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Circles holding a regular triangular prism

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Abstract

A regular triangular prism with all edges of length 1 can be held by a circle.

1 Introduction

Problems on a net or a box holding a unit sphere have been considered by A. S. Besicovitch [1]. H. S. M. Coxeter proposed a problem on a cage holding a unit sphere in [3]: Find a cage of minimum sum of edge lengths holding a unit sphere (not permitting it to slide out). Coxeter conjectured that it is a right triangular prism all of whose edges are equal to $\sqrt{3}$, so that the total length of all edges is $9\sqrt{3} = 15.59...$. But this conjecture was false, as Besicovitch proved in [2] that the greatest lower bound of the sum of edges of a cage to hold a unit-sphere is at most $\gamma = \frac{8\pi}{3} + 2\sqrt{3} = 11.88...$.

Are there any convex bodies which can be held using a circle? T. Zamfirescu [6] showed not only that these convex bodies do exist, but also that they form a large majority. More precisely, he showed that the convex bodies which cannot be held by a circle form a nowhere dense subset.

What about the prisms, pyramids, cylinders or cones? Let $\mathcal{C}$ be the space of all circles in $\mathbb{R}^3$, endowed with the Hausdorff metric δ.

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Following [6], for a convex body $\mathsf{B} \subset \mathbb{R}^3$ to be *held* by the circle C means that $\mathrm{int}\mathsf{B} \cap \mathsf{C} = \emptyset$ and, for some natural number m, there is no continuous mapping, $f : [0,\ 1] \to \mathcal{C}$ such that $f(0) = \mathsf{C}$, $\delta(f(0),\ f(1)) > m$ and, for all $t \in [0,\ 1]$, $f(t)$ is congruent with C and $f(t) \cap \mathrm{int}\mathsf{B} = \emptyset$. That is, the circle C cannot move or can move only slightly. If the circle C cannot move at all, we say that B is held by C *rigidly*.

For example, balls and circular cylinders cannot be held by a circle; on the other hand regular tetrahedra can. For a related question, see [4]. Whether a regular triangular pyramid can be held or not depends on its height [5].

In this paper, we will investigate the non-obvious question whether a prism B can be held by a circle. We reach the following main conclusion.

Theorem 1.1. *A regular triangular prism with all edges of length 1 can be held by a circle.*

2 Preliminaries

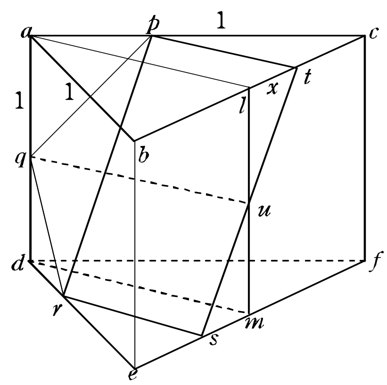

Figure 1:

We consider the regular triangular prism *abcdef* with all edges of length 1 (see Fig. 1). Let l be the orthogonal projection of a on bc and m be the orthogonal projection of d on ef. Take points t on the line-segment lc, and p on the side ac with $pt \parallel al$. Let x be the distance $|lt|$ from l to t $(0 \leqslant x < \frac{1}{2})$. Denote by q, u the mid points of ad, lm respectively. Let π be the plane through the points p, q, t, as in Figure 1. The intersection $\pi \cap \mathsf{B}$ is a pentagon $pqrst$ symmetric with respect to qu. The quadrilateral $prst$ is a rectangle. We denote the radius of the circle circumscribed to the rectangle $prst$ by $R(x)$. From elementary calculation, it follows that

$$R(x) = \frac{1}{2}\sqrt{7x^2 - 3x + \frac{7}{4}}. \tag{2.1}$$

$R(x)$ has a (single) minimum at $x = \tau = \frac{3}{14} = 0.24...$

We denote the radius of the circle circumscribed to the triangle tqs by $T(x)$, and find out that

$$T(x) = \frac{1}{\sqrt{3}}\left(x^2 + 1\right). \tag{2.2}$$

Since $x \geqslant 0$, $T(x)$ is an increasing function. The triangle tqs is acute. We denote the value of x which realizes $R(x)$=$T(x)$ by σ. We define the function

$$M(x) = \max\{R(x), T(x)\}. \tag{2.3}$$

Lemma 2.1. *$M(x)$ has a unique minimum at x=σ.*

Proof. The value of σ is the solution of the equation

$$\frac{1}{2}\sqrt{7x^2 - 3x + \frac{7}{4}} = \frac{1}{\sqrt{3}}\left(x^2 + 1\right), \tag{2.4}$$

i.e.

$$16x^4 - 52x^2 + 36x - 5 = 0. \tag{2.5}$$

We find

$$\sigma = \frac{1}{2}\left(-\sqrt{\xi} + \sqrt{\xi - \frac{9 - (4\xi - 13)\sqrt{\xi}}{2\sqrt{\xi}}}\right) = 0.192..... \tag{2.6}$$

where,

$$\xi = \frac{\sqrt{109}}{6}\cos\frac{\alpha}{3} + \frac{13}{6} \qquad \text{and} \qquad \cos\alpha = -\frac{163}{\sqrt{109^3}}. \tag{2.7}$$

Hence, $\sigma < \tau$. Since $R(x)$ is decreasing and $T(x)$ is increasing in $[0, \sigma]$, $M(x)$ $(= R(x))$ is decreasing there. Since both $R(x)$ and $T(x)$ are increasing in $[\sigma, \frac{1}{2}]$, $M(x)$ $(= T(x))$ is also increasing there. Therefore, $M(x)$ has a unique minimum at σ.

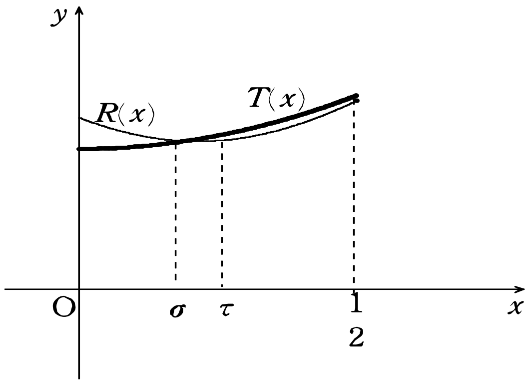

Figure 2:

Next, we consider the case of moving the plane π in a neighborhood of $x = \sigma$, in various directions. We prepare now the proof of Theorem 1.1.

Lemma 2.2. *Let π' be the plane obtained from π after a rotation of angle θ around ts. We denote by p', r' the intersections of π' with the edges ac, de respectively. The intersection $\pi' \cap \mathsf{B}$ becomes the trapezoid $p'r'st$, and denote by ρ, ρ' the radii of the circles circumscribed to the rectangle $prst$ and the trapezoid $p'r'st$. Then $\rho \leqslant \rho'$ if $0 \leqslant x \leqslant \frac{3}{10}$.*

Proof. Assume $\theta > 0$, *i.e.* a counterclockwise rotation. Then

$$\rho'^2 = \frac{x^2 + \frac{1}{4}}{4x^2\cos^2\theta + 1}\left(\frac{3\left(\frac{1}{2} - x\right)^2}{4\sin^2\left(\frac{\pi}{6} - \theta\right)} + 4\left(x^2 + \frac{1}{4}\right) - \frac{2\sqrt{3}x\left(\frac{1}{2} - x\right)\sin\theta}{\sin\left(\frac{\pi}{6} - \theta\right)}\right). \tag{2.8}$$

We have
$$\lim_{\theta\to 0}\left(\frac{d\rho\prime^2}{d\theta}\right) = \frac{\sqrt{3}}{2}\left(\frac{1}{2}-x\right)\left(\frac{3}{2}-5x\right). \tag{2.9}$$

Therefore $\lim_{\theta\to 0}\left(\dfrac{d\rho\prime^2}{d\theta}\right) \geqslant 0$ if $0 \leqslant x \leqslant \frac{3}{10}$. This happens indeed around σ, because $0 < \sigma < \frac{3}{10}$.
@ By symmetry, $\rho'(\theta) = \rho'(-\theta)$, hence ρ' attains a minimum at $\theta = 0$.

Lemma 2.3. *Consider the points p'' on the line pt and r'' on the line rs. If $|pt| > |p''t|$ and $|rs| < |r''s|$ then the radius ρ of the circle circumscribed to the rectangle $prst$ is smaller than the radius ρ'' of the circle circumscribed to the trapezoid $p''r''st$. If $|pt| < |p''t|$ and $|rs| > |r''s|$ then we get the same conclusion.*

Proof. Clearly, $\rho = \frac{|rt|}{2}$. Now, in case $|pt| > |p''t|$ and $|rs| < |r''s|$, the radius of the circle circumscribed to $p''r''st$ is $\frac{|r''t|}{2}$ and so $\rho'' = \frac{|r''t|}{2} \geqslant \frac{|rt|}{2} = \rho$. The case $|pt| < |p''t|$ and $|rs| > |r''s|$ is analogous.

Lemma 2.4. *Let $h_0 > 0$. Among all triangles having fixed side and the corresponding height $h \geqslant h_0$, the isosceles one of height h_0 has smallest circumscribed circle.*

Proof. The easy proof is left to the reader.

Lemma 2.5. *Any plane in some neighborhood of the plane π can be obtained from π by a rotation around uq, a rotation around st, and a translation along lcD*

Proof. This is common knowledge.

Lemma 2.6. *Let $abcd$ be a trapezoid with $ad \parallel bc$, $|ab| = |dc| = w$, $|ad| = u, |bc| = v$ $(u > v)$, and $\tilde{a}\tilde{b}\tilde{c}\tilde{d}$ be a rectangle with $|\tilde{a}\tilde{d}| = |\tilde{b}\tilde{c}| = \frac{u+v}{2}, |\tilde{a}\tilde{b}| = |\tilde{c}\tilde{d}| = w$. We denote the radii of the circles circumscribed to $abcd$, $\tilde{a}\tilde{b}\tilde{c}\tilde{d}$ by $\rho, \tilde{\rho}$. If $w > \frac{u-v}{2}$, then $\rho > \tilde{\rho}$.*

Proof. From direct calculation,

$$\rho^2 = \frac{w^2(uv+w^2)}{\{2w+(u-v)\}\{2w-(u-v)\}}, \quad \tilde{\rho}^2 = \frac{1}{16}\left\{(u+v)^2+4w^2\right\},$$

and

$$\rho^2 - \tilde{\rho}^2 = \frac{(u+v)^2(u-v)^2}{16\{2w+(u-v)\}\{2w-(u-v)\}} > 0. \tag{2.10}$$

3 Proof of Theorem

We now proceed to the proof of our main result.

Proof of Theorem 1.1.
By Lemma 2.5, we have to consider the following three transformations of the plane cutting B. First, a rotation around the axis qu, next, a rotation about the axis st, next, a translation along bc. Suppose π_0 is the position of the plane for $x = \sigma$, π is the position of the plane after the first rotation, π' the position after the second rotation, and π'' the position after the translation. The intersection $\pi' \cap \mathsf{B}$ is a pentagon $p'q'r'st$, the triangle $tq's$ becomes non-isosceles and the quadrilateral $p'r'st$ becomes a trapezoid.
In case $\sigma < x < \frac{1}{2}$, we will prove that $T(\sigma)$ is less than the radius of the circle circumscribed to all intersections with B of planes in a neighborhood of π. Indeed, this is clear by Lemma 2.4.
In case $0 < x < \sigma$, we will prove that $R(\sigma)$ is the smallest radius of the circle circumscribed to the quadrilateral $p''r''s''t''$ determined by π''. We denote the radii of the circles circumscribed to the trapezoids $p'r'st$, $p''r''s''t''$ by $R'(x,\theta)$, $R''(x,\theta,\Delta x)$ respectively. For $0 < x < \frac{3}{10}$, by Lemma 2.2, the radius $R'(x,\theta)$ increases with θ. But, by the translation parallel to π', $R''(x,\theta,\Delta x)$ decreases in the following two cases. We must consider these two cases carefully.
(i) The case of a counterclockwise rotation ($\theta > 0$) and of a translation $\Delta x > 0$ (in direction bc).
(ii) The case of a clockwise rotation ($\theta < 0$) and of a translation $\Delta x < 0$ (in direction cb).
The two cases are symmetric, so we shall discuss only case (i).

In case (i), the diagonal $p's$ of the trapezoid $p'r'st$ is longer than the diagonal $r't$. But the following translation Δx yields

$$|p''s''| < |p's| \quad \text{and} \quad |r''t''| > |r't|.$$

Thus, when $\Delta x = \mu$, the trapezoid $p''_{(\mu)}r''_{(\mu)}s''_{(\mu)}t''_{(\mu)}$ satisfies $|p''_{(\mu)}s''_{(\mu)}| = |r''_{(\mu)}t''_{(\mu)}|$. Then $|p''_{(\mu)}r''_{(\mu)}| = |s''_{(\mu)}t''_{(\mu)}|$, the trapezoid is isosceles. We show now that $R''(x,\theta,\mu)$ is the smallest radius of the circle circumscribed to all the quadrilaterals $p''r''s''t''$, for Δx in a neighborhood of μ.
From direct calculation,

$$\mu = \frac{-\frac{3\sqrt{3}}{4}\frac{\sin\theta}{\sin(\frac{\pi}{6}+\theta)\sin(\frac{\pi}{6}-\theta)} - 2\sqrt{3}\sin\theta}{\frac{3}{4}\frac{\cos\theta}{\sin(\frac{\pi}{6}+\theta)\sin(\frac{\pi}{6}-\theta)}}x + \frac{\sqrt{3}}{2}\tan\theta. \tag{3.1}$$

We have

$$|p''_{(\mu)}t''_{(\mu)}| = \frac{\sqrt{3}}{4}\big(\frac{1}{2}-(x+\mu)\big)\frac{1}{\sin\left(\frac{\pi}{6}-\theta\right)}, \quad |r''_{(\mu)}s''_{(\mu)}| = \frac{\sqrt{3}}{4}\big(\frac{1}{2}-(x-\mu)\big)\frac{1}{\sin\left(\frac{\pi}{6}+\theta\right)}$$

and

$$\begin{aligned} m &= \frac{|p''_{(\mu)}t''_{(\mu)}| + |r''_{(\mu)}s''_{(\mu)}|}{2} \\ &= \frac{\sqrt{3}}{4\sin\left(\frac{\pi}{6}-\theta\right)\sin\left(\frac{\pi}{6}+\theta\right)}\left\{\left(\frac{1}{2}-x\right)\left\{\sin\left(\frac{\pi}{6}+\theta\right)+\sin\left(\frac{\pi}{6}-\theta\right)\right\}\right. \\ &\qquad \left. -\mu\left\{\sin\left(\frac{\pi}{6}+\theta\right)-\sin\left(\frac{\pi}{6}-\theta\right)\right\}\right\} \\ &= \frac{\sqrt{3}}{2\cos 2\theta - 1}\left\{\left(\frac{1}{2}-x\right)\cos\theta - \sqrt{3}\mu\sin\theta\right\}. \end{aligned} \tag{3.2}$$

We will prove that the radius $\tilde{\rho}(\mu)$ of the circle circumscribed to the rectangle $\tilde{p}_{(\mu)}\tilde{r}_{(\mu)}\tilde{s}_{(\mu)}\tilde{t}_{(\mu)}$ is larger than the radius $R(x)$ (see the notation of Lemma 2.2). As $|pr| = |st| = |\tilde{p}_{(\mu)}\tilde{r}_{(\mu)}| = |\tilde{s}_{(\mu)}\tilde{t}_{(\mu)}|$, we will compare $|\tilde{p}_{(\mu)}\tilde{t}_{(\mu)}|$ with $|pt|$. Let f be the function defined by

$$f(x,\theta) = m - |pt| = \frac{\sqrt{3}}{2\cos 2\theta - 1}\left\{\left(\frac{1}{2}-x\right)\cos\theta - \sqrt{3}\mu\sin\theta\right\} - \sqrt{3}\left(\frac{1}{2}-x\right). \tag{3.3}$$

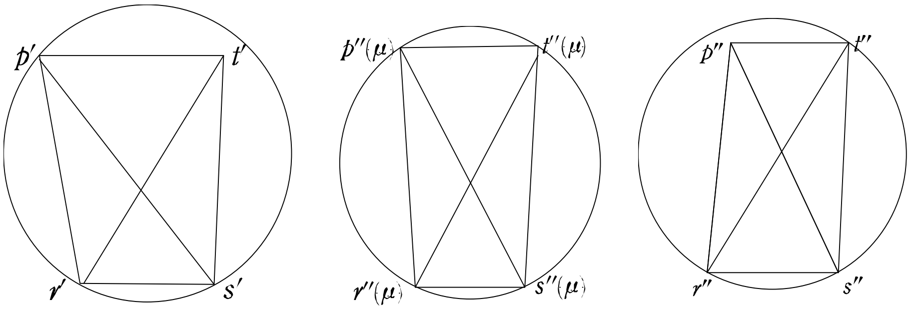

Figure 3:

From direct calculation,

$$f(x,0)=0, \qquad \lim_{\theta\to 0}\frac{df(x,\theta)}{d\theta}=0, \qquad \lim_{\theta\to 0}\frac{d^2f(x,\theta)}{d\theta^2}=\frac{\sqrt{3}}{2}(6x+1)>0. \quad (3.4)$$

Therefore, for fixed x, $f(x,\theta)$ has a local minimum at $\theta=0$, and $f(x,\theta)>0$ in a whole neighborhood of 0 except $\{0\}$.

From Lemma 2.6, we have $R''(x,\theta,\mu)>\tilde{\rho}(\mu)$. Clearly, $\tilde{\rho}(\mu)>R(x)$, and $R(x)>R(\sigma)$, so $R(\sigma)<R''(x,\theta,\mu)$. The theorem is proven.

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