Position vector of spacelike biharmonic curves in the Lorentzian Heisenberg group $Heis^3$

Essin TURHAN, Talat KÖRPINAR

Abstract

In this paper, we study spacelike biharmonic curves in the Lorentzian Heisenberg group $Heis^3$. We show that spacelike biharmonic curves are general helices. We characterize position vector of spacelike biharmonic general helices in terms of their curvature and torsion.

1 Introduction

Let (M, g) and (N, h) be Lorentzian manifolds and $\phi : M \longrightarrow N$ a smooth map. Denote by ∇^{ϕ} the connection of the vector bundle $\phi^{*}TN$ induced from the Levi-Civita connection ∇^{h} of (N, h). The *second fundamental form* $\nabla d\phi$ is defined by

$$(\nabla d\phi)(X, Y) = \nabla^{\phi}_{X} d\phi (Y) - d\phi (\nabla_X Y), \quad X, Y \in \Gamma (TM).$$

Here ∇ is the Levi-Civita connection of (M, g). The *tension field* $\tau(\phi)$ is a section of $\phi^{*}TN$ defined by

$$\tau(\phi) = tr\nabla d\phi. \tag{1.1}$$

A smooth map ϕ is said to be *harmonic* if its tension field vanishes. It is well known that ϕ is harmonic if and only if ϕ is a critical point of the *energy*:

$$E(\phi) = \frac{1}{2}\int h(d\phi, d\phi)\, dv_g$$

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over every compact region of M. Now let $\phi : M \longrightarrow N$ be a harmonic map. Then the Hessian $\mathcal{H}$ of E is given by

$$\mathcal{H}_{\phi}(V,W) = \int h(\mathcal{J}_{\phi}(V), W)\, dv_g, \quad V,W \in \Gamma(\phi^* TN).$$

Here the *Jacobi operator* $\mathcal{J}_{\phi}$ is defined by

$$\mathcal{J}_{\phi}(V) := \overline{\Delta}_{\phi} V - \mathcal{R}_{\phi}(V), \quad V \in \Gamma(\phi^* TN), \tag{1.2}$$

$$\overline{\Delta}_{\phi} := \sum_{i=1}^{m} \left(\nabla^{\phi}_{e_i} \nabla^{\phi}_{e_i} - \nabla^{\phi}_{\nabla_{e_i} e_i} \right), \mathcal{R}_{\phi}(V) = \sum_{i=1}^{m} R^N (V, d\phi(e_i))\, d\phi(e_i), \tag{1.3}$$

where R^N and $\{e_i\}$ are the Riemannian curvature of N, and a local orthonormal frame field of M, respectively.

Let $\phi : (M,g) \to (N,h)$ be a smooth map between two Lorentzian manifolds. The *bienergy* $E_2(\phi)$ of ϕ over compact domain $\Omega \subset M$ is defined by

$$E_2(\phi) = \int_{\Omega} h(\tau(\phi), \tau(\phi))\, dv_g.$$

A smooth map $\phi : (M,g) \to (N,h)$ is said to be *biharmonic* if it is a critical point of the $E_2(\phi)$.

The section $\tau_2(\phi)$ is called the *bitension field* of ϕ and the Euler-Lagrange equation of E_2 is

$$\tau_2(\phi) := -\mathcal{J}_{\phi}(\tau(\phi)) = 0. \tag{1.4}$$

Recently, there have been a growing interest in the theory of biharmonic maps which can be divided into two main research directions. On the one side, the differential geometric aspect has driven attention to the construction of examples and classification results. The other side is the analytic aspect from the point of view of PDE: biharmonic maps are solutions of a fourth order strongly elliptic semilinear PDE.

Biharmonic functions are utilized in many physical situations, particularly in fluid dynamics and elasticity problems. Most important applications of the theory of functions of a complex variable were obtained in the plane theory of elasticity and in the approximate theory of plates subject to normal loading. That is, in cases when the solutions are biharmonic functions or functions associated with them. In linear elasticity, if the equations are formulated in terms of displacements for two-dimensional problems then the introduction of a stress function leads to a fourth-order equation of biharmonic type. For instance, the stress function is proved to be biharmonic for an elastically isotropic crystal undergoing phase transition, which follows spontaneous dilatation. Biharmonic functions also arise when dealing with transverse displacements of plates and shells. They can describe the deflection of a thin plate subjected to uniform loading over its surface with fixed edges. Biharmonic functions arise in fluid dynamics, particularly in Stokes flow problems (i.e., low-Reynolds-number flows). There are many applications for Stokes flow such as in engineering and biological transport phenomena (for details, see [8, 13]). Fluid flow through a narrow pipe or channel, such as that used in micro-fluidics, involves low Reynolds number. Seepage flow through cracks and pulmonary alveolar blood flow can also be approximated by Stokes flow. Stokes flow also arises in flow through porous media, which have been long applied by civil engineers to groundwater movement. The industrial applications include the fabrication of microelectronic components, the effect of surface roughness on lubrication, the design of polymer dies and the development of peristaltic pumps for sensitive viscous materials. In natural systems, creeping flows are important in biomedical applications and studies of animal locomotion.

Non-geodesic biharmonic curves are called proper biharmonic curves. Obviously geodesics are biharmonic. Caddeo, Montaldo and Piu showed that on a surface with non-positive Gaussian curvature, any biharmonic curve is a geodesic of the surface [6]. So they gave a positive answer to generalized Chen's conjecture. Caddeo et al. in [5] studied biharmonic curves in the unit 3-sphere. More precisely, they showed that proper biharmonic curves in $\mathbb{S}^3$ are circles of geodesic curvature 1 or helices which are geodesics in the Clifford minimal torus.

In this paper, we first write down the conditions that any spacelike biharmonic curve in the Lorentzian Heisenberg group $Heis^3$ must satisfy. Then, we prove that the non-geodesic biharmonic curves in the Lorentzian Heisenberg group $Heis^3$ are general helices. Finally, we characterize position vector of spacelike biharmonic general helices in terms of their curvature and torsion.

2 Preliminaries

The Heisenberg group $Heis^3$ is a Lie group which is diffeomorphic to $\mathbb{R}^3$ and the group operation is defined as

$$(x, y, z) * (\overline{x}, \overline{y}, \overline{z}) = (x + \overline{x}, y + \overline{y}, z + \overline{z} - \overline{x}y + x\overline{y}).$$

The identity of the group is $(0, 0, 0)$ and the inverse of (x, y, z) is given by $(-x, -y, -z)$. The left-invariant Lorentz metric on $Heis^3$ is

$$g = -dx^2 + dy^2 + (xdy + dz)^2.$$

The following set of left-invariant vector fields forms an orthonormal basis for the corresponding Lie algebra:

$$\left\{ e_1 = \frac{\partial}{\partial z},\ e_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z},\ e_3 = \frac{\partial}{\partial x} \right\}. \tag{2.1}$$

The characterising properties of this algebra are the following commutation relations:

$$[e_2, e_3] = 2e_1, \quad [e_3, e_1] = 0, \quad [e_2, e_1] = 0,$$

with

$$g(e_1, e_1) = g(e_2, e_2) = 1, \quad g(e_3, e_3) = -1. \tag{2.2}$$

Proposition 2.1. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g, defined above the following is true:*

$$\nabla = \begin{pmatrix} 0 & e_3 & e_2 \\ e_3 & 0 & e_1 \\ e_2 & -e_1 & 0 \end{pmatrix}, \tag{2.3}$$

where the (i, j)-element in the table above equals $\nabla_{e_i} e_j$ for our basis

$$\{e_k, k = 1, 2, 3\} = \{e_1, e_2, e_3\}.$$

We adopt the following notation and sign convention for Riemannian curvature operator:

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The Riemannian curvature tensor is given by

$$R(X, Y, Z, W) = g(R(X, Y)W, Z).$$

Moreover, we put

$$R_{abc} = R(e_a, e_b)e_c,\ R_{abcd} = R(e_a, e_b, e_c, e_d),$$

where the indices a, b, c and d take the values $1, 2$ and 3.

Then the non-zero components of the Riemannian curvature tensor field and of the Riemannian curvature tensor are, respectively,

$$R_{121} = e_2, \quad R_{131} = e_3, \quad R_{232} = -3e_3,$$

and

$$R_{1212} = -1, \quad R_{1313} = 1, \quad R_{2323} = -3. \tag{2.4}$$

3 Position Vectors of Spacelike Biharmonic Curves In Lorentzian Heisenberg Group $Heis^3$

Let $\gamma : I \longrightarrow Heis^3$ be a non geodesic curve on the Lorentzian Heisenberg group $Heis^3$ parametrized by arc length. A non geodesic curve γ is called spacelike curve if $g(\gamma', \gamma') > 0$. Caddeo et al. [2], used the Frenet formulas in the Riemannian case.

Let $\{T, N, B\}$ be the Frenet frame fields tangent to the Lorentzian Heisenberg group $Heis^3$ along γ defined as follows:

T is the unit vector field γ' tangent to γ, N is the unit vector field in the direction of $\nabla_T T$ (normal to γ) and B is chosen so that $\{T, N, B\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned} \nabla_T T &= \kappa N \\ \nabla_T N &= \kappa T + \tau B \\ \nabla_T B &= \tau N, \end{aligned}$$

where $\kappa(s) = |\tau(\gamma)| = |\nabla_T T|$ is the curvature of γ, $\tau(s)$ is its torsion and

$$\begin{aligned} g(T,T) &= 1,\ g(N,N) = -1,\ g(B,B) = 1, \\ g(T,N) &= g(T,B) = g(N,B) = 0. \end{aligned}$$

If we write this curve in the another parametric representation $\tilde{\gamma} = \tilde{\gamma}(\theta)$, where $\theta = \int \kappa(s)\, ds$. We have new Frenet equations as follows:

$$\begin{aligned} \nabla_{\tilde{T}(\theta)} \tilde{T}(\theta) &= \tilde{N}(\theta) \\ \nabla_{\tilde{T}(\theta)} \tilde{N}(\theta) &= \tilde{T}(\theta) + f(\theta) \tilde{B}(\theta) \\ \nabla_{\tilde{T}(\theta)} \tilde{B}(\theta) &= f(\theta) \tilde{N}(\theta), \end{aligned} \tag{3.1}$$

where $f(\theta) = \frac{\tau(\theta)}{\kappa(\theta)}$.

With respect to the orthonormal basis $\{e_1, e_2, e_3\}$, we can write

$$\begin{aligned} \tilde{T}(\theta) &= \tilde{T}_1(\theta) e_1 + \tilde{T}_2(\theta) e_2 + \tilde{T}_3(\theta) e_3, \\ \tilde{N}(\theta) &= \tilde{N}_1(\theta) e_1 + \tilde{N}_2(\theta) e_2 + \tilde{N}_3(\theta) e_3, \\ \tilde{B}(\theta) &= \tilde{T}(\theta) \times \tilde{N}(\theta) = \tilde{B}_1(\theta) e_1 + \tilde{B}_2(\theta) e_2 + \tilde{B}_3(\theta) e_3. \end{aligned} \tag{3.2}$$

Theorem 3.1. *$\tilde{\gamma} = \tilde{\gamma}(\theta)$ is a non geodesic spacelike biharmonic curve in the Lorentzian Heisenberg group $Heis^3$ if and only if*

$$\begin{aligned} f^2(\theta) &= -2 + 4\tilde{B}_1^2(\theta), \\ f'(\theta) &= 2\tilde{N}_1(\theta) \tilde{B}_1(\theta). \end{aligned} \tag{3.3}$$

Proof. Using (3.1), we have

$$\begin{aligned}\tau_2(\widetilde{\gamma}) &= \nabla^3_{\tilde{T}(\theta)}\tilde{T}(\theta) - R(\tilde{T}(\theta), \nabla_{\tilde{T}(\theta)}\tilde{T}(\theta))\tilde{T}(\theta)\\ &= (1+f^2(\theta))\tilde{N}(\theta) + f'(\theta)\tilde{B}(\theta) - R(\tilde{T}(\theta), \tilde{N}(\theta))\tilde{T}(\theta).\end{aligned}$$

Using (1.2), we see that $\tilde{\gamma}$ is a biharmonic curve if and only if

$$\begin{aligned}1+f^2(\theta) &= -R(\tilde{T}(\theta), \tilde{N}(\theta), \tilde{T}(\theta), \tilde{N}(\theta)),\\ f'(\theta) &= R(\tilde{T}(\theta), \tilde{N}(\theta), \tilde{T}(\theta), \tilde{B}(\theta)).\end{aligned} \tag{3.4}$$

A direct computation using (2.4), yields

$$\begin{aligned}R(\tilde{T}(\theta), \tilde{N}(\theta), \tilde{T}(\theta), \tilde{N}(\theta)) &= 1-4\tilde{B}_1^2(\theta)\\ R(\tilde{T}(\theta), \tilde{N}(\theta), \tilde{T}(\theta), \tilde{B}(\theta)) &= 2\tilde{N}_1(\theta)\tilde{B}_1(\theta),\end{aligned} \tag{3.5}$$

These, together with (3.4), complete the proof of the theorem.

Theorem 3.2. *Let $\tilde{\gamma} = \tilde{\gamma}(\theta)$ is a non-geodesic spacelike biharmonic curve in the Lorentzian Heisenberg group $Heis^3$. Then, $\tilde{\gamma} = \tilde{\gamma}(\theta)$ is a general helix.*

Proof. Suppose that $f'(\theta) = 2\tilde{N}_1(\theta)\tilde{B}_1(\theta) \neq 0$. We shall derive a contradiction by showing that must be $f(\theta)$ =constant.

We can use (2.3) to compute the covariant derivatives of the vector fields $\tilde{T}(\theta), \tilde{N}(\theta)$ and $\tilde{B}(\theta)$ as:

$$\begin{aligned}\nabla_{\tilde{T}(\theta)}\tilde{T}(\theta) &= \tilde{T}_1'(\theta)e_1 + (\tilde{T}_2'(\theta) + 2\tilde{T}_1(\theta)\tilde{T}_3(\theta))e_2\\ &\quad +(\tilde{T}_3'(\theta) + 2\tilde{T}_1(\theta)\tilde{T}_2(\theta))e_3,\\ \nabla_{\tilde{T}(\theta)}\tilde{N}(\theta) &= (\tilde{N}_1'(\theta) + \tilde{T}_2(\theta)\tilde{N}_3(\theta) - \tilde{T}_3(\theta)\tilde{N}_2(\theta))e_1\\ &\quad +(\tilde{N}_2'(\theta) + \tilde{T}_1(\theta)\tilde{N}_3(\theta) - \tilde{T}_3(\theta)\tilde{N}_1(\theta))e_2\\ &\quad +(N_3'(\theta) + \tilde{T}_2(\theta)\tilde{N}_1(\theta) - \tilde{T}_1(\theta)\tilde{N}_2(\theta))e_3,\\ \nabla_{\tilde{T}(\theta)}\tilde{B}(\theta) &= (\tilde{B}_1'(\theta) + \tilde{T}_2(\theta)\tilde{B}_3(\theta) - \tilde{T}_3(\theta)\tilde{B}_2(\theta))e_1\\ &\quad +(B_2'(\theta) + \tilde{T}_1(\theta)\tilde{B}_3(\theta) - \tilde{T}_3(\theta)\tilde{B}_1(\theta))e_2\\ &\quad +(\tilde{B}_3'(\theta) + \tilde{T}_2(\theta)\tilde{B}_1(\theta) - \tilde{T}_1(\theta)\tilde{B}_2(\theta))e_3.\end{aligned} \tag{3.6}$$

It follows that the first components of these vectors are given by

$$\begin{aligned}&< \nabla_{\tilde{T}(\theta)}\tilde{T}(\theta), e_1 >= \tilde{T}_1'(\theta),\\ &< \nabla_{\tilde{T}(\theta)}N(\theta), e_1 >= \tilde{N}_1'(\theta) + \tilde{T}_2(\theta)\tilde{N}_3(\theta) - \tilde{T}_3(\theta)\tilde{N}_2(\theta),\\ &< \nabla_{\tilde{T}(\theta)}\tilde{B}(\theta), e_1 >= \tilde{B}_1'(\theta) + \tilde{T}_2(\theta)\tilde{B}_3(\theta) - \tilde{T}_3(\theta)\tilde{B}_2(\theta).\end{aligned} \tag{3.7}$$

On the other hand, using Frenet formulas (3.1), we have,

$$\begin{aligned} &< \nabla_{\tilde{T}(\theta)}\tilde{T}(\theta), e_1 >= \tilde{N}_1(\theta), \\ &< \nabla_{\tilde{T}(\theta)}\tilde{N}(\theta), e_1 >= \tilde{T}_1(\theta) + f(\theta)\tilde{B}_1(\theta), \\ &< \nabla_{\tilde{T}(\theta)}\tilde{B}(\theta), e_1 >= f(\theta)\tilde{N}_1(\theta). \end{aligned} \tag{3.8}$$

These, together with (3.7) and (3.8), give

$$\begin{aligned} \tilde{T}_1'(\theta) &= \tilde{N}_1(\theta), \\ \tilde{N}_1'(\theta) + \tilde{B}_1(\theta) &= \tilde{T}_1(\theta) + f(\theta)\tilde{B}_1(\theta), \\ \tilde{B}_1'(\theta) + \tilde{T}_2(\theta)\tilde{B}_3(\theta) - \tilde{T}_3(\theta)\tilde{B}_2(\theta) &= f(\theta)\tilde{N}_1(\theta). \end{aligned} \tag{3.9}$$

Assume now that $\tilde{\gamma}$ is biharmonic.Differentiating (3.2) with respect to θ, we obtain

Using $f'(\theta) = 2\tilde{N}_1(\theta)\tilde{B}_1(\theta) \neq 0$ and (3.3), we obtain

$$2f(\theta)f'(\theta) = 8\tilde{B}_1(\theta)\tilde{B}_1'(\theta).$$

We substitute $f'(\theta) = 2\tilde{N}_1(\theta)\tilde{B}_1(\theta)$ above equation, give

$$f(\theta)\tilde{N}_1(\theta)\tilde{B}_1(\theta) = 2\tilde{B}_1(\theta)\tilde{B}_1'(\theta).$$

Then

$$f(\theta) = \frac{2\tilde{B}_1'(\theta)}{\tilde{N}_1(\theta)}. \tag{3.10}$$

If we use $\tilde{T}_2(\theta)\tilde{B}_3(\theta) - \tilde{T}_3(\theta)\tilde{B}_2(\theta) = -\tilde{N}_1(\theta)$ and (3.9), we get

$$\tilde{B}_1'(\theta) = (1 + f(\theta))\tilde{N}_1(\theta).$$

We substitute $\tilde{B}_1'(\theta)$ in equation (3.10):

$$f(\theta) = -\frac{2}{3} = \text{constant}.$$

Therefore also $f(\theta)$ is constant and we have a contradiction that is $f'(\theta) \neq 0$. Therefore $\tilde{\gamma}(\theta)$ is not biharmonic.

Corollary 3.3. *$\tilde{\gamma} = \tilde{\gamma}(\theta)$ is a spacelike biharmonic curve in the Lorentzian Heisenberg group $Heis^3$ if and only if*

$$\begin{aligned} f(\theta) &= \text{constant}, \\ \tilde{N}_1(\theta)\tilde{B}_1(\theta) &= 0, \\ f^2(\theta) &= -2 + 4\tilde{B}_1^2(\theta). \end{aligned} \tag{3.11}$$

Theorem 3.4. *The position vector of the spacelike biharmonic curve* $\tilde{\gamma} = \tilde{\gamma}(\theta)$ *in the Lorentzian Heisenberg group* $Heis^3$ *is given by*

$$\begin{aligned}\tilde{\gamma}(\theta) &= \left(\left(1-\frac{1}{\mu}\right)\theta - \frac{c_1}{\mu}e^{\mu\theta} + \frac{c_2}{\mu}e^{-\mu\theta} + c_3\right)\tilde{T}(\theta) \\ &\quad + \left(c_1 e^{\mu\theta} + c_2 e^{-\mu\theta} + \frac{1}{\mu^2}\right)\tilde{N}(\theta) \\ &\quad + \left(\frac{\mu^2-1}{\mu}\theta + c_1\frac{\mu^2-1}{\mu}e^{\mu\theta} - c_2\frac{\mu^2-1}{\mu}e^{-\mu\theta} + c_4\right)\tilde{B}(\theta),\end{aligned} \tag{3.12}$$

where $\mu = \sqrt{1+f^2(\theta)}$ *and* c_1, c_2, c_3, c_4 *are constants of integration.*

Proof. If $\tilde{\gamma}(\theta)$ is a non-geodesic biharmonic curve in the Lorentzian Heisenberg group $Heis^3$, then we can write its position vector as follows:

$$\tilde{\gamma}(\theta) = \xi(\theta)\tilde{T}(\theta) + \eta(\theta)\tilde{N}(\theta) + \rho(\theta)\tilde{B}(\theta) \tag{3.13}$$

for some differentiable functions ξ, η and ρ of $\theta \in I \subset \mathbb{R}$. These functions are called component functions (or simply components) of the position vector.

Differentiating (3.9) with respect to θ and by using the corresponding Frenet equation (3.1), we find

$$\begin{aligned}\xi'(\theta) + \eta(\theta) &= 1, \\ \eta'(\theta) + \xi(\theta) + f(\theta)\rho(\theta) &= 0, \\ \rho'(\theta) + f(\theta)\eta(\theta) &= 0.\end{aligned} \tag{3.14}$$

From (3.14), we get the following differential equation:

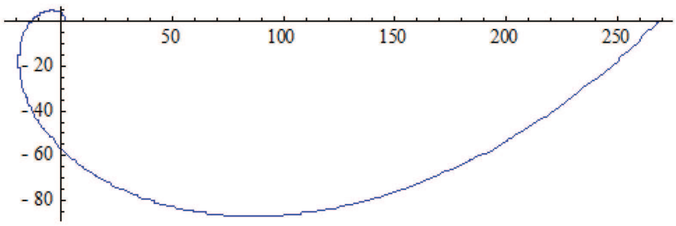
$$\eta''(\theta) - (1+f^2(\theta))\eta(\theta) + 1 = 0. \tag{3.15}$$

The solution of (3.15) is

$$\begin{aligned}\eta(\theta) &= c_1 e^{\sqrt{1+f^2(\theta)}\theta} + c_2 e^{-\sqrt{1+f^2(\theta)}\theta} + \frac{1}{1+f^2(\theta)} \\ &= c_1 e^{\mu\theta} + c_2 e^{-\mu\theta} + \frac{1}{\mu^2},\end{aligned} \tag{3.16}$$

where $c_1, c_2 \in \mathbb{R}$.

The picture of $\eta(\theta)$ at $c_1 = c_2 = f(\theta) = 1$: From $\xi'(\theta) = 1 - \eta(\theta)$ and

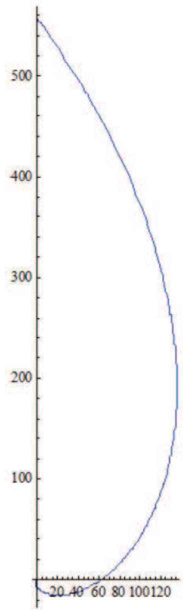


using (3.16), we find the solution of this equation as follows:

$$\begin{aligned}\xi(\theta) &= \left(1-\frac{1}{\sqrt{1+f^2(\theta)}}\right)\theta+\frac{c_1}{\sqrt{1+f^2(\theta)}}e^{\sqrt{1+f^2(\theta)}\theta} \\ &\quad -\frac{c_2}{\sqrt{1+f^2(\theta)}}e^{-\sqrt{1+f^2(\theta)}\theta}+c_3 \\ &= \left(1-\frac{1}{\mu}\right)\theta-\frac{c_1}{\mu}e^{\mu\theta}+\frac{c_2}{\mu}e^{-\mu\theta}+c_3,\end{aligned} \tag{3.17}$$

where $c_1, c_2, c_3 \in \mathbb{R}$

The picture of $\xi(\theta)$ at $c_1 = c_2 = c_3 = f(\theta) = 1$: By using (3.16), we find

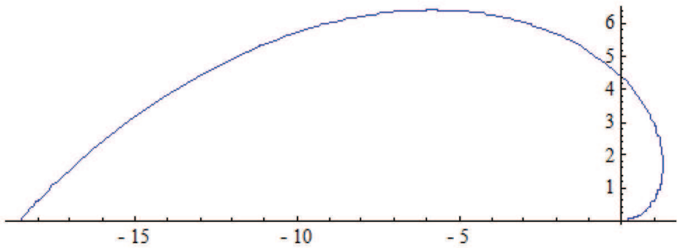


the solution of $\rho'(\theta) = \eta(\theta) f(\theta)$ as follows:

$$\begin{aligned}\rho(\theta) &= -\frac{f(\theta)}{\sqrt{1+f^2(\theta)}}\theta - \frac{f(\theta)c_1}{\sqrt{1+f^2(\theta)}}e^{\sqrt{1+f^2(\theta)}\theta} \\ &\quad + \frac{f(\theta)c_2}{\sqrt{1+f^2(\theta)}}e^{-\sqrt{1+f^2(\theta)}\theta} + c_4 \qquad (3.18) \\ &= \frac{\mu^2-1}{\mu}\theta + c_1\frac{\mu^2-1}{\mu}e^{\mu\theta} - c_2\frac{\mu^2-1}{\mu}e^{-\mu\theta} + c_4,\end{aligned}$$

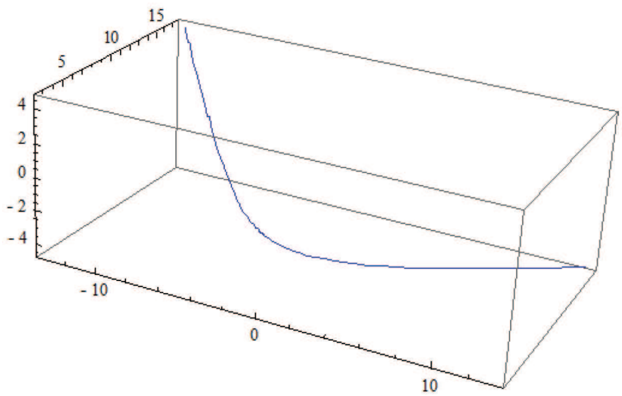
where $c_1, c_2, c_3, c_4 \in \mathbb{R}$

The picture of $\rho(\theta)$ at $c_1 = c_2 = c_4 = f(\theta) = 1$:



Substituting (3.16), (3.17) and (3.18) into the equation in (3.13), we have (3.12). This concludes the proof of Theorem 3.4.

The picture of $\tilde{\gamma}(\theta)$ at $c_1 = c_2 = c_3 = c_4 = f(\theta) = 1$:



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Fırat University, Department of Mathematics
e-mail: essin.turhan@gmail.com