_n S\xi_n - u_n\| \leq \|g_{n+1} - x^*\| + \|p_n - x^*\| \\ &\leq \|g_{n+1} - x^*\| + (1 - \frac{1}{2}\mu a)\|g_n - x^*\| + B_n \to 0\end{aligned}$$

as $n \to \infty$. That is, $\lim_{n\to\infty} \varepsilon_n = 0$. Hence, (III) holds. This completes the proof.

Remark 2.2. Theorem 2.1 improves and extends Theorem 2.1 of [5] in the following aspects:
(1) The mapping $N(T(\cdot), A(\cdot)) : X \to X$ is replaced by $N(T(\cdot), A(\cdot), B(\cdot)) : X \times X \times X \to X$.
(2) $\varphi : X^* \to R \cup \{+\infty\}$ is replaced by $\varphi : X^* \times X \to R \cup \{+\infty\}$, where for each fixed $y \in X, \varphi(\cdot, y)$ is a proper convex lower semicontinuous function with η-subdifferential $\partial_\eta\varphi$.
(3) The condition $L(L+1)(\alpha_n+\beta_n)+L(L^2-L)\alpha_n\beta_n \leq 1-t-r$ of Theorem 2.1 in [5] is replaced by the more general

$$\beta_n L\left(1 + L\left(1 + \gamma_n L\right)\right) + \alpha_n L\left(1 + L + \beta_n L\left(L - 1 + \gamma_n L^2\right)\right) \leq 1 + k - \mu,$$

where $n \geq 0, t = \max\{0, -k\} \in (0,1), r \in (0,\ 1-t), L \geq 1, \mu \in (0,\ 1+k)$ and $-1 < k < 1$.
(4) The $u_n = u'_n + u''_n$, $\|u'_n\| = o(\alpha_n)$ and $\sum_{n=0}^{\infty} \|u''_n\| < \infty$ are replaced by the $\|u_n\| \to 0\, (n \to \infty)$.
(5) General convergence rate estimates are given in our results.
(6) Extend Ishikawa iterative process with errors to the more general Noor iterative process with errors.
(7) It is proved that the Noor iterative process with errors is S-stable in Theorem 2.1.

Example 2.3. Let $X, T, A, N(\cdot,\cdot,\cdot), g, \eta, \varphi$ be as in Theorem 2.1 and $N(x,y,z) = N(x,y), \varphi(x_1,y) = \varphi(x_1), \forall x,y,z \in X, \forall x_1 \in X^*$, $0 < k < 1$, $L \geq 2$, $\alpha_n = \dfrac{1}{L(1+L)}$, $\beta_n = \gamma_n = 0, u_n = v_n = \dfrac{1}{n+1}, w_n = 0$ for all $n \geq 0$. Then the conditions of the Theorem 2.1 are satisfied with $\mu = k$ and $0 < \mu < 1+k$. But the Theorem 2.1 in [5] cannot be applied, since $\alpha_n L(1+L) = \dfrac{1}{L(1+L)} \cdot L(1+L) = 1 > 1-r$ for all $r \in (0,1)$.

Remark 2.4. Theorem 2.1 improves and extends Theorem 2.1 of [6] in the following aspects:
(1) The mapping $N(T(\cdot),A(\cdot)) : X \to X$ is replaced by $N(T(\cdot),A(\cdot),B(\cdot)) : X \times X \times X \to X$.
(2) $\varphi : X^* \to R \cup \{+\infty\}$ is replaced by $\varphi : X^* \times X \to R \cup \{+\infty\}$, where for each fixed $y \in X, \varphi(\cdot,y)$ is a proper convex lower semicontinuous function with η-subdifferential $\partial_\eta \varphi$.
(3) It abolishes the restriction that $\{\partial\varphi(g(x_n))\}$ and $\{x_n\}$ are bounded.
(4) Sequences $\{\alpha_n\}$ and $\{\beta_n\}$ need not converge to zero.
(5) It abolishes the restriction that the mapping $x \to N(x,y)$ is μ-Lipschitz continuous with respect to T and the mapping $y \to N(x,y)$ is ξ-Lipschitz continuous with respect to A.
(6) The $\|u_n'\| \to 0$ as $n \to \infty$ and $\sum\limits_{n=0}^{\infty} \|u_n''\| < \infty$ is replaced by $\|u_n\| \to 0$ as $n \to \infty$.
(7) General convergence rate estimates are given in our results.
(8) Extend Ishikawa iterative process with errors to the more general Noor iterative process with errors.
(9) It is proved that the iterative process with mixed errors is S-stable in Theorem 2.1.

Example 2.5. Let $N, X, T, A, B, \eta, \varphi, L$ be as in Example 2.3 and $\alpha_n = \beta_n = \dfrac{1+k}{4L(L+1)}$, $\gamma_n = 0, v_n = u_n = \dfrac{1}{n+1}, w_n = 0$ for all $n \geq 0$ and $-1 < k < 1$. Then the conditions of the Theorem 2.1 are satisfied with $\mu = \frac{7(1+k)}{16}$. But the Theorem 2.1 in [6] cannot be applied, since $\{\alpha_n\}$ and $\{\beta_n\}$ do not converge to 0.

Remark 2.6. Theorem 2.1 also extends and improves the corresponding results of Chang[1, 2], Ding[3], Gu[7], Hassouni and Moudafi [8], Kazmi [9], Noor [11, 12], Siddiqi and Ansari [13], Siddiqi, Ansari and Kazmi [14] and Zeng [16].

Remark 2.7. If $\varphi = 0$ in Theorem 2.1, then we obtain the corresponding result of the variational inequality.

Remark 2.8. If $k = 0$ and $0 < k < 1$ in Theorem 2.1, then we get the corresponding results of the accretive and k-strongly accretive, respectively.

Remark 2.9. In Theorem 2.1, if $\gamma_n = 0, w_n = 0, n \geq 0$, then $z_n = x_n, \eta_n = g_n, n \geq 0$, this implies the corresponding results of Ishikawa iterative sequences, we omit it here.

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Shuyi Zhang,
Department of Mathematics,
BoHai University,
Jinzhou, Liaoning, 121013, China.
Email: E-mail: jzzhangshuyi@126. com

Xinqi Guo,
Dalian City No.37 Middle School,
Dalian, Liaoning, 116011, China.
Email: libby27@163. com

Jun Wang,
Department of Mathematics,
BoHai University,
Jinzhou, Liaoning, 121013, China.
Email: E-mail: wangjun0134@126. com