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Iterative methods for zero points of accretive operators in Banach spaces

Sheng Hua Wang, Sun Young Cho, Xiao Long Qin

Abstract

The purpose of this paper is to consider the problem of approximating zero points of accretive operators. We introduce and analysis Mann-type iterative algorithm with errors and Halpern-type iterative algorithms with errors. Weak and strong convergence theorems are established in a real Banach space. As applications, we consider the problem of approximating a minimizer of a proper lower semicontinuous convex function in a real Hilbert space.

1 Introduction-Preliminaries

Let C be a nonempty closed and convex subset of a Banach space E and E^* the dual space of E. Let $\langle \cdot, \cdot \rangle$ denote the pairing between E and E^*. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}$$

for all $x \in E$. In the sequel, we use j to denote the single-valued normalized duality mapping. Let $U = \{x \in E : \|x\| = 1\}$. E is said to be smooth or said to be have a Gâteaux differentiable norm if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

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exists for each $x, y \in U$. E is said to have a uniformly Gâteaux differentiable norm if for each $y \in U$, the limit is attained uniformly for all $x \in U$. E is said to be uniformly smooth or said to be have a uniformly Fréchet differentiable norm if the limit is attained uniformly for $x, y \in U$. It is known that if the norm of E is uniformly Gâteaux differentiable, then the duality mapping J is single valued and uniformly norm to weak* continuous on each bounded subset of E.

The modulus of convexity of E is defined by

$$\delta(\epsilon) = \inf\{1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon\}$$

for every ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. If E is uniformly convex, then

$$\left\|\frac{x + y}{2}\right\| \leq r\Big(1 - \delta(\frac{\epsilon}{r})\Big)$$

for every $x, y \in E$ with $\|x\| \leq r$, $\|y\| \leq r$ and $\|x - y\| \geq \epsilon$.

In this paper, $\to$ and $\rightharpoonup$ denote strong and weak convergence, respectively.

A Banach space E is said to satisfy Opial's condition [13] if for any sequence $\{x_n\} \subset E$, $x_n \rightharpoonup y$ implies that

$$\liminf_{n\to\infty} \|x_n - y\| < \liminf_{n\to\infty} \|x_n - z\|$$

for all $z \in E$ with $z \neq y$.

Recall that a mapping $T : C \to C$ is said to be nonexpanisve if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

In this paper, we use $F(T)$ to denote the set of fixed points of T. A closed convex subset C of E is said to have the fixed point property for nonexpansive mappings if every nonexpansive mapping of a bounded closed convex subset D of C into itself has a fixed point in D.

A mapping P of C into itself is called a retraction if $P^2 = P$. If a mapping P of C into itself is a retraction, then $Pz = z$ for all $z \in R(P)$, where $R(P)$ is the range of P. A subset D of C is called a nonexpansive retract of C if there exists a nonexpansive retraction from C onto D.

Let I denote the identity operator on E. An operator $A \subset E \times E$ with domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \cup\{Az : z \in D(A)\}$ is said to be accretive if for each $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$, there exists $j(x_1 - x_2) \in J(x_1 - x_2)$ such that

$$\langle y_1 - y_2, j(x_1 - x_2)\rangle \geq 0.$$

An accretive operator A is said to satisfy the range condition if

$$\overline{D(A)} \subset \cap_{r>0} R(I + rA),$$

where $\overline{D(A)}$ denote the closure of $D(A)$. An accretive operator A is said to be m-accretive if $R(I + rA) = E$ for all $r > 0$. In a real Hilbert space, an operator A is m-accretive if and only if A is maximal monotone.

For an accretive operator A, we can define a nonexpansive single-valued mapping $J_r : R(I + rA) \to D(A)$ by

$$J_r = (I + rA)^{-1}$$

for each $r > 0$, which is called the resolvent of A. We also define the Yosida approximation A_r by

$$A_r = \frac{1}{r}(I - J_r).$$

It is known that $A_r x \in AJ_r x$ for all $x \in R(I + rA)$ and $\|A_r x\| \leq \inf\{\|y\| : y \in Ax\}$ for all $x \in D(A) \cap R(I + rA)$.

One of classical methods of studying the problem $0 \in Ax$, where $A \subset E \times E$ is an accretive operator, is the following:

$$x_0 \in E, \quad x_{n+1} = J_{r_n} x_n, \quad n \geq 0, \tag{Δ}$$

where $J_{r_n} = (I + r_n A)^{-1}$ and $\{r_n\}$ is a sequence of positive real numbers.

The convergence of (Δ) has been studied by many authors; see, for example, Benavides, Acedo and Xu [1], Brézis and Lions [2], Bruck [3], Bruck and Passty [4], Bruck and Reich [5], Cho, Zhou and Kim [7], Ceng, Wu and Yao [8], Kamimur and Takahashi [10,11], Pazy [14], Qin, Kang and Cho [15], Qin and Su [16], Rockafellar [17], Reich [19-22], Takahashi and Ueda [23], Takahashi [24], Xu [26] and Zhou [27].

In this paper, motivated by the research work going on in this direction, we introduce and analysis Mann-type iterative algorithms with errors and Halpern-type iterative algorithms with errors. Weak and strong convergence theorems are established in a real Banach space.

In order to prove our main results, we need the following lemmas.

Lemma 1.1 ([21],[23]). *Let E be a real reflexive Banach space whose norm is uniformly Gâteaux differentiable and $A \subset E \times E$ be an accretive operator. Suppose that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings. Let C be a nonempty, closed and convex subset of E such that $\overline{D(A)} \subset C \subset \cap_{t>0} R(I + tA)$. If $A^{-1}(0) \neq \emptyset$, then the strong limit $\lim_{t\to\infty} J_t x$ exists and belongs to $A^{-1}(0)$ for all $x \in C$, where $J_t = (I + tA)^{-1}$ is the resolvent of A for all $t > 0$.*

Lemma 1.2 ([12]). *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three nonnegative real sequences satisfying*

$$a_{n+1} \leq (1-t_n)a_n + b_n + c_n, \quad n \geq 0,$$

where $\{t_n\}$ is a sequence in $[0,1]$. Assume that the following conditions are satisfied

(a) $\sum_{n=0}^{\infty} t_n = \infty$ *and* $b_n = o(t_n)$*;*

(b) $\sum_{n=0}^{\infty} c_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 1.3 ([6]). *Let C be a nonempty closed and convex subset of a uniformly convex Banach space E and $T : C \to C$ a nonexpansive mapping. If a sequence $\{x_n\}$ in C converges weakly to $z \in C$ and $\{x_n - Tx_n\}$ converges strongly to 0 as $n \to \infty$, then $Tz = z$.*

Lemma 1.4 ([25]). *Let $\{a_n\}$ and $\{b_n\}$ be sequences of positive numbers satisfying*

$$a_{n+1} \leq a_n + b_n, \quad n \geq 0.$$

If $\sum_{n=0}^{\infty} b_n < \infty$, then the limit of $\{a_n\}$ exists.

Lemma 1.5 ([9]). In a Banach space E, there holds the inequality

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad x, y \in E,$$

where $j(x+y) \in J(x+y)$.

2 Main results

Theorem 2.1. *Let E be a real reflexive Banach space with a uniformly Gâteaux differentiable norm and C a nonempty closed and convex subset of E. Let P be a nonexpansive retraction of E onto C and $A \subset E \times E$ an accretive operator with $A^{-1}(0) \neq \emptyset$. Assume that $\overline{D(A)} \subset C \subset \cap_{r>0} R(I + rA)$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$x_0 \in E, \quad x_{n+1} = \alpha_n u + \beta_n J_{r_n}(x_n + e_{n+1}) + \gamma_n P f_n, \quad n \geq 0, \qquad (\Upsilon)$$

where $u \in C$ is a fixed point, $\{f_n\} \subset E$ is a bounded sequence, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$, $\{e_n\}$ is a sequence in E, $\{r_n\} \subset (0,\infty)$ and $J_{r_n} = (I + r_n A)^{-1}$. Suppose that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings. Assume that the following conditions are satisfied

(a) $\alpha_n + \beta_n + \gamma_n = 1$;

(b) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(c) $\sum_{n=0}^{\infty} \gamma_n < \infty$ *and* $\sum_{n=1}^{\infty} \|e_n\| < \infty$;

(d) $r_n \to \infty$ *as* $n \to \infty$.

Then the sequence $\{x_n\}$ *generated by* (Υ) *converges strongly to a zero of* A.

Proof. First, we show that the sequence $\{x_n\}$ is bounded. Fixing $p \in A^{-1}(0)$, we have

$$\begin{aligned} \|x_1 - p\| &= \|\alpha_0 u + \beta_0 J_{r_0}(x_0 + e_1) + \gamma_0 P f_0 - p\| \\ &\leq \alpha_0\|u - p\| + \beta_0\|J_{r_0}(x_0 + e_1) - p\| + \gamma_0\|Pf_0 - p\| \\ &\leq \alpha_0\|u - p\| + \beta_0\|(x_0 + e_1) - p\| + \gamma_0\|f_0 - p\| \\ &\leq \alpha_0\|u - p\| + \beta_0(\|x_0 - p\| + \|e_1\|) + \gamma_0\|f_0 - p\| \\ &\leq K, \end{aligned}$$

where $K = \|u - p\| + \|x_0 - p\| + \|e_1\| + \|f_0 - p\| < \infty$. Putting

$$M = \max\{K, \sup_{n\geq 0} \|f_n - p\|\},$$

we prove that

$$\|x_n - p\| \leq M + \sum_{i=1}^{n} \|e_i\|, \quad \forall n \geq 1. \tag{2.1}$$

It is easy to see that the result holds for $n = 1$. We assume that the result holds for some n. It follows that

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n u + \beta_n J_{r_n}(x_n + e_{n+1}) + \gamma_n P f_n - p\| \\ &\leq \alpha_n\|u - p\| + \beta_n\|J_{r_n}(x_n + e_{n+1}) - p\| + \gamma_n\|Pf_n - p\| \\ &\leq \alpha_n\|u - p\| + \beta_n\|(x_n + e_{n+1}) - p\| + \gamma_n\|f_n - p\| \\ &\leq \alpha_n\|u - p\| + \beta_n\|x_n - p\| + \|e_{n+1}\| + \gamma_n\|f_n - p\| \\ &\leq \alpha_n M + \beta_n(M + \textstyle\sum_{i=0}^{n} \|e_i\|) + \|e_{n+1}\| + \gamma_n M \\ &= M + \textstyle\sum_{i=1}^{n+1} \|e_i\|. \end{aligned}$$

This shows that (2.1) holds. From the condition $\sum_{i=1}^{\infty} \|e_i\| < \infty$, we see that the sequence $\{x_n\}$ is bounded.

Next, we show that $\limsup_{n\to\infty}\langle u - z, J(x_{n+1} - z)\rangle \leq 0$, where $z = \lim_{t\to\infty} J_t u$, which is guaranteed by Lemma 1.1. Note that $\frac{u - J_t u}{t} \in AJ_t u$, $A_{r_n} x_n \in AJ_{r_n} x_n$ and A is accretive. It follows that

$$\langle A_{r_n} x_n - \frac{u - J_t u}{t}, J(J_{r_n} x_n - J_t u)\rangle \geq 0.$$

This implies that

$$\langle u - J_t u, J(J_{r_n} x_n - J_t u)\rangle \leq \langle t A_{r_n} x_n, J(J_{r_n} x_n - J_t u)\rangle. \tag{2.2}$$

On the other hand, we have

$$\lim_{n\to\infty} \|A_{r_n} x_n\| = \lim_{n\to\infty} \|\frac{x_n - J_{r_n} x_n}{r_n}\| = 0.$$

In view of (2.2), we arrive at

$$\limsup_{n\to\infty} \langle u - J_t u, J(J_{r_n} x_n - J_t u)\rangle \leq 0, \quad \forall t \geq 0. \tag{2.3}$$

Since $z = \lim_{t\to\infty} J_t u$ and the norm of E is uniformly Gâteaux differentiable, for any $\epsilon > 0$, there exists $t_0 > 0$ such that

$$|\langle z - J_t u, J(J_{r_n} x_n - J_t u)\rangle| \leq \frac{\epsilon}{2}$$

and

$$|\langle u - z, J(J_{r_n} x_n - J_t u) - J(J_{r_n} x_n - z)\rangle| \leq \frac{\epsilon}{2}$$

for all $t \geq t_0$ and $n \geq 0$. It follows that

$$\begin{aligned}
&|\langle u - J_t u, J(J_{r_n} x_n - J_t u)\rangle - \langle u - z, J(J_{r_n} x_n - z)\rangle| \\
&\leq |\langle u - J_t u, J(J_{r_n} x_n - J_t u)\rangle - \langle u - z, J(J_{r_n} x_n - J_t u)\rangle| \\
&\quad + |\langle u - z, J(J_{r_n} x_n - J_t u)\rangle - \langle u - z, J(J_{r_n} x_n - z)\rangle| \\
&= |\langle z - J_t u, J(J_{r_n} x_n - J_t u)\rangle| + |\langle u - z, J(J_{r_n} x_n - J_t u) - J(J_{r_n} x_n - z)\rangle| \\
&\leq \epsilon
\end{aligned} \tag{2.4}$$

for all $t \geq t_0$ and $n \geq 0$. It follows from (2.3) and (2.4) that

$$\limsup_{n\to\infty} \langle u - z, J(J_{r_n} x_n - z)\rangle \leq \limsup_{n\to\infty} \langle u - J_t u, J(J_{r_n} x_n - J_t u)\rangle + \epsilon \leq \epsilon.$$

Since ϵ is arbitrary, we see that

$$\limsup_{n\to\infty} \langle u - z, J(J_{r_n} x_n - z)\rangle \leq 0. \tag{2.5}$$

Note that

$$\|J_{r_n} x_n - J_{r_n}(x_n + e_{n+1})\| \leq \|e_{n+1}\|.$$

This implies that

$$\lim_{n\to\infty} \|J_{r_n} x_n - J_{r_n}(x_n + e_{n+1})\| = 0.$$

Since E has a uniformly Gâteaux differentiable norm, we arrive at

$$\limsup_{n\to\infty}\langle u-z, J(J_{r_n}(x_n+e_{n+1})-z)\rangle \le 0. \tag{2.6}$$

On the other hand, , we see from the iterative (Υ) that

$$x_{n+1}-J_{r_n}(x_n+e_{n+1}) = \alpha_n[u-J_{r_n}(x_n+e_{n+1})]+\gamma_n[Pf_n-J_{r_n}(x_n+e_{n+1})].$$

That is,

$$\|x_{n+1}-J_{r_n}(x_n+e_{n+1})\| \le \alpha_n\|u-J_{r_n}(x_n+e_{n+1})\|+\gamma_n\|Pf_n-J_{r_n}(x_n+e_{n+1})\|.$$

From the conditions (b) and (c), we obtain that

$$\limsup_{n\to\infty}\|x_{n+1}-J_{r_n}(x_n+e_{n+1})\| = 0,$$

which combines with (2.6) yields that

$$\limsup_{n\to\infty}\langle u-z, J(x_{n+1}-z)\rangle \le 0. \tag{2.7}$$

From the algorithm (Υ), we see that

$$\begin{aligned} x_{n+1}-z &= \alpha_n(u-z)+\beta_n[J_{r_n}(x_n+e_{n+1})-z]+\gamma_n(Pf_n-z)\\ &= (1-\alpha_n)[J_{r_n}(x_n+e_{n+1})-z]+\alpha_n(u-z)+\gamma_n[Pf_n-J_{r_n}(x_n+e_{n+1})]. \end{aligned}$$

It follows from Lemma 1.5 that

$$\begin{aligned} &\|x_{n+1}-z\|^2\\ &\le (1-\alpha_n)^2\|J_{r_n}(x_n+e_{n+1})-z\|^2+2\alpha_n\langle u-z, J(x_{n+1}-z)\rangle\\ &\quad +2\gamma_n\langle Pf_n-J_{r_n}(x_n+e_{n+1}), J(x_{n+1}-z)\rangle\\ &\le (1-\alpha_n)\|(x_n+e_{n+1})-z\|^2+2\alpha_n\langle u-z, J(x_{n+1}-z)\rangle\\ &\quad +2\gamma_n\|Pf_n-J_{r_n}(x_n+e_{n+1})\|\|x_{n+1}-z\|\\ &\le (1-\alpha_n)(\|x_n-z\|^2-2\langle e_{n+1}, J[(x_n+e_{n+1})-z]\rangle)+2\alpha_n\langle u-z, J(x_{n+1}-z)\rangle\\ &\quad +2\gamma_n\|f_n-J_{r_n}(x_n+e_{n+1})\|\|x_{n+1}-z\|\\ &\le (1-\alpha_n)(\|x_n-z\|^2+2\|e_{n+1}\|\|(x_n+e_{n+1})-z\|)+2\alpha_n\langle u-z, J(x_{n+1}-z)\rangle\\ &\quad +2\gamma_n\|f_n-J_{r_n}(x_n+e_{n+1})\|\|x_{n+1}-z\|\\ &\le (1-\alpha_n)\|x_n-z\|^2+2\alpha_n\langle u-z, J(x_{n+1}-z)\rangle\\ &\quad +2\gamma_n\|f_n-J_{r_n}(x_n+e_{n+1})\|\|x_{n+1}-z\|+2\|e_{n+1}\|\|(x_n+e_{n+1})-z\|\\ &\le (1-\alpha_n)\|x_n-z\|^2+2\alpha_n\langle u-z, J(x_{n+1}-z)\rangle+(\gamma_n+\|e_{n+1}\|)B, \end{aligned}$$

where B is an appropriate constant such that

$$B \ge \max\{\sup_{n\ge 0}\{2\|f_n-J_{r_n}(x_n+e_{n+1})\|\|x_{n+1}-z\|\}, \sup_{n\ge 0}\{2\|(x_n+e_{n+1})-z\|\}\}$$

Let $\lambda_n = \max\{\langle u - z, J(x_{n+1} - z)\rangle, 0\}$. Next, we show that $\lim_{n\to\infty} \lambda_n = 0$. Indeed, from (2.7), for any give $\epsilon > 0$, there exists a positive integer n_1 such that

$$\langle u - z, J(x_{n+1} - z)\rangle < \epsilon, \quad \forall n \geq n_1.$$

This implies that $0 \leq \lambda_n < \epsilon \ \forall n \geq n_1$. Since $\epsilon > 0$ is arbitrary, we see that $\lim_{n\to\infty} \lambda_n = 0$. Put $a_n = \|x_n - z\|$, $b_n = 2\alpha_n\lambda_n$, $c_n = (\gamma_n + \|e_{n+1}\|)B$ and $t_n = \alpha_n$. In view of Lemma 1.2, we can obtain the desired conclusion immediately. This completes the proof.

In a real Hilbert space, Theorem 2.1 is reduced to the following.

Corollary 2.2. *Let H be a real Hilbert space and C a nonempty, closed and convex subset of H. Let P be a metric projection of H onto C and $A \subset H \times H$ a monotone operator with $A^{-1}(0) \neq \emptyset$. Assume that $\overline{D(A)} \subset C \subset \cap_{r>0} R(I + rA)$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$x_0 \in H, \quad x_{n+1} = \alpha_n u + \beta_n J_{r_n}(x_n + e_{n+1}) + \gamma_n P f_n, \quad n \geq 0,$$

where $u \in C$ is a fixed point, $\{f_n\} \subset H$ is a bounded sequence, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$, $\{e_n\}$ is a sequence in H, $\{r_n\} \subset (0,\infty)$ and $J_{r_n} = (I + r_nA)^{-1}$. Assume that the following conditions are satisfied

(a) $\alpha_n + \beta_n + \gamma_n = 1$;

(b) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(c) $\sum_{n=0}^{\infty} \gamma_n < \infty$ *and* $\sum_{n=1}^{\infty} \|e_n\| < \infty$;

(d) $r_n \to \infty$ *as* $n \to \infty$.

Then the sequence $\{x_n\}$ converges strongly to a zero of A.

Theorem 2.3. *Let E be a real uniformly convex Banach space which satisfies Opial's condition and C a nonempty closed and convex subset of E. Let P be a nonexpansive retraction of E onto C and $A \subset E \times E$ an accretive operator with $A^{-1}(0) \neq \emptyset$. Assume that $\overline{D(A)} \subset C \subset \cap_{r>0} R(I + rA)$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$x_0 \in C, \quad x_{n+1} = \alpha_n x_n + \beta_n J_{r_n}(x_n + e_{n+1}) + \gamma_n P f_n, \quad n \geq 0, \qquad (\Upsilon\Upsilon)$$

where $\{f_n\} \subset E$ is a bounded sequence, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequence in $(0,1)$, $\{e_n\}$ is a sequence in E, $\{r_n\} \subset (0,\infty)$ and $J_{r_n} = (I + r_nA)^{-1}$. Assume that the following conditions are satisfied

(a) $\alpha_n + \beta_n + \gamma_n = 1$;

(b) $\limsup_{n\to\infty} \alpha_n < 1$;

(c) $\sum_{n=0}^{\infty} \gamma_n < \infty$ *and* $\sum_{n=1}^{\infty} \|e_n\| < \infty$;

(d) $\liminf_{n\to\infty} r_n > 0$.

Then the sequence $\{x_n\}$ *generated by* $(\Upsilon\Upsilon)$ *converges weakly to a zero of* A.

Proof. First, we show that the sequence $\{x_n\}$ is bounded. Fixing $p \in A^{-1}(0)$, we have

$$\begin{aligned} \|x_1 - p\| &= \|\alpha_0 x_0 + \beta_0 J_{r_0}(x_0 + e_1) + \gamma_0 P f_0 - p\| \\ &\leq \alpha_0\|x_0 - p\| + \beta_0\|J_{r_0}(x_0 + e_1) - p\| + \gamma_0\|Pf_0 - p\| \\ &\leq \alpha_0\|x_0 - p\| + \beta_0\|(x_0 + e_1) - p\| + \gamma_0\|f_0 - p\| \\ &\leq \alpha_0\|x_0 - p\| + \beta_0(\|x_0 - p\| + \|e_1\|) + \gamma_0\|f_0 - p\| \\ &\leq K', \end{aligned}$$

where $K' = \|x_0 - p\| + \|e_1\| + \|f_0 - p\| < \infty$. Putting

$$M' = \max\{K, \sup_{n\geq 0} \|f_n - p\|\},$$

we prove that

$$\|x_n - p\| \leq M' + \sum_{i=1}^{n} \|e_i\|, \quad \forall n \geq 1. \tag{2.8}$$

It is easy to see that the result holds for $n = 1$. We assume that the result holds for some n. It follows that

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n x_n + \beta_n J_{r_n}(x_n + e_{n+1}) + \gamma_n P f_n - p\| \\ &\leq \alpha_n\|x_n - p\| + \beta_n\|J_{r_n}(x_n + e_{n+1}) - p\| + \gamma_n\|Pf_n - p\| \\ &\leq \alpha_n\|x_n - p\| + \beta_n\|(x_n + e_{n+1}) - p\| + \gamma_n\|f_n - p\| \\ &\leq \alpha_n\|x_n - p\| + \beta_n\|x_n - p\| + \|e_{n+1}\| + \gamma_n\|f_n - p\| \\ &\leq \alpha_n M + \beta_n(M + \textstyle\sum_{i=0}^{n} \|e_i\|) + \|e_{n+1}\| + \gamma_n M \\ &= M + \textstyle\sum_{i=1}^{n+1} \|e_i\|. \end{aligned}$$

This shows that (2.8) holds. From the condition $\sum_{i=1}^{\infty} \|e_i\| < \infty$, we see that the sequence $\{x_n\}$ is bounded.

Next, we show that $\lim_{n\to\infty} \|x_n - x^*\|$ exists for any $x^* \in A^{-1}(0)$. In fact, we have

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|\alpha_n x_n + \beta_n J_{r_n}(x_n + e_{n+1}) + \gamma_n P f_n - x^*\| \\ &\leq \alpha_n\|x_n - x^*\| + \beta_n\|J_{r_n}(x_n + e_{n+1}) - x^*\| + \gamma_n\|Pf_n - x^*\| \\ &\leq \alpha_n\|x_n - x^*\| + \beta_n\|(x_n + e_{n+1}) - x^*\| + \gamma_n\|f_n - x^*\| \\ &\leq \|x_n - x^*\| + \lambda_n, \end{aligned}$$

where $\lambda_n = \|e_{n+1}\| + \gamma_n\|f_n - x^*\|$ for each $n \geq 0$. From the assumption, we see that $\sum_{n=0}^{\infty} \lambda_n < \infty$. It follows from Lemma 1.4 that $\lim_{n\to\infty} \|x_n - x^*\|$ exists for any $x^* \in A^{-1}(0)$. Put $d = \lim_{n\to\infty} \|x_n - x^*\|$ for any $x^* \in A^{-1}(0)$. We may, without loss of generality, assume that $d > 0$. Since A is accretive and E is uniformly convex, we have

$$\begin{aligned} \|J_{r_n}x_n - x^*\| &\leq \|J_{r_n}x_n - x^* + \tfrac{r_n}{2}(A_{r_n}x_n - 0)\| \\ &= \|J_{r_n}x_n - x^* + \tfrac{1}{2}(x_n - J_{r_n}x_n)\| \\ &= \|\tfrac{x_n + J_{r_n}x_n}{2} - x^*\| \\ &\leq \|x_n - x^*\|[1 - \delta(\tfrac{\|x_n - J_{r_n}x_n\|}{\|x_n - x^*\|})]. \end{aligned} \tag{2.9}$$

Note that

$$\begin{aligned} &\|x_{n+1} - x^*\| \\ &= \|\alpha_n x_n + \beta_n J_{r_n}(x_n + e_{n+1}) + \gamma_n Pf_n - x^*\| \\ &\leq \alpha_n\|x_n - x^*\| + \beta_n\|J_{r_n}(x_n + e_{n+1}) - x^*\| + \gamma_n\|Pf_n - x^*\| \\ &\leq \alpha_n\|x_n - x^*\| + \beta_n\|J_{r_n}(x_n + e_{n+1}) - J_{r_n}x_n\| + \beta_n\|J_{r_n}x_n - x^*\| + \gamma_n\|Pf_n - x^*\| \\ &\leq \alpha_n\|x_n - x^*\| + \beta_n\|e_{n+1}\| + \beta_n\|J_{r_n}x_n - x^*\| + \gamma_n\|Pf_n - x^*\| \\ &\leq \alpha_n\|x_n - x^*\| + \|e_{n+1}\| + (1 - \alpha_n)\|J_{r_n}x_n - x^*\| + \gamma_n\|Pf_n - x^*\|. \end{aligned}$$

This is,

$$-(\alpha_n\|x_n - x^*\| + \|e_{n+1}\| + (1-\alpha_n)\|J_{r_n}x_n - x^*\| + \gamma_n\|Pf_n - x^*\|) \leq -\|x_{n+1} - x^*\|. \tag{2.10}$$

It follows from (2.9) and (2.10) that

$$\begin{aligned} &(1 - \alpha_n)\|x_n - x^*\|\delta(\tfrac{\|x_n - J_{r_n}x_n\|}{\|x_n - x^*\|}) \\ &\leq (1 - \alpha_n)(\|x_n - x^*\| - \|J_{r_n}x_n - x^*\|) \\ &= \|x_n - x^*\| - (\alpha_n\|x_n - x^*\| + (1 - \alpha_n)\|J_{r_n}x_n - x^*\|) \\ &= \|x_n - x^*\| - (\alpha_n\|x_n - x^*\| + \|e_{n+1}\| + (1 - \alpha_n)\|J_{r_n}x_n - x^*\| + \gamma_n\|Pf_n - x^*\|) \\ &\quad + \|e_{n+1}\| + \gamma_n\|Pf_n - x^*\| \\ &\leq \|x_n - x^*\| - \|x_{n+1} - x^*\| + \|e_{n+1}\| + \gamma_n\|Pf_n - x^*\|. \end{aligned}$$

From the conditions (b), (c) and $\lim_{n\to\infty} \|x_n - x^*\| = d > 0$, we arrive at

$$\delta(\frac{\|x_n - J_{r_n}x_n\|}{\|x_n - x^*\|}) \to 0$$

as $n \to \infty$. This implies that

$$\lim_{n\to\infty} \|x_n - J_{r_n}x_n\| = 0. \tag{2.11}$$

On the other hand, we have

$$\begin{aligned}\|J_{r_n}x_n - J_1 J_{r_n}x_n\| &= \|(I - J_1)J_{r_n}x_n\| \\ &= \|A_1 J_{r_n}x_n\| \\ &\leq \inf\{\|u\| : u \in AJ_{r_n}x_n\} \\ &\leq \|A_{r_n}x_n\| \\ &= \|\tfrac{x_n - J_{r_n}x_n}{r_n}\|.\end{aligned}$$

From (2.11) and the condition (d), we obtain that

$$\lim_{n\to\infty} \|J_{r_n}x_n - J_1 J_{r_n}x_n\| = 0. \tag{2.12}$$

Letting $v \in C$ be a weak subsequential limit of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v$. From (2.11), we see that $J_{r_{n_i}}x_{n_i} \rightharpoonup v$. In view of Lemma 1.3, we obtain that $v \in F(J_1) = A^{-1}(0)$. Since the space satisfies Opial's condition (see [18]), we see that the desired conclusion holds. This completes the proof.

In a real Hilbert space, Theorem 2.3 is reduced to the following.

Corollary 2.4. *Let H be a real Hilbert space and C a nonempty, closed and convex subset of E. Let P be a metric projection of E onto C and $A \subset H \times H$ a monotone operator with $A^{-1}(0) \neq \emptyset$. Assume that $\overline{D(A)} \subset C \subset \cap_{r>0}R(I + rA)$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$x_0 \in C, \quad x_{n+1} = \alpha_n x_n + \beta_n J_{r_n}(x_n + e_{n+1}) + \gamma_n P f_n, \quad n \geq 0,$$

where $\{f_n\} \subset H$ is a bounded sequence, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequence in $(0,1)$, $\{e_n\}$ is a sequence in H, $\{r_n\} \subset (0,\infty)$ and $J_{r_n} = (I + r_n A)^{-1}$. Assume that the following conditions are satisfied

(a) $\alpha_n + \beta_n + \gamma_n = 1$;

(b) $\limsup_{n\to\infty} \alpha_n < 1$;

(c) $\sum_{n=0}^{\infty} \gamma_n < \infty$ *and* $\sum_{n=1}^{\infty} \|e_n\| < \infty$;

(d) $\liminf_{n\to\infty} r_n > 0$.

Then the sequence $\{x_n\}$ converges weakly to a zero of A.

3 Applications

In this section, as applications of main Theorems 2.1 and 2.3, we consider the problem of finding a minimizer of a convex function f.

Let H be a Hilbert space and $h : H \to (-\infty, +\infty]$ be a proper convex lower semi-continuous function. Then the subdifferential ∂h of h is defined as follows:

$$\partial h(x) = \{y \in H : h(z) \geq h(x) + \langle z - x, y\rangle, \quad z \in H\}, \quad \forall x \in H.$$

Theorem 3.1. *Let H be a real Hilbert space and $h : H \to (-\infty, +\infty]$ a proper convex lower semi-continuous function such that $\partial h(0) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{cases} x_0 \in H, \\ y_n = \arg\min_{x\in H}\{h(x) + \frac{1}{2r_n}\|x - x_n - e_{n+1}\|^2\}, \\ x_{n+1} = \alpha_n u + \beta_n y_n + \gamma_n f_n, \quad n \geq 0, \end{cases}$$

where $u \in H$ is a fixed point, $\{f_n\} \subset H$ is a bounded sequence, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$, $\{e_n\}$ is a sequence in H and $\{r_n\} \subset (0, \infty)$. Assume that the following conditions are satisfied

(a) $\alpha_n + \beta_n + \gamma_n = 1$;

(b) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(c) $\sum_{n=0}^{\infty} \gamma_n < \infty$ *and* $\sum_{n=1}^{\infty} \|e_n\| < \infty$;

(d) $r_n \to \infty$ *as* $n \to \infty$.

Then the sequence $\{x_n\}$ converges strongly to a minimizer of h.

Proof. Since $h : H \to (-\infty, +\infty]$ is a proper convex lower semi-continuous function, we have that the subdifferential ∂h of h is maximal monotone by Rockafellar [18]. Notice that

$$y_n = \arg\min_{x\in H}\{h(x) + \frac{1}{2r_n}\|x - x_n - e_{n+1}\|^2\}$$

is equivalent to the following

$$0 \in \partial h(y_n) + \frac{1}{r_n}(y_n - x_n - e_{n+1}).$$

It follows that

$$x_n + e_{n+1} \in y_n + r_n\partial h(y_n), \quad \forall n \geq 0.$$

By Theorem 2.1, we can obtain the desired conclusion immediately.

Theorem 3.2. *Let H be a real Hilbert space and $h : H \to (-\infty, +\infty]$ a proper convex lower semi-continuous function such that $\partial h(0) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{cases} x_0 \in H, \\ y_n = \arg\min_{x \in H}\{h(x) + \frac{1}{2r_n}\|x - x_n - e_{n+1}\|^2\}, \\ x_{n+1} = \alpha_n x_n + \beta_n y_n + \gamma_n f_n, \quad n \geq 0, \end{cases}$$

where $\{f_n\} \subset H$ is a bounded sequence, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequence in $(0,1)$, $\{e_n\}$ is a sequence in H and $\{r_n\} \subset (0, \infty)$. Assume that the following conditions are satisfied

(a) $\alpha_n + \beta_n + \gamma_n = 1$;

(b) $\limsup_{n\to\infty} \alpha_n < 1$;

(c) $\sum_{n=0}^{\infty} \gamma_n < \infty$ *and* $\sum_{n=1}^{\infty} \|e_n\| < \infty$;

(d) $\liminf_{n\to\infty} r_n > 0$.

Then the sequence $\{x_n\}$ converges weakly to a minimizer of h.

Proof. We can easily obtain from the proof of Theorem 2.3 and Theorem 3.1 the desired conclusion.

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Sheng Hua Wang,
Department of Mathematics and Physics,
North China Electric Power University,
Baoding 071003, China.
Email: sheng-huawang@hotmail.com

Sun Young Cho,
Department of Mathematics,
Gyeongsang National University,
Jinju 660-701, Korea.
Email: ooly61@yahoo.co.kr

Xiao Long Qin,
Department of Mathematics,
Hangzhou Normal University,
Hangzhou 310036, China.
Email: qxlxajh@163.com