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Intuitionistic Fuzzy Normal subrings over a non-associative ring

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Abstract

N. Palaniappan et. al [20, 28] have investigated the concept of intuitionistic fuzzy normal subrings in associative rings. In this study we extend these notions for a class of non-associative rings.

1 Introduction

In 1972, left almost semigroups (LA-semigroups) have been introduced by Kazim and Naseerudin [14]. A groupoid S is called an LA-semigroup if it satisfies the left invertive law: $(ab)c = (cb)a$ for all $a, b, c \in S$. This structure is also known as Abel-Grassmann's groupoid (abbreviated as AG-groupoid) [21, 22]. Holgate [11], has called the same structure as left invertive groupoid. An AG-groupoid is the midway structure between a commutative semigroup and a groupoid. Actually an LA-semigroup is non-commutative and non-associative structure. In [13], a groupoid S is called medial if $(ab)(cd) = (ac)(bd)$ for all $a, b, c, d \in S$, and S is paramedial if $(ab)(cd) = (db)(ca)$ (see [5, line 37]). Naturally every AG-groupoid satisfies medial law. In general an AG-groupoid needs not to be a paramedial. However, by [21] every AG-groupoid with left identity is paramedial. Ideals in AG-groupoids have been discussed in [21, 22].

In [15], Kamran extended the notion of LA-semigroup to left almost group (LA-group). A groupoid G is called a left almost group (LA-group), if there

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exists left identity $e \in G$ (that is $ea = a$ for all $a \in G$), for $a \in G$ there exists $b \in G$ such that $ba = e$ and left invertive law holds in G.

Shah and Rehman [25], have discussed left almost ring (LA-ring) of finitely nonzero functions which is in fact a generalization of a commutative semigroup ring. By a left almost ring, we mean a non-empty set R with at least two elements such that $(R, +)$ is an LA-group, $(R, \cdot)$ is an LA-semigroup, both left and right distributive laws hold. For example, from a commutative ring $(R, +, \cdot)$, we can always obtain an LA-ring $(R, \oplus, \cdot)$ by defining for all $a, b \in R$, $a \oplus b = b - a$ and $a \cdot b$ is same as in the ring. An LA-ring is in fact a class of non-associative and non-commutative rings. Recently Shah and Rehman [26], investigated some properties of LA-rings through their ideals and intuitively ideal thoery would be a gate way for fuzzy concepts in LA-rings.

After the introduction of fuzzy set by Zadeh [31], several researchers explored on the generalization of the notion of fuzzy set. The concept of intuitionistic fuzzy set was introduced by Atanassov [1, 2], as a generalization of the notion of fuzzy set.

Fuzzy rings and fuzzy ideals have been discussed in [9]. In [8], Dib and Youssef have examined the properties of fuzzy cartesian product, fuzzy relations and fuzzy functions. Volf, has investigated the properties of fuzzy subfield in [29].

Intuitionistic fuzzy subrings and intuitionistic fuzzy ideals of a ring have been defined in [3, 4]. Palaniappan et. al [20] explored the notion of homomorphism, antihomomorphism of intuitionistic fuzzy normal subrings. Moreover intuitionistic fuzzy ring and its homomorphism image have been investigated by Yan [30]. Recently, some properties of intuitionistic fuzzy normal subrings have been discussed in [28].

In this study we followed lines as adopted in [20, 28] and established the notion of intuitionistic fuzzy normal LA-subrings of LA-rings. Specifically we show that: (1) An IFS $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy normal LA-subring of an LA-ring R if and only if the fuzzy sets μ_A and $\overline{\gamma_A}$ are fuzzy normal LA-subrings of R. (2) An IFS $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy normal LA-subring of an LA-ring R if and only if the fuzzy sets $\overline{\mu_A}$ and γ_A are anti-fuzzy normal LA-subrings of R.

2 Preliminaries

Let R be an LA-ring. By an LA-subring of R we mean a non-empty subset A of R such that $a - b$ and $ab \in A$, for all $a, b \in A$, and by a left (right) ideal of R we mean an LA-subring A of R such that $RA \subseteq A$ ($AR \subseteq A$), respectively. By two-sided ideal or simply ideal, we mean a non-empty subset A of R which is both a left and a right ideal of R.

In the following we are giving an example of a finite LA-ring from [27].
Example 2.1. [27, Example 1] An LA-ring of order 5 :

+	0	1	2	3	4
0	0	1	2	3	4
1	4	0	1	2	3
2	3	4	0	1	2
3	2	3	4	0	1
4	1	2	3	4	0

and

·	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

As in [31], by a fuzzy set μ in a non-empty set X we mean a function $\mu : X \to [0,1]$ and we denote the complement of μ by $\overline{\mu}(x) = 1 - \mu(x)$ for all $x \in X$.

An intuitionistic fuzzy set (briefly, IFS) A in a non-empty set X is an object having the form $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$, where the functions $\mu_A\colon X \to [0,1]$ and $\gamma_A\colon X \to [0,1]$ denote the degree of membership and the degree of nonmembership respectively, and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for all $x \in X$ (see [1, 2]).

An intuitionistic fuzzy set $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$ in X can be identified to an ordered pair (μ_A, γ_A) in $I^X \times I^X$, where I^X is the set of all functions from X to $[0,1]$. For the sake of simplicity, we shall use the symbol $A = (\mu_A, \gamma_A)$ for the IFS $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$.

3 Intuitionistic fuzzy normal LA-subrings

A fuzzy subset μ of an LA-ring R is called a fuzzy LA-subring of R if $\mu(x-y) \geq min\{\mu_A(x), \mu_A(y)\}$ and $\mu(xy) \geq min\{\mu_A(x), \mu_A(y)\}$ for all $x, y \in R$ [23]. A fuzzy subset μ of an LA-ring R is called an anti fuzzy LA-subring of R if $\mu(x-y) \leq max\{\mu_A(x), \mu_A(y)\}$ and $\mu(xy) \leq max\{\mu_A(x), \mu_A(y)\}$ for all $x, y \in R$. A fuzzy LA-subring of an LA-ring R is called a fuzzy normal LA-subring of R if $\mu(xy) = \mu(yx)$ for all $x, y \in R$. Similarly for anti-fuzzy normal LA-subring.

Definition 3.1. [24] An IFS $A = (\mu_A, \gamma_A)$ in R is called an intuitionistic fuzzy LA-subring IFLS(IFLSR) of R if

(a) $\mu_A(x-y) \geq min\{\mu_A(x), \mu_A(y)\}$,
(b) $\gamma_A(x-y) \leq max\{\gamma_A(x), \gamma_A(y)\}$,
(c) $\mu_A(xy) \geq min\{\mu_A(x), \mu_A(y)\}$,
(d) $\gamma_A(xy) \leq max\{\gamma_A(x), \gamma_A(y)\}$,
for all $x, y \in R$.

Intuitionistic fuzzy LA-subring is an extension of fuzzy LA-subring.

Let R be an LA-ring. An intuitionistic fuzzy LA-subring $A = (\mu_A, \gamma_A)$ of R is said to be an intuitionistic fuzzy normal LA-subring (IFNLSR) of R if $\mu_A(xy) = \mu_A(yx)$ and $\gamma_A(xy) = \gamma_A(yx)$ for all $x, y \in R$.

Example 3.2. Let $R = \{0, 1, 2, 3\}$. Define $+$ and $\cdot$ in R as follows :

$+$	0	1	2	3
0	0	1	2	3
1	3	0	1	2
2	2	3	0	1
3	1	2	3	0

and

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

Then R is an LA-ring. Let an IFS $A = (\mu_A, \gamma_A)$ of R. We define $\mu_A(0) = \mu_A(2) = 0.7$, $\mu_A(1) = \mu_A(3) = 0$ and $\gamma_A(0) = \gamma_A(2) = 0$, $\gamma_A(1) = \gamma_A(3) = 0.7$. Then $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy normal LA-subring of R.

Remark 3.3. $S = \{0, 2\}$ is an LA-subring of R.

Definition 3.4. [24] Let R be an LA-ring and A be a non-empty subset of R. The intuitionistic characteristic function of A is denoted by $\chi_A = \langle \mu_{\chi_A}, \gamma_{\chi_A} \rangle$ and is defined by

$$\mu_{\chi_A} : R \to [0,1] \mid x \to \mu_{\chi_A}(x) : = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$$

and

$$\gamma_{\chi_A} : R \to [0,1] \mid x \to \gamma_{\chi_A}(x) : = \begin{cases} 0 \text{ if } x \in A \\ 1 \text{ if } x \notin A \end{cases}$$

Lemma 3.5. If A is a subset of an LA-ring R, then A is an LA-subring of R if and only if the intuitionistic characteristic function $\chi_A = \langle \mu_{\chi_A}, \gamma_{\chi_A} \rangle$ of A is an intuitionistic fuzzy normal LA-subring of R.

Proof. Let A be an LA-subring of R and $a, b \in R$. If $a, b \in A$, then by definition of characteristic function $\mu_{\chi_A}(a) = 1 = \mu_{\chi_A}(b)$ and $\gamma_{\chi_A}(a) = 0 = \gamma_{\chi_A}(b)$. A being an LA-subring, $a-b$ and $ab \in A$. It follows that $\mu_{\chi_A}(a-b) = 1 = 1 \wedge 1 = \mu_{\chi_A}(a) \wedge \mu_{\chi_A}(b)$ and $\mu_{\chi_A}(ab) = 1 = 1 \wedge 1 = \mu_{\chi_A}(a) \wedge \mu_{\chi_A}(b)$. This imply that $\mu_{\chi_A}(a-b) \geq min\{\mu_{\chi_A}(a), \mu_{\chi_A}(b)\}$ and $\mu_{\chi_A}(ab) \geq min\{\mu_{\chi_A}(a), \mu_{\chi_A}(b)\}$. Now $\gamma_{\chi_A}(a-b) = 0 = 0 \vee 0 = \gamma_{\chi_A}(a) \vee \gamma_{\chi_A}(b)$ and $\gamma_{\chi_A}(ab) = 0 = 0 \vee 0 = \gamma_{\chi_A}(a) \vee \gamma_{\chi_A}(b)$. This imply that $\gamma_{\chi_A}(a-b) \leq max\{\gamma_{\chi_A}(a), \gamma_{\chi_A}(b)\}$ and $\gamma_{\chi_A}(ab) \leq max\{\gamma_{\chi_A}(a), \gamma_{\chi_A}(b)\}$. Since ab and $ba \in A$, it follows that $\mu_{\chi_A}(ab) = 1 = \mu_{\chi_A}(ba)$ and $\gamma_{\chi_A}(ab) = 0 = \gamma_{\chi_A}(ba)$. Consequently $\mu_{\chi_A}(ab) = \mu_{\chi_A}(ba)$ and $\gamma_{\chi_A}(ab) = \gamma_{\chi_A}(ba)$. Similarly we can prove that

$$\begin{aligned} \mu_{\chi_A}(a-b) &\geq min\{\mu_{\chi_A}(a), \mu_{\chi_A}(b)\},\ \mu_{\chi_A}(ab) \geq min\{\mu_{\chi_A}(a), \mu_{\chi_A}(b)\}, \\ \gamma_{\chi_A}(a-b) &\leq max\{\gamma_{\chi_A}(a), \gamma_{\chi_A}(b)\},\ \gamma_{\chi_A}(ab) \leq max\{\gamma_{\chi_A}(a), \gamma_{\chi_A}(b)\}, \\ \mu_{\chi_A}(ab) &= \mu_{\chi_A}(ba) \text{ and } \gamma_{\chi_A}(ab) = \gamma_{\chi_A}(ba), \end{aligned}$$

when $a, b \notin A$. Thus the intuitionistic characteristic function $\chi_A = \langle \mu_{\chi_A}, \gamma_{\chi_A} \rangle$ of A is an intuitionistic fuzzy normal LA-subring of R.

Conversely, assume that the intuitionistic characteristic function $\chi_A = \langle \mu_{\chi_A}, \gamma_{\chi_A} \rangle$ of A is an intuitionistic fuzzy normal LA-subring of R. Let $a, b \in A$. By definition $\mu_{\chi_A}(a) = 1 = \mu_{\chi_A}(b)$ and $\gamma_{\chi_A}(a) = 0 = \gamma_{\chi_A}(b)$. By hypothesis

$$\begin{aligned} \mu_{\chi_A}(a-b) &\geq \mu_{\chi_A}(a) \wedge \mu_{\chi_A}(b) = 1 \wedge 1 = 1, \\ \mu_{\chi_A}(ab) &\geq \mu_{\chi_A}(a) \wedge \mu_{\chi_A}(b) = 1 \wedge 1 = 1, \\ \gamma_{\chi_A}(a-b) &\leq \gamma_{\chi_A}(a) \vee \gamma_{\chi_A}(b) = 0 \vee 0 = 0, \\ \gamma_{\chi_A}(ab) &\leq \gamma_{\chi_A}(a) \vee \gamma_{\chi_A}(b) = 0 \vee 0 = 0, \end{aligned}$$

This imply that $\mu_{\chi_A}(a-b) = 1, \mu_{\chi_A}(ab) = 1$ and $\gamma_{\chi_A}(a-b) = 0, \gamma_{\chi_A}(ab) = 0$. Thus $a-b$ and $ab \in A$. Hence A is an LA-subring of R. □

If A and B are two LA-subrings of an LA-ring R, then their intersection $A \cap B$ is also an LA-subring of R.

Lemma 3.6. If A and B are two LA-subrings of an LA-ring R, then their intersection $A \cap B$ is an LA-subring of R if and only if the intuitionistic characteristic function $\chi_C = \langle \mu_{\chi_c}, \gamma_{\chi_c} \rangle$ of $C = A \cap B$ is an intuitionistic fuzzy normal LA-subring of R.

Proof. Let $C = A \cap B$ be an LA-subring of R and $a, b \in R$. If $a, b \in C = A \cap B$, then by definition of characteristic function $\mu_{\chi_C}(a) = 1 = \mu_{\chi_C}(b)$ and $\gamma_{\chi_C}(a) = 0 = \gamma_{\chi_C}(b)$. Since $a-b, ab \in A$ and B, it follows that $a-b$ and $ab \in C$. Thus $\mu_{\chi_C}(a-b) = 1 = 1 \wedge 1 = \mu_{\chi_C}(a) \wedge \mu_{\chi_C}(b)$ and $\mu_{\chi_C}(ab) = 1 = 1 \wedge 1 = \mu_{\chi_C}(a) \wedge \mu_{\chi_C}(b)$. Thus $\mu_{\chi_C}(a-b) \geq min\{\mu_{\chi_C}(a), \mu_{\chi_C}(b)\}$ and $\mu_{\chi_C}(ab) \geq min\{\mu_{\chi_C}(a), \mu_{\chi_C}(b)\}$. Now $\gamma_{\chi_C}(a-b) = 0 = 0 \vee 0 = \gamma_{\chi_C}(a) \vee \gamma_{\chi_C}(b)$ and $\gamma_{\chi_C}(ab) = 0 = 0 \vee 0 = \gamma_{\chi_C}(a) \vee \gamma_{\chi_C}(b)$. Thus $\gamma_{\chi_C}(a-b) \leq max\{\gamma_{\chi_C}(a), \gamma_{\chi_C}(b)\}$ and $\gamma_{\chi_C}(ab) \leq max\{\gamma_{\chi_C}(a), \gamma_{\chi_C}(b)\}$. As ab and $ba \in C$, so $\mu_{\chi_C}(ab) = 1 = \mu_{\chi_C}(ba)$ and $\gamma_{\chi_C}(ab) = 0 = \gamma_{\chi_C}(ba)$. Accordingly $\mu_{\chi_C}(ab) = \mu_{\chi_C}(ba)$ and $\gamma_{\chi_C}(ab) = \gamma_{\chi_C}(ba)$. Similarly we have

$$\begin{aligned} \mu_{\chi_C}(a-b) &\geq min\{\mu_{\chi_C}(a), \mu_{\chi_C}(b)\},\ \mu_{\chi_C}(ab) \geq min\{\mu_{\chi_C}(a), \mu_{\chi_C}(b)\}, \\ \gamma_{\chi_C}(a-b) &\leq max\{\gamma_{\chi_C}(a), \gamma_{\chi_C}(b)\},\ \gamma_{\chi_C}(ab) \leq max\{\gamma_{\chi_C}(a), \gamma_{\chi_C}(b)\}, \\ \gamma_{\chi_C}(ab) &= \gamma_{\chi_C}(ba) \text{ and } \gamma_{\chi_C}(ab) = \gamma_{\chi_C}(ba), \end{aligned}$$

when $a, b \notin C$. Hence the intuitionistic characteristic function $\chi_C = \langle \mu_{\chi_C}, \gamma_{\chi_C} \rangle$ of C is an intuitionistic fuzzy normal LA-subring of R.

Conversely, assume that the intuitionistic characteristic function $\chi_C = \langle \mu_{\chi_C}, \gamma_{\chi_C} \rangle$ of $C = A \cap B$ is an intuitionistic fuzzy normal LA-subring of

R. Let $a, b \in C = A \cap B$. This imply that $\mu_{\chi_C}(a) = 1 = \mu_{\chi_C}(b)$ and $\gamma_{\chi_C}(a) = 0 = \gamma_{\chi_C}(b)$. By our supposition

$$\begin{aligned}
\mu_{\chi_C}(a-b) &\geq \mu_{\chi_C}(a) \wedge \mu_{\chi_C}(b) = 1 \wedge 1 = 1, \\
\mu_{\chi_C}(ab) &\geq \mu_{\chi_C}(a) \wedge \mu_{\chi_C}(b) = 1 \wedge 1 = 1, \\
\gamma_{\chi_C}(a-b) &\leq \gamma_{\chi_C}(a) \vee \gamma_{\chi_C}(b) = 0 \vee 0 = 0, \\
\gamma_{\chi_C}(ab) &\leq \gamma_{\chi_C}(a) \vee \gamma_{\chi_C}(b) = 0 \vee 0 = 0,
\end{aligned}$$

This imply that $\mu_{\chi_C}(a-b) = 1, \mu_{\chi_C}(ab) = 1$ and $\gamma_{\chi_C}(a-b) = 0, \gamma_{\chi_C}(ab) = 0$. Thus $a - b$ and $ab \in C$. Hence C is an LA-subring of R. □

Corollary 3.7. If $\{A_i\}_{i \in I}$ is a family of LA-subrings of R, then $C = \cap A_i$ is an LA-subring of R, where $\cap A_i = (\wedge \mu_{A_i}, \vee \gamma_{A_i})$ and

$$\begin{aligned}
\wedge \mu_{A_i}(x) &= inf \left\{ \mu_{A_i}(x) : i \in I,\ x \in R \right\}, \\
\vee \gamma_{A_i}(x) &= sup \left\{ \gamma_{A_i}(x) : i \in I,\ x \in R \right\},
\end{aligned}$$

if and only if the intuitionistic characteristic function $\chi_C = \langle \mu_{\chi_c}, \gamma_{\chi_c} \rangle$ of $C = \cap A_i$ is an intuitionistic fuzzy normal LA-subring of R.

Theorem 3.8. If A and B are two intuitionistic fuzzy normal LA-subrings of an LA-ring R, then their intersection $A \cap B$ is an intuitionistic fuzzy normal LA-subring of R.

Proof. Let $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in R\}$ and $B = \{(x, \mu_B(x), \gamma_B(x)) : x \in R\}$ be intuitionistic fuzzy normal LA-subrings of an LA-ring R. Let $C = A \cap B$ and $C = \{(x, \mu_C(x), \gamma_C(x)) \mid x \in R\}$, where $\mu_C(x) = min\{\mu_A(x), \mu_B(x)\}$ and

$\gamma_C(x) = max\{\gamma_A(x), \gamma_B(x)\}$. Now

$$\begin{aligned}
\mu_C(x-y) &= min\{\mu_A(x-y), \mu_B(x-y)\} \\
&= \mu_A(x-y) \wedge \mu_B(x-y) \\
&\geq \{\mu_A(x) \wedge \mu_A(y)\} \wedge \{\mu_B(x) \wedge \mu_B(y)\} \\
&= \mu_A(x) \wedge \{\mu_A(y) \wedge \mu_B(x)\} \wedge \mu_B(y) \\
&= \mu_A(x) \wedge \{\mu_B(x) \wedge \mu_A(y)\} \wedge \mu_B(y) \\
&= \{\mu_A(x) \wedge \mu_B(x)\} \wedge \{\mu_A(y) \wedge \mu_B(y)\} \\
&= \mu_C(x) \wedge \mu_C(y). \\
\mu_C(xy) &= min\{\mu_A(xy), \mu_B(xy)\} \\
&= \mu_A(xy) \wedge \mu_B(xy) \\
&\geq \{\mu_A(x) \wedge \mu_A(y)\} \wedge \{\mu_B(x) \wedge \mu_B(y)\} \\
&= \mu_A(x) \wedge \{\mu_A(y) \wedge \mu_B(x)\} \wedge \mu_B(y) \\
&= \mu_A(x) \wedge \{\mu_B(x) \wedge \mu_A(y)\} \wedge \mu_B(y) \\
&= \{\mu_A(x) \wedge \mu_B(x)\} \wedge \{\mu_A(y) \wedge \mu_B(y)\} \\
&= \mu_C(x) \wedge \mu_C(y).
\end{aligned}$$

Similarly $\gamma_C(x-y) \leq \gamma_C(x) \vee \gamma_C(y)$ and $\gamma_C(xy) \leq \gamma_C(x) \vee \gamma_C(y)$. Thus C is an intuitionistic fuzzy LA-subring of an LA-ring R. Now $\mu_C(xy) = min\{\mu_A(xy), \mu_B(xy)\} = min\{\mu_A(yx), \mu_B(yx)\} = \mu_C(yx)$. Similarly $\gamma_C(xy) = \gamma_C(yx)$. Hence $A \cap B$ is an intuitionistic fuzzy normal LA-subring of R. □

Proposition 3.9. If A is an intuitionistic fuzzy normal LA-subring of an LA-ring R, then $\Box A\colon = (\mu_A, \overline{\mu_A})$ is an intuitionistic fuzzy normal LA-subring of an LA-ring R.

Proof. We have to show that $\Box A\colon = (\mu_A, \overline{\mu_A})$ is an intuitionistic fuzzy normal LA-subring of R.

$$\begin{aligned}
\overline{\mu_A}(x-y) &= 1 - \mu_A(x-y) \leq 1 - min\,\{\mu_A(x), \mu_A(y)\} \\
&= max\,\{1-\mu_A(x), 1-\mu_A(y)\} = max\{\overline{\mu_A}(x), \overline{\mu_A}(y)\} \\
&\Rightarrow \overline{\mu_A}(x-y) \leq max\{\overline{\mu_A}(x), \overline{\mu_A}(y)\}. \\
\overline{\mu_A}(xy) &= 1 - \mu_A(xy) \leq 1 - min\,\{\mu_A(x), \mu_A(y)\} \\
&= max\,\{1-\mu_A(x), 1-\mu_A(y)\} = max\{\overline{\mu_A}(x), \overline{\mu_A}(y)\} \\
&\Rightarrow \overline{\mu_A}(xy) \leq max\{\overline{\mu_A}(x), \overline{\mu_A}(y)\}. \\
\overline{\mu_A}(xy) &= 1 - \mu_A(xy) = 1 - \mu_A(yx) = \overline{\mu_A}(yx).
\end{aligned}$$

□

Proposition 3.10. If A is an intuitionistic fuzzy normal LA-subring of an LA-ring R, then $\Diamond A = (\overline{\gamma_A}, \gamma_A)$ is an intuitionistic fuzzy normal LA-subring of an LA-ring R.

Proof. We have to show that $\Diamond A = (\overline{\gamma_A}, \gamma_A)$ is an intuitionistic fuzzy normal LA-subring of R.

$$\begin{aligned}
\overline{\gamma_A}(x-y) &= 1-\gamma_A(x-y) \geq 1 - max\,\{\gamma_A(x), \gamma_A(y)\} \\
&= min\,\{1-\gamma_A(x), 1-\gamma_A(y)\} = min\{\overline{\gamma_A}(x), \overline{\gamma_A}(y)\} \\
&\Rightarrow \overline{\gamma_A}(x-y) \geq min\{\overline{\gamma_A}(x), \overline{\gamma_A}(y)\}. \\
\overline{\gamma_A}(xy) &= 1-\gamma_A(xy) \geq 1 - max\,\{\gamma_A(x), \gamma_A(y)\} \\
&= min\,\{1-\gamma_A(x), 1-\gamma_A(y)\} = min\{\overline{\gamma_A}(x), \overline{\gamma_A}(y)\} \\
&\Rightarrow \overline{\gamma_A}(xy) \geq min\{\overline{\gamma_A}(x), \overline{\gamma_A}(y)\}. \\
\overline{\gamma_A}(xy) &= 1-\gamma_A(xy) = 1-\gamma_A(yx) = \overline{\gamma_A}(yx).
\end{aligned}$$

□

Theorem 3.11. An IFS $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy normal LA-subring of an LA-ring R if and only if the fuzzy subsets μ_A and $\overline{\gamma_A}$ are fuzzy normal LA-subrings of R.

Proof. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy normal LA-subring of R. Then clearly μ_A is fuzzy normal LA-subring of R. Now

$$\begin{aligned}
\overline{\gamma_A}(x-y) &= 1-\gamma_A(x-y) \\
&\geq 1 - max\{\gamma_A(x), \gamma_A(y)\} \\
&= min\{1-\gamma_A(x), 1-\gamma_A(y)\} \\
&= min\{\overline{\gamma_A}(x), \overline{\gamma_A}(y)\}. \\
\overline{\gamma_A}(xy) &= 1-\gamma_A(xy) \\
&\geq 1 - max\{\gamma_A(x), \gamma_A(y)\} \\
&= min\{1-\gamma_A(x), 1-\gamma_A(y)\} \\
&= min\{\overline{\gamma_A}(x), \overline{\gamma_A}(y)\}. \\
\overline{\gamma_A}(xy) &= 1-\gamma_A(xy) = 1-\gamma_A(yx) = \overline{\gamma_A}(yx).
\end{aligned}$$

Thus $\overline{\gamma_A}$ is a fuzzy normal LA-subring of R.

Conversely, μ_A and $\overline{\gamma_A}$ are fuzzy normal LA-subrings of R.

$$\begin{aligned}
1-\gamma_A(x-y) &= \overline{\gamma_A}(x-y) \geq min\{\overline{\gamma_A}(x), \overline{\gamma_A}(y)\} \\
&= min\{(1-\gamma_A(x)), (1-\gamma_A(y))\} \\
&= 1-max\{\gamma_A(x), \gamma_A(y)\} \\
&\Rightarrow \gamma_A(x-y) \leq max\{\gamma_A(x), \gamma_A(y)\}. \\
1-\gamma_A(xy) &= \overline{\gamma_A}(xy) \geq min\{\overline{\gamma_A}(x), \overline{\gamma_A}(y)\} \\
&= min\{(1-\gamma_A(x)), (1-\gamma_A(y))\} \\
&= 1-max\{\gamma_A(x), \gamma_A(y)\} \\
&\Rightarrow \gamma_A(xy) \leq max\{\gamma_A(x), \gamma_A(y)\}. \\
1-\gamma_A(xy) &= \overline{\gamma_A}(xy) = \overline{\gamma_A}(yx) = 1-\gamma_A(yx) \\
&\Rightarrow \gamma_A(xy) = \gamma_A(yx).
\end{aligned}$$

Thus $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy normal LA-subring of an LA-ring R. □

Theorem 3.12. An IFS $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy normal LA-subring of an LA-ring R if and only if the fuzzy subsets $\overline{\mu_A}$ and γ_A are anti-fuzzy normal LA-subrings of R.

Proof. Suppose $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy normal LA-subring of an LA-ring R. Clear γ_A is an anti fuzzy normal LA-subring of R. Now we have to show that $\overline{\mu_A}$ is also an anti fuzzy normal LA-subring of R.

$$\begin{aligned}
\overline{\mu_A}(x-y) &= 1-\mu_A(x-y) \leq 1-min\{\mu_A(x), \mu_A(y)\} \\
&= max\{1-\mu_A(x), 1-\mu_A(y)\} \\
&= max\{\overline{\mu_A}(x), \overline{\mu_A}(y)\} \\
&\Rightarrow \overline{\mu_A}(x-y) \leq max\{\overline{\mu_A}(x), \overline{\mu_A}(y)\}. \\
\overline{\mu_A}(xy) &= 1-\mu_A(xy) \leq 1-min\{\mu_A(x), \mu_A(y)\} \\
&= max\{1-\mu_A(x), 1-\mu_A(y)\} \\
&= max\{\overline{\mu_A}(x), \overline{\mu_A}(y)\} \\
&\Rightarrow \overline{\mu_A}(xy) \leq max\{\overline{\mu_A}(x), \overline{\mu_A}(y)\}. \\
\overline{\mu_A}(xy) &= 1-\mu_A(xy) = 1-\mu_A(yx) = \overline{\mu_A}(yx).
\end{aligned}$$

Hence $\overline{\mu_A}$ and γ_A are anti fuzzy normal LA-subrings of R.

Conversely, suppose that $\overline{\mu_A}$ and γ_A are anti fuzzy normal LA-subrings of R. Now we have to show that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy normal

LA-subring of an LA-ring R.

$$\begin{aligned}
1-\mu_A(x-y) &= \overline{\mu_A}(x-y) \leq max\{\overline{\mu_A}(x), \overline{\mu_A}(y)\} \\
&= max\{1-\mu_A(x), 1-\mu_A(y)\} \\
&= 1-min\{\mu_A(x), \mu_A(y)\} \\
&\Rightarrow \mu_A(x-y) \geq min\{\mu_A(x), \mu_A(y)\}. \\
1-\mu_A(xy) &= \overline{\mu_A}(xy) \leq max\{\overline{\mu_A}(x), \overline{\mu_A}(y)\} \\
&= max\{1-\mu_A(x), 1-\mu_A(y)\} \\
&= 1-min\{\mu_A(x), \mu_A(y)\} \\
&\Rightarrow \mu_A(xy) \geq min\{\mu_A(x), \mu_A(y)\}. \\
1-\mu_A(xy) &= \overline{\mu_A}(xy) = \overline{\mu_A}(yx) = 1-\mu_A(yx) \\
&\Rightarrow \mu_A(xy) = \mu_A(yx).
\end{aligned}$$

Thus $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy normal LA-subring of an LA-ring R. □

4 Direct Product of LA-rings

In this section we discuss the direct product of LA-rings. Specifically, we show that if A and B are two LA-subrings of LA-rings R_1 and R_2 respectively, then $A \times B$ is an LA-subring of $R_1 \times R_2$ if and only if the intuitionistic characteristic function $\chi_C = \langle \mu_{\chi_C}, \gamma_{\chi_C} \rangle$ of $C = A \times B$ is an intuitionistic fuzzy normal LA-subring of $R_1 \times R_2$.

If R_1, R_2 are LA-rings, then direct product $R_1 \times R_2$ of R_1 and R_2 is an LA-ring with pointwise addition '+' and multiplication '$\circ$' defined as $(a,b) + (c,d) = (a+c, b+d)$ and $(a,b) \circ (c,d) = (ac, bd)$, respectively for every $(a,b), (c,d)$ in $R_1 \times R_2$. Likewise the direct product $R = \times_{i \in \Omega} R_i$ of a family of LA-rings $\{R_i : i \in \Omega\}$ has the structure of an LA-ring with the operations of addition and multiplication defined as

$$\begin{aligned}
a+b &= (a_1, a_2, a_3, ...) + (b_1, b_2, b_3, ...) \\
&= (a_1+b_1, a_2+b_2, a_3+b_3, ...) \\
\text{and } a \circ b &= (a_1, a_2, a_3, ...) \circ (b_1, b_2, b_3, ...) \\
&= (a_1 b_1, a_2 b_2, a_3 b_3, ...)
\end{aligned}$$

for all $a, b \in R$.

Example 4.1. Let $R_1 = \{0,1,2,3,4\}$ and $R_2 = \{0,1,2,3\}$. Define addition and multiplication in R_1 and R_2 as in Example 2 and Example 3, respectively. Then $R_1 \times R_2$ is an LA-ring with pointwise addition '+' and multiplication '$\circ$' defined as $(a,b)+(c,d)=(a+c,b+d)$ and $(a,b)\circ(c,d)=(ac,bd)$, respectively for every $(a,b),(c,d)$ in $R_1\times R_2$.

$A=R_1=\{0,1,2,3,4\}$ and $B=\{0,2\}$ being LA-subrings of R_1 and R_2, then $A\times B$ is an LA-subring of $R_1\times R_2$ under the same operations defined as in $R_1\times R_2$.

Lemma 4.2. If A and B are two LA-subrings of LA-rings R_1 and R_2 respectively, then $A\times B$ is also an LA-subring of $R_1\times R_2$ under the same operations defined as in $R_1\times R_2$.

Proof. Straight forward. □

Let A and B be two intuitionistic fuzzy subsets of LA-rings R_1 and R_2, respectively. The direct product of A and B, is denoted by $A\times B$, is defined as $A\times B=\{((x,y),\mu_{A\times B}((x,y)),\gamma_{A\times B}((x,y)))\mid \text{for all } x\in R_1 \text{ and } y\in R_2\}$, where $\mu_{A\times B}((x,y))=min\{\mu_A(x),\mu_B(y)\}$ and $\gamma_{A\times B}((x,y))=max\{\gamma_A(x),\gamma_B(y)\}$.

Theorem 4.3. Let A and B be two LA-subrings of LA-rings R_1 and R_2, respectively. The $A\times B$ is an LA-subring of $R_1\times R_2$ if and only if the intuitionistic characteristic function $\chi_C=\langle\mu_{\chi_C},\gamma_{\chi_C}\rangle$ of $C=A\times B$ is an intuitionistic fuzzy normal LA-subring of $R_1\times R_2$.

Proof. Let $C=A\times B$ be an LA-subring of $R_1\times R_2$ and $a,b\in R_1\times R_2$. If $a,b\in C=A\times B$, then by definition of characteristic function $\mu_{\chi_C}(a)=1=\mu_{\chi_C}(b)$ and $\gamma_{\chi_C}(a)=0=\gamma_{\chi_C}(b)$. Since $a-b$ and $ab\in C$, C being an LA-subring. It follows that $\mu_{\chi_C}(a-b)=1=1\wedge 1=\mu_{\chi_C}(a)\wedge\mu_{\chi_C}(b)$ and $\mu_{\chi_C}(ab)=1=1\wedge 1=\mu_{\chi_C}(a)\wedge\mu_{\chi_C}(b)$. Thus $\mu_{\chi_C}(a-b)\geq min\{\mu_{\chi_C}(a),\mu_{\chi_C}(b)\}$ and $\mu_{\chi_C}(ab)\geq min\{\mu_{\chi_C}(a),\mu_{\chi_C}(b)\}$. Now $\gamma_{\chi_C}(a-b)=0=0\vee 0=\gamma_{\chi_C}(a)\vee\gamma_{\chi_C}(b)$ and $\gamma_{\chi_C}(ab)=0=0\vee 0=\gamma_{\chi_C}(a)\vee\gamma_{\chi_C}(b)$. Thus $\gamma_{\chi_C}(a-b)\leq max\{\gamma_{\chi_C}(a),\gamma_{\chi_C}(b)\}$ and $\gamma_{\chi_C}(ab)\leq max\{\gamma_{\chi_C}(a),\gamma_{\chi_C}(b)\}$. As ab and $ba\in C$, so $\mu_{\chi_C}(ab)=1=\mu_{\chi_C}(ba)$ and $\gamma_{\chi_C}(ab)=0=\gamma_{\chi_C}(ba)$. This imply that $\mu_{\chi_C}(ab)=\mu_{\chi_C}(ba)$ and $\gamma_{\chi_C}(ab)=\gamma_{\chi_C}(ba)$. Similarly we have

$$\begin{aligned}
\mu_{\chi_C}(a-b) &\geq min\{\mu_{\chi_C}(a),\mu_{\chi_C}(b)\},\ \mu_{\chi_C}(ab)\geq min\{\mu_{\chi_C}(a),\mu_{\chi_C}(b)\},\\
\gamma_{\chi_C}(a-b) &\leq max\{\gamma_{\chi_C}(a),\gamma_{\chi_C}(b)\},\ \gamma_{\chi_C}(ab)\leq max\{\gamma_{\chi_C}(a),\gamma_{\chi_C}(b)\},\\
\mu_{\chi_C}(ab) &= \mu_{\chi_C}(ba) \text{ and } \gamma_{\chi_C}(ab)=\gamma_{\chi_C}(ba),
\end{aligned}$$

when $a,b\notin C$. Hence the intuitionistic characteristic function $\chi_C=\langle\mu_{\chi_C},\gamma_{\chi_C}\rangle$ of $C=A\times B$ is an intuitionistic fuzzy normal LA-subring of $R_1\times R_2$.

Conversely, assume that the intuitionistic characteristic function $\chi_C=\langle\mu_{\chi_C},\gamma_{\chi_C}\rangle$ of $C=A\times B$ is an intuitionistic fuzzy normal LA-subring of $R_1\times R_2$. Now we have to show that $C=A\times B$ is an LA-subring of R. Let

$a,\ b \in C$, where $a = \left(a^{'}, b^{'}\right)$ and $b = \left(a^{''}, b^{''}\right)$, $a', a'' \in A$, $b', b'' \in B$. By definition $\mu_{\chi_C}(a) = 1 = \mu_{\chi_C}(b)$ and $\gamma_{\chi_C}(a) = 0 = \gamma_{\chi_C}(b)$. By our supposition

$$\begin{array}{rcl}
\mu_{\chi_C}(a-b) & \geq & \mu_{\chi_C}(a) \wedge \mu_{\chi_C}(b) = 1 \wedge 1 = 1, \\
\mu_{\chi_C}(ab) & \geq & \mu_{\chi_C}(a) \wedge \mu_{\chi_C}(b) = 1 \wedge 1 = 1, \\
\gamma_{\chi_C}(a-b) & \leq & \gamma_{\chi_C}(a) \vee \gamma_{\chi_C}(b) = 0 \vee 0 = 0, \\
\gamma_{\chi_C}(ab) & \leq & \gamma_{\chi_C}(a) \vee \gamma_{\chi_C}(b) = 0 \vee 0 = 0,
\end{array}$$

This imply that $\mu_{\chi_C}(a-b) = 1, \mu_{\chi_C}(ab) = 1$ and $\gamma_{\chi_C}(a-b) = 0, \gamma_{\chi_C}(ab) = 0$. Thus $a-b$ and $ab \in C$. Hence $C = A \times B$ is an LA-subring of $R_1 \times R_2$. □

Theorem 4.4. If A and B are two intuitionistic fuzzy normal LA-subrings of LA-rings R_1 and R_2 respectively, then $A \times B$ is an intuitionistic fuzzy normal LA-subring of $R_1 \times R_2$.

Proof. Let $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in R_1\}$ and $B = \{(y, \mu_B(y), \gamma_B(y)) \mid y \in R_2\}$ be intuitionistic fuzzy normal LA-subrings of LA-rings R_1 and R_2, respectively. Now $A \times B = \{((x,y), \mu_{A\times B}\left((x,y)\right), \gamma_{A\times B}\left((x,y)\right)) \mid$ for all $x \in R_1$ and $y \in R_2\}$, where $\mu_{A\times B}((x,y)) = min\{\mu_A(x), \mu_B(y)\}$ and $\gamma_{A\times B}((x,y)) = max\{\gamma_A(x), \gamma_B(y)\}$. We have to show that $A \times B$ is an intuitionistic fuzzy normal LA-subring of $R_1 \times R_2$. Let $(a,b), (c,d) \in R_1 \times R_2$. Now

$$\begin{array}{rcl}
\mu_{A\times B}((a,b)-(c,d)) & = & \mu_{A\times B}((a-c, b-d)) \\
& = & min\{\mu_A(a-c), \mu_B(b-d)\} \\
& = & \mu_A(a-c) \wedge \mu_B(b-d) \\
& \geq & \{\mu_A(a) \wedge \mu_A(c)\} \wedge \{\mu_B(b) \wedge \mu_B(d)\} \\
& = & \mu_A(a) \wedge \{\mu_A(c) \wedge \mu_B(b)\} \wedge \mu_B(d) \\
& = & \mu_A(a) \wedge \{\mu_B(b) \wedge \mu_A(c)\} \wedge \mu_B(d) \\
& = & \{\mu_A(a) \wedge \mu_B(b)\} \wedge \{\mu_A(c) \wedge \mu_B(d)\} \\
& = & \mu_{A\times B}((a,b)) \wedge \mu_{A\times B}((c,d)).
\end{array}$$

and

$$\begin{aligned}
\mu_{A\times B}((a,b)\circ(c,d)) &= \mu_{A\times B}((ac,bd)) \\
&= min\{\mu_A(ac),\mu_B(bd)\} \\
&= \mu_A(ac)\wedge\mu_B(bd) \\
&\geq \{\mu_A(a)\wedge\mu_A(c)\}\wedge\{\mu_B(b)\wedge\mu_B(d)\} \\
&= \mu_A(a)\wedge\{\mu_A(c)\wedge\mu_B(b)\}\wedge\mu_B(d) \\
&= \mu_A(a)\wedge\{\mu_B(b)\wedge\mu_A(c)\}\wedge\mu_B(d) \\
&= \{\mu_A(a)\wedge\mu_B(b)\}\wedge\{\mu_A(c)\wedge\mu_B(d)\} \\
&= \mu_{A\times B}((a,b))\wedge\mu_{A\times B}((c,d)).
\end{aligned}$$

Thus $A\times B$ is an intuitionistic fuzzy LA-subring. Now

$$\begin{aligned}
\mu_{A\times B}((a,b)\circ(c,d)) &= \mu_{A\times B}((ac,bd)) = min\{\mu_A(ac),\mu_B(bd)\} \\
&= min\{\mu_A(ca),\mu_B(db)\},\ \text{since } A \text{ and } B \text{ are IFNLSRs} \\
&= \mu_{A\times B}((ca,db)) = \mu_{A\times B}((c,d)\circ(a,b)).
\end{aligned}$$

Similarly, $\gamma_{A\times B}((a,b)-(c,d))\leq\gamma_{A\times B}((a,b))\vee\gamma_{A\times B}((c,d))$, $\gamma_{A\times B}((a,b)\circ(c,d))\leq\gamma_{A\times B}((a,b))\vee\gamma_{A\times B}((c,d))$ and $\gamma_{A\times B}((a,b)\circ(c,d))=\gamma_{A\times B}((c,d)\circ(a,b))$. Hence $A\times B$ is an intuitionistic fuzzy normal LA-subring of $R_1\times R_2$. □

Theorem 4.5. Let A and B be intuitionistic fuzzy subsets of LA-rings R_1 and R_2 with left identity e_1 and e_2, respectively and $A\times B$ is an intuitionistic fuzzy LA-subring of $R_1\times R_2$. Then the following are true.

(1) If $\mu_A(x)\leq\mu_B(e_2)$ and $\gamma_A(x)\geq\gamma_B(e_2)$, for all $x\in R_1$, then A is an intuitionistic fuzzy LA-subring of R_1.

(2) If $\mu_B(x)\leq\mu_A(e_1)$ and $\gamma_B(x)\geq\gamma_A(e_1)$, for all $x\in R_2$, then B is an intuitionistic fuzzy LA-subring of R_2.

Proof. (1) Let $\mu_A(x)\leq\mu_B(e_2)$ and $\gamma_A(x)\geq\gamma_B(e_2)$ for all $x\in R_1$, and $y\in R_1$. We have to show that A is an intuitionistic fuzzy LA-subring of R_1.

Now

$$\begin{aligned}
\mu_A(x-y) &= \mu_A(x+(-y)) = min\{\mu_A(x+(-y)), \mu_B(e_2+(-e_2)\} \\
&= \mu_{A\times B}((x+(-y), e_2+(-e_2)) \\
&= \mu_{A\times B}((x,e_2)+(-y,-e_2)) \\
&= \mu_{A\times B}((x,e_2)-(y,e_2)) \\
&= \mu_{A\times B}((x,e_2)-(y,e_2)) \\
&\geq \mu_{A\times B}((x,e_2)) \wedge \mu_{A\times B}((y,e_2)), \text{ since } A\times B \text{ is IFLSR} \\
&= min\{\mu_A(x), \mu_B(e_2)\} \wedge min\{\mu_A(y), \mu_B(e_2)\} \\
&= \mu_A(x) \wedge \mu_A(y) \\
\text{and } \mu_A(xy) &= min\{\mu_A(xy), \mu_B(e_2e_2)\} \\
&= \mu_{A\times B}((xy, e_2e_2)) \\
&= \mu_{A\times B}((x,e_2)\circ(y,e_2)) \\
&\geq \mu_{A\times B}((x,e_2)) \wedge \mu_{A\times B}((y,e_2)) \text{ since } A\times B \text{ is IFLSR} \\
&= min\{\mu_A(x), \mu_B(e_2)\} \wedge min\{\mu_A(y), \mu_B(e_2)\} \\
&= \mu_A(x) \wedge \mu_A(y)
\end{aligned}$$

Similarly, we can prove that $\gamma_A(x-y) \leq max\{\gamma_A(x), \gamma_A(y)\}$ and $\gamma_A(xy) \leq max\{\gamma_A(x), \gamma_A(y)\}$ for all $x, y \in R_1$. Thus A is an intuitionistic fuzzy LA-subring of R_1. (2) , is same as (1). □

Theorem 4.6. Let A and B be intuitionistic fuzzy subsets of LA-rings R_1 and R_2 with left identity e_1 and e_2, respectively and $A\times B$ is an intuitionistic fuzzy normal LA-subring of $R_1 \times R_2$. Then the following are true.

(1) If $\mu_A(x) \leq \mu_B(e_2)$ and $\gamma_A(x) \geq \gamma_B(e_2)$, for all x in R_1, then A is an intuitionistic fuzzy normal LA-subring of R_1.

(2) If $\mu_B(x) \leq \mu_A(e_1)$ and $\gamma_B(x) \geq \gamma_A(e_1)$, for all x in R_2, then B is an intuitionistic fuzzy normal LA-subring of R_2.

Proof. Let $A\times B$ be an intuitionistic fuzzy normal LA-subring of $R_1 \times R_2$.

(1) Let $\mu_A(x) \leq \mu_B(e_2)$ and $\gamma_A(x) \geq \gamma_B(e_2)$ for all x in R_1, and let $y \in R_1$. Now we have to show that A is an intuitionistic fuzzy normal LA-subring of R_1. Since A is an intuitionistic fuzzy LA-subring of R_1, by Theorem 4 (1).

Now

$$\begin{aligned}
\mu_A(xy) &= min\{\mu_A(xy), \mu_B(e_2e_2)\} \\
&= \mu_{A\times B}\left((xy, e_2e_2)\right) \\
&= \mu_{A\times B}\left((x, e_2)\circ(y, e_2)\right) \\
&= \mu_{A\times B}\left((y, e_2)\circ(x, e_2)\right), \text{ since } A\times B \text{ is an IFNLSR} \\
&= \mu_{A\times B}\left((yx, e_2e_2)\right) \\
&= min\{\mu_A(yx), \mu_B(e_2e_2)\} \\
&= \mu_A(yx)
\end{aligned}$$

Similarly $\gamma_A(xy) = \gamma_A(yx)$. Hence A is an intuitionistic fuzzy normal LA-subring of R_1.

(2) Let $\mu_B(x) \leq \mu_A(e_1)$ and $\gamma_B(x) \geq \gamma_A(e_1)$ for all x in R_2, and let $y \in R_2$. Now we have to show that B is an intuitionistic fuzzy normal LA-subring of R_2. Since B is an intuitionistic fuzzy LA-subring of R_2, by Theorem 4 (2). Now

$$\begin{aligned}
\mu_B(xy) &= min\{\mu_A(e_1e_1), \mu_B(xy)\} \\
&= \mu_{A\times B}((e_1e_1, xy)) \\
&= \mu_{A\times B}\left((e_1, x)\circ(e_1, y)\right) \\
&= \mu_{A\times B}\left((e_1, y)\circ(e_1, x)\right), \text{ since } A\times B \text{ is an IFNLSR} \\
&= \mu_{A\times B}\left((e_1e_1, yx)\right) \\
&= min\{\mu_A(e_1e_1), \mu_B(yx)\} \\
&= \mu_B(yx).
\end{aligned}$$

Similarly, $\gamma_B(xy) = \gamma_B(yx)$. Hence B is an intuitionistic fuzzy normal LA-subring of R_2. □

Conclusion

Though the study of fuzzy sets, where the base crisp set is a commutative ring, has attracted the attention of many researchers over many years. Even then, many sets are naturally endowed with two compatible operations forming a non-commutative and non-associative ring. In this context, we can find examples showing that the fuzzy properties must not be restricted for commutative rings. Thus it seems natural to study fuzzy sets over non-commutative and non-associative rings. In this paper, we initiated the concept of intuitionistic fuzzy normal LA-subrings of LA-rings. We extended the notion of intuitionistic fuzzy normal subrings to a non-associative class of LA-rings. Also we established the direct product of LA-rings and derived some related properties. Though LA-ring is a non-associative and non-commutative structure, but due to its peculiar characteristics, it possesses properties which we usually encounter in associative algebraic structures. In future we hope that, this concept would have a useful contribution in the application of non-associative algebraic structures.

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