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ON KUROSH-AMITSUR RADICALS OF FINITE GROUPS

Jan Krempa and Izabela Agata Malinowska

Abstract

Between 1952 and 1954 Amitsur and Kurosh initiated the general theory of radicals in various contexts. In the case of all groups some interesting results concerning radicals were obtained by Kurosh, Shchukin, Ryabukhin, Gardner and others.

In this paper we are going to examine radical theory in the class of all finite groups. This strong restriction gives chance to obtain stronger results, then in the case of all groups. For example, we obtain a complete description of hereditary and of strongly hereditary radicals in the class of all finite groups.

1 Preliminaries

All groups considered in this paper are finite. For basics on such groups we refer to [3, 9]. In this section we recall some notation and quote some results on groups, for making this paper easily accessible not only for experts in group theory.

Let G be a group. If $H \leq G$ is a normal subgroup then we write $H \lhd G$ or $G \rhd H$. By $(H_i)_{i=0,\dots,n}$ we denote a subgroup series:

$$1 = H_0 \leqslant H_1 \leqslant \cdots \leqslant H_{i-1} \leqslant H_i \leqslant \cdots \leqslant H_{n-1} \leqslant H_n = G \qquad (1)$$

of length n of G. A series $(H_i)_{i=0,1,\dots,n}$ is a *normal* series, if $H_i \lhd G$ and a *subnormal series*, if $H_{i-1} \lhd H_i$ for all $i = 1, \dots, n$. A normal series $(H_i)_{i=0,\dots,n}$

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is a *chief series*, if each H_{i-1} is maximal among the normal subgroups of G that are properly contained in H_i. A subnormal series $(H_i)_{i=0,\dots,n}$ is a *composition series* if each H_{i-1} is maximal among the proper normal subgroups of H_i. A subgroup H of a group G is *subnormal* in G if H is a member of a subnormal series for G. Cyclic groups of prime order are simple in this paper.

Theorem 1.1. *Let G be a group. Then every minimal normal subgroup of G is a direct product of isomorphic simple groups, while every minimal subnormal subgroup is a simple group.*

As a consequence of Theorem 1.1, for every $1 \leq i \leq n$, the *composition factor* H_i/H_{i-1} of a composition series (1) is a simple group, and the *chief factor* H_i/H_{i-1} of a chief series (1) is a direct product of isomorphic simple groups.

Theorem 1.2. (Jordan-Hölder) *Let G be a group. Then any subnormal (normal) series of G can be extended to a composition (chief) series, every two composition (chief) series have the same length and the same set of factors.*

The set of all primes is denoted by $\mathbb{P}$, p will always denote a prime, π will always denote some subset of $\mathbb{P}$ and $\pi' = \mathbb{P} \setminus \pi$.

Let G be a group. Then G is a π-*group* if all prime divisors of the order of G belong to π. Thus, G is a π'-group if none of prime divisors of the order of G belongs to π.

For further information on series of subgroups, subnormal subgroups, and π-groups we refer to [9].

In the sequel $\mathfrak{F}$ will denote the class of all finite groups. All considered classes of groups are subclasses of $\mathfrak{F}$ and are closed under taking isomorphic images and contain the trivial group 1. For example, $\mathfrak{F}_\pi$ denotes the class of all π-groups. In this way we are considering a set of classes of cardinality $2^{\aleph_0}$. Hence our further operations on classes will be operations on sets.

If a group G belongs to a class $\mathfrak{X}$, then G will be often called an $\mathfrak{X}$-group. If $\mathfrak{X}$ is a family of groups, we use $(\mathfrak{X})$ to denote the smallest class of groups containing $\mathfrak{X}$. Hence every group from $(\mathfrak{X})$ is either isomorphic to a group from $\mathfrak{X}$, or is trivial.

Further $\mathfrak{S}$ will denote the family of all simple groups. If, $\mathfrak{T} \subseteq \mathfrak{S}$ is a famly of simple groups then $(\mathfrak{T})$ will be named a class of simple groups.

We will use here some standard operators with values in classes of groups. If $\mathfrak{X}$ is a family of groups then: $\mathcal{Q}\mathfrak{X}$ denotes the homomorphic closure of groups from $\mathfrak{X}$, $\mathcal{H}\mathfrak{X}$ denotes the class of all groups isomorphic to subnormal subgroups of groups from $\mathfrak{X}$, $\mathcal{H}^s\mathfrak{X}$ denotes the class of all groups isomorphic to subgroups of groups from $\mathfrak{X}$ and $\mathcal{E}\mathfrak{X} = \mathfrak{X}\mathfrak{X}$ denotes the class of all extensions of groups from $\mathfrak{X}$ by groups from $\mathfrak{X}$. By $\bar{\mathcal{E}}\mathfrak{X}$ we denote the smallest class containing $\mathfrak{X}$ and closed under extensions. We have $\bar{\mathcal{E}}\mathfrak{X} = \bigcup\limits_{n=1}^{\infty} \mathcal{E}^n\mathfrak{X}$.

Following terms used in radical theory let us agree that a class $\mathfrak{X}$ is *hereditary* if $\mathfrak{X} = \mathcal{H}\mathfrak{X}$ and $\mathfrak{X}$ is *strongly hereditary* if $\mathfrak{X} = \mathcal{H}^s\mathfrak{X}$. For some further results on classes of groups we refer to [1, 3].

2 Radicals

All radicals in this paper are understood in the sense of Kurosh and Amitsur. For details on such radicals of rings and some other universal classes see for example [5, 7]. For the development of radicals of all groups we refer to [8, 5, 6]. A.G. Kurosh (see [8], page 272) posed the following problem:

Problem 1. Give the complete description of all radicals in the class of all finite groups.

The aim of this paper is to contribute to Problem 1. Hence, our universal class will be the class $\mathfrak{F}$. So this paper can be considered as a work parallel to [6]. Some of our results will be similar in spirit to those from [11].

Using ideas from the case of associative rings and of all groups we are going to show, that in the class $\mathfrak{F}$ situation is much simpler than in those classes, but still not completely solved.

Let us start from the very beginning. A class $\mathfrak{R}$ of groups is a *radical class* if it has the following three properties:

(I) $\mathfrak{R} = \mathcal{Q}\mathfrak{R}$;

(II) For every group G, the join $\mathfrak{R}(G) = \langle H \lhd G \mid H \in \mathfrak{R}\rangle$ is in $\mathfrak{R}$;

(III) $\mathfrak{R}\big(G/\mathfrak{R}(G)\big) = 1$ for every group G.

The subgroup $\mathfrak{R}(G)$ is called the $\mathfrak{R}$-*radical* (or simply *a radical*) of G and $G/\mathfrak{R}(G)$ is called an $\mathfrak{R}$-*semisimple (*semisimple*)* image of G. In this way to any radical class $\mathfrak{R}$ corresponds the *semisimple class*

$$\mathcal{S}\mathfrak{R} = \{G \mid \mathfrak{R}(G) = 1\}. \tag{2}$$

As an immediate consequence of condition (II) we obtain

Proposition 2.1 (ADS Property)**.** *Let $\mathfrak{R}$ be a radical class. Then $\mathfrak{R}(G)$ is a characteristic subgroup of G for any group G. Thus, if $N \lhd G$ then $\mathfrak{R}(N) \lhd G$ and $\mathfrak{R}(N) \leqslant \mathfrak{R}(G)$. In particular the class $\mathcal{S}\mathfrak{R}$ is hereditary.*

The above proposition is valid in the class of all groups. However, in our class $\mathfrak{F}$, this proposition allows to give a simple intrinsic characterization as radical classes, as semisimple classes. First let us agree for a class $\mathfrak{C}$ that $\mathcal{U}\mathfrak{C}$ is the class of all groups G having no homomorphisms onto a nontrivial $\mathfrak{C}$-group. Immediately from the definition we have

Lemma 2.2. *If $\mathfrak{C} = \mathcal{H}\mathfrak{C}$ then the class $\mathcal{U}\mathfrak{C}$ is a radical class, called the* upper radical determined by $\mathfrak{C}$.

Theorem 2.3. *Let $\mathfrak{C}$ be a class of groups.*

(i) *$\mathfrak{C}$ is a radical class if and only if $\mathfrak{C} = \mathcal{Q}\mathfrak{C} = \mathcal{E}\mathfrak{C}$.*

(ii) *$\mathfrak{C}$ is an $\mathfrak{R}$-semisimple class for a radical class $\mathfrak{R}$ if and only if $\mathfrak{C} = \mathcal{H}\mathfrak{C} = \mathcal{E}\mathfrak{C}$.*

Proof. (i) Let $\mathfrak{C}$ be a radical class. Then, by definition, $\mathfrak{C} = \mathcal{Q}\mathfrak{C}$. Now let $N \lhd G$ be such a normal subgroup that N and G/N are $\mathfrak{C}$-groups. By definition $N \leq \mathfrak{C}(G)$, hence the $\mathfrak{C}$-semisimple group $G/\mathfrak{C}(G)$ is a homomorphic image of a $\mathfrak{C}$-group G/N. This means that $G/\mathfrak{C}(G) = 1$ and $G \in \mathfrak{C}$. Thus we have $\mathfrak{C} = \mathcal{E}\mathfrak{C}$.

Conversly, let $\mathfrak{C} = \mathcal{Q}\mathfrak{C} = \mathcal{E}\mathfrak{C}$. If G is a group then as the $\mathfrak{C}$-radical of G put a normal $\mathfrak{C}$-subgroup of G of maximal order. It can be checked using the assumption, that $\mathfrak{C}(G)$ is the largest normal $\mathfrak{C}$-subgroup of G; Moreover, $G/\mathfrak{C}(G)$ has trivial $\mathfrak{C}$-radical. Hence, by definition, $\mathfrak{C}$ is a radical class.

(ii) Let $\mathfrak{C} = \mathcal{S}\mathfrak{R}$ for a radical class $\mathfrak{R}$. If $G \in \mathfrak{C}$ and $N \lhd G$ then, by Proposition 2.1, $N \in \mathfrak{C}$. Hence, $\mathfrak{C} = \mathcal{H}\mathfrak{C}$.

Let G be a group and $N \lhd G$. If N and G/N belong to $\mathfrak{C}$, then

$$(\mathfrak{R}(G)N)/N \subseteq \mathfrak{R}(G/N) = 1.$$

Thus $\mathfrak{R}(G) \leq \mathfrak{R}(N) = 1$ and $\mathfrak{R}(G) = 1$. This means that $\mathfrak{C} = \mathcal{E}\mathfrak{C}$.

Now let $\mathfrak{C} = \mathcal{H}\mathfrak{C} = \mathcal{E}\mathfrak{C}$. By the above lemma $\mathcal{U}\mathfrak{C}$ is a radical class and all groups from $\mathfrak{C}$ are $\mathcal{U}\mathfrak{C}$-semismple. Let G be $\mathcal{U}\mathfrak{C}$-semisimple. Using the induction on $|G|$ one can check, that $G \in \mathfrak{C}$. Hence $\mathfrak{C} = \mathcal{S}\mathcal{U}\mathfrak{C}$ is a semisimple class for the radical class $\mathcal{U}\mathfrak{C}$. $\square$

In our opinion, for any class $\mathfrak{C}$, natural are the following classes: *the lower radical class* $\mathcal{L}\mathfrak{C}$ containing $\mathfrak{C}$, *the lower hereditary radical class* $\mathcal{L}^h\mathfrak{C}$ containing $\mathfrak{C}$, *the lower strongly hereditary radical class* $\mathcal{L}^{sh}\mathfrak{C}$ containing $\mathfrak{C}$ and *the lower semisimple class,* say $\mathcal{M}\mathfrak{C}$, containing $\mathfrak{C}$. Here, according to radical tradition, *the lower* means *the smallest.*

Now, with the help of intersection of classes and Theorem 2.3 we immediately obtain

Theorem 2.4. *Let $\mathfrak{C}$ be a class of groups. Then the classes $\mathcal{L}\mathfrak{C}$, $L^h\mathfrak{C}$, $L^{sh}\mathfrak{C}$ and $\mathcal{M}\mathfrak{C}$ exist.*

Every family of groups, even empty one, say $\mathfrak{C}$, is contained in the smallest class $(\mathfrak{C})$. As a consequence, we can extend operators $\mathcal{L}$, $\mathcal{L}^h$, $\mathcal{L}^{sh}$ and $\mathcal{M}$ to arbitrary families of groups in a natural way. Then, the operators $\mathcal{L}$, $\mathcal{L}^h$, $\mathcal{L}^{sh}$ and $\mathcal{M}$ became closure operators. The operators $\mathcal{U}$ and $\mathcal{S}$ are also extendable to any family of groups, being equal to the composition $\mathcal{U}\mathcal{M}$ and $\mathcal{S}\mathcal{L}$, respectively.

As another consequence of Theorem 2.3 one can obtain a Galois correspondence between radical and semisimple classes. Namely, with the help of Formula (2) and Lemma 2.2 one can prove the following result

Theorem 2.5. *For any semisimple class $\mathfrak{S}$ and radical class $\mathfrak{R}$ we have*

$$\mathcal{S}\mathcal{U}\mathfrak{S} = \mathfrak{S} \quad \textit{and} \quad \mathcal{U}\mathcal{S}\mathfrak{R} = \mathfrak{R}.$$

In view of Theorem 2.5 we shall say that $\mathfrak{R}$ and $\mathfrak{S}$ are *corresponding radical and semisimple classes* if $\mathfrak{R} = \mathcal{U}\mathfrak{S}$ and $\mathfrak{S} = \mathcal{S}\mathfrak{R}$.

Now let's characterize groups belonging to lower radical and lower semisimple classes.

Lemma 2.6. *Let $\mathfrak{C}$ be a class of groups. For a group G the following conditions are equivalent:*

(i) $G \in \mathcal{L}\mathfrak{C}$;

(ii) *G has a subnormal series with factors in $\mathcal{Q}\mathfrak{C}$;*

(iii) *Every nontrivial homomorphic image of G contains a nontrivial subnormal subgroup from $\mathcal{Q}\mathfrak{C}$;*

(iv) $G \in \bar{\mathcal{E}}\mathcal{Q}\mathfrak{C}$.

Proof. Clearly (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii), because of Theorem 2.3 and Proposition 2.1.

On the other hand, if G satisfies condition (ii) or (iii) then, by Theorem 2.3 and Proposition 2.1, G has no nontrivial $\mathcal{L}\mathfrak{C}$-semisimple homomorphic image. Hence $G \in \mathcal{L}\mathfrak{C}$.

It is evident that (ii) $\Leftrightarrow$ (iv). □

Lemma 2.7. *Let $\mathfrak{C}$ be a class of groups. For a group G the following conditions are equivalent:*

(i) $G \in \mathcal{M}\mathfrak{C}$;

(ii) *G has a subnormal series with factors in* $\mathcal{H}\mathfrak{C}$;

(iii) *Every nontrivial normal subgroup of G has a nontrivial homomorphic image in $\mathcal{H}\mathfrak{C}$;*

(iv) *Every nontrivial subnormal subgroup of G has a nontrivial homomorphic image in $\mathcal{H}\mathfrak{C}$;*

(v) $G \in \bar{\mathcal{E}}\mathcal{H}\mathfrak{C}$.

Proof. The above lemma, similarly to the previous one, is an easy consequence of Theorem 2.3 and Proposition 2.1. □

3 Hereditary radicals

An analogue of Lemma 2.6 for hereditary and for strongly hereditary radicals one can easily obtain by replacing in this lemma the class $\mathfrak{C}$ by $\mathcal{H}\mathfrak{C}$ or by $\mathcal{H}^s\mathfrak{C}$ respectively, and using the following observation:

Lemma 3.1. *Let $\mathfrak{C}$ be a class of groups.*

(i) *If $\mathfrak{C}$ is (strongly) hereditary, then $\mathcal{E}\mathfrak{C}$ is (strongly) hereditary too;*

(ii) *If $\mathfrak{C}$ is closed under homomorphisms, then $\mathcal{E}\mathfrak{C}$ is closed under homomorphisms too.*

As a consequence of Proposition 2.1 and isomorphism theorems we have the following characterization of hereditary and strongly hereditary radical classes:

Proposition 3.2. *A radical class $\mathfrak{R}$ is hereditary if and only if $H \cap \mathfrak{R}(G) = \mathfrak{R}(H)$ for every normal subgroup H of every group G.*

Proposition 3.3. *A radical class $\mathfrak{R}$ is strongly hereditary if and only if $H \cap \mathfrak{R}(G) \subseteqq \mathfrak{R}(H)$ for every subgroup H of every group G.*

Now we give a full solution of Problem 1 for hereditary and for strongly hereditary radicals. For this purpose, for a class $\mathfrak{T}$ of simple groups let $\mathfrak{F}_{\mathfrak{T}}$ denotes the class of all groups with all composition factors in $\mathfrak{T}$.

Theorem 3.4. *Let $\mathfrak{R}$ be a radical class. Then the following conditions are equivalent:*

(i) $\mathfrak{R}$ *is a hereditary radical;*

(ii) $\mathfrak{R}$ *is the lower radical of a hereditary class;*

(iii) $\mathfrak{R}$ *is the lower radical of a class of simple groups;*

(iv) $\mathfrak{R} = \mathfrak{F}_{\mathfrak{T}}$ *where* $\mathfrak{T}$ *is the class of all simple* $\mathfrak{R}$*-groups.*

Proof. (i) $\Rightarrow$ (iv). Let $\mathfrak{R}$ be a hereditary radical and let $G \in \mathfrak{R}$. Then, by induction on $|G|$, one can check that $G \in \mathfrak{F}_{\mathfrak{T}}$, where $\mathfrak{T}$ is the class of all simple $\mathfrak{R}$-groups. The inclusion $\mathfrak{F}_{\mathfrak{T}} \subseteq \mathfrak{R}$ is evident.

The implications (iv) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (ii) are obvious.

(ii) $\Rightarrow$ (i). Let $\mathfrak{X}$ be a hereditary class and $\mathfrak{R} = \mathcal{L}\mathfrak{X}$. Then $\mathfrak{R}$ is hereditary by Theorem 2.3 and Lemma 3.1. $\square$

For strongly hereditary radicals we have a characterization that needs some more notions. If H is a subgroup of a group G and $N \triangleleft H$ then the factor group H/N is called *a section of* G and this section is *simple* if H/N is a simple group. Let us agree that a class $\mathfrak{T}$ of groups is *full* if for every group $G \in \mathfrak{T}$ every simple section of G also belongs to $\mathfrak{T}$.

Theorem 3.5. *Let* $\mathfrak{R}$ *be a radical class. The following conditions are equivalent:*

(i) $\mathfrak{R}$ *is a strongly hereditary radical;*

(ii) $\mathfrak{R}$ *is a lower radical of a strongly hereditary class of groups;*

(iii) $\mathfrak{R} = \mathfrak{F}_{\mathfrak{T}}$ *where* $\mathfrak{T}$ *is a full class of simple groups;*

(iv) *The class* $\mathfrak{T}$ *of all simple* $\mathfrak{R}$*-groups is full and* $\mathfrak{R} = \mathfrak{F}_{\mathfrak{T}}$.

Proof. (i) $\Rightarrow$ (iv). Let $\mathfrak{T}$ be the class of all simple $\mathfrak{R}$-groups. By Theorem 3.4 $\mathfrak{R} = \mathfrak{F}_{\mathfrak{T}}$.

Let $G \in \mathfrak{T}$, $H \leq G$ be a subgroup and H/N be a simple section of G for a maximal normal subgroup N of H. Then, by assumption, H and then H/N are $\mathfrak{R}$-groups, and $H/N \in \mathfrak{T}$. This means that the class $\mathfrak{T}$ is full.

(iv) $\Rightarrow$ (iii) is an obvious implication.

(iii) $\Rightarrow$ (ii) Let a class $\mathfrak{T}$ satisfies the assumed conditions and let $\overline{\mathfrak{T}} = \mathcal{H}^s\mathfrak{T}$. Then $\overline{\mathfrak{T}}$ is strongly hereditary. Using the assumption on fullness of $\mathfrak{T}$ and composition series it is not hard to check that $\overline{\mathfrak{T}} \subseteq \mathfrak{F}_{\mathfrak{T}} = \mathfrak{R}$. Hence $\mathcal{L}(\overline{\mathfrak{T}}) = \mathfrak{R}$, as required.

(ii) $\Rightarrow$ (i) This implication is true even on categorical level and follows from Theorem 2.3 and Lemma 3.1. $\square$

In connection with the above result the following problem seems to be interesting.

Problem 2. Is a class of simple groups $\mathfrak{T}$ full if and only if for any $G \in \mathfrak{T}$ all simple subgroups of G also belong to $\mathfrak{T}$? In other words, let G be a simple group and $G_1 = H/N$ be simple section of G for $G \neq H$. Is G_1 isomorphic to a subgroup of G ?

If, under the above notation, $H/N \simeq C_p$ then the answer is yes by a theorem of Cauchy. Some other special cases were in fact considered in [2].

Example 3.6. The class $\mathfrak{D}$ of all solvable groups is a strongly hereditary radical class. It is also a semisimple class closed under taking quotient images. If G is a nonabelian simple group and H is a nontrivial cyclic subgroup of G, then $1 = H \cap \mathfrak{D}(G) \neq \mathfrak{D}(H) = H$. Hence, in Proposition 3.3 only inequality can be used.

Example 3.7. Let π be a set of prime numbers. Then the class $\mathfrak{F}_\pi$ of all π-groups is a strongly hereditary radical class. The class $\mathfrak{F}_\pi$ is also a semisimple class closed under taking quotient images. If, in particular, $\pi = \mathbb{P}$ then $\mathfrak{F}_\pi = \mathfrak{F}$.

A group G is called *π-separable* if every composition factor of G is either a π-group or a π'-group. A group G is called *π-solvable* if every composition factor of G is either a p-group with $p \in \pi$ or a π'-group. These classes are usually considered outside of radical theory.

Example 3.8. Let π be a set of prime numbers. Then the class of all π-separable (π-solvable) groups is a strongly hereditary radical class. It is also a semisimple class closed under taking homomorphic images.

Recall that a group G is called *perfect* if it coincides with its derived group G'.

Example 3.9. The class $\mathfrak{P}$ of all perfect groups is a radical class (see [4]) and $\mathfrak{P} = \mathcal{U}(\mathfrak{D})$. It is not a hereditary radical class since the special linear group $SL(2,5)$ is perfect but $Z\big(SL(2,5)\big)$ is cyclic of order 2, hence nonperfect.

4 Some constructions

In the sequel we'll concentrate on problems, which were extensively studied and led to interesting results in the class of associative rings and of all groups.

Let's begin by the Tangeman-Kreiling lower radical construction. Let $\mathfrak{C}$ be a class of groups. We define $\mathcal{L}_{[0]}\mathfrak{C} = \mathcal{Q}\mathfrak{C}$. If $\mathcal{L}_{[n-1]}\mathfrak{C}$ has been defined for $0 < n < \infty$, then we put $\mathcal{L}_{[n]}\mathfrak{C} = \mathcal{E}(\mathcal{L}_{[n-1]}\mathfrak{C})$.

Now we recall Kurosh's construction of lower radicals. For a class $\mathfrak{C}$ of groups let $\mathcal{L}_0\mathfrak{C} = \mathfrak{Q}\mathfrak{C}$. If $\mathcal{L}_{n-1}\mathfrak{C}$ has been defined for $0 < n < \infty$, than let $\mathcal{L}_n\mathfrak{C}$ be the class of all groups G such that every nontrivial homomorphic image of G contains a nontrivial normal subgroup from the class $\mathcal{L}_{n-1}\mathfrak{C}$. By induction on n one can check that $\mathcal{L}_{[n]}\mathfrak{C} \subseteq \mathcal{L}_n\mathfrak{C} \subseteq \mathcal{L}\mathfrak{C}$ for every n. From this observation, Theorem 2.3 and Lemma 2.6 one can prove

Theorem 4.1. *For any class $\mathfrak{C}$ we have $\mathcal{L}\mathfrak{C} = \cup_{n=0}^{\infty}\mathcal{L}_{[n]}(\mathfrak{C}) = \cup_{n=0}^{\infty}\mathcal{L}_n(\mathfrak{C})$.*

The dual constructions are also possible. For any class $\mathfrak{C}$ let $\mathcal{M}_{[0]}\mathfrak{C} = \mathcal{H}\mathfrak{C}$, and $\mathcal{M}_{[n]}\mathfrak{C} = \mathcal{E}(\mathcal{M}_{[n-1]}\mathfrak{C})$ for $n > 0$. Then, with the help of Lemma 2.7, it can be checked that $\mathcal{M}\mathfrak{C} = \cup_{n=0}^{\infty}\mathcal{M}_{[n]}\mathfrak{C}$.

Now let $\mathcal{M}_0\mathfrak{C} = \mathcal{H}\mathfrak{C}$ and for every $n > 0$ let $\mathcal{M}_n\mathfrak{C}$ be the class of all groups G such that every nontrivial normal subgroup of G can be homomorphically mapped onto a nontrivial group from the class $\mathcal{M}_{n-1}\mathfrak{C}$. In this case, from Lemma 2.7, we in fact have $\mathcal{M}\mathfrak{C} = \mathcal{M}_1\mathfrak{C}$.

Example 4.2. For a prime p let $\mathfrak{C} = (C_p)$. Then $\mathcal{L}\mathfrak{C}$ is the class of all p-groups. If G_n is an elementary Abelian p-group of rank 2^n then, by induction on n, one can check that $G_n \in \mathcal{L}_{[n]}\mathfrak{C}$ and n is minimal with this property. On the other hand, $G_n \in \mathcal{L}_1\mathfrak{C}$ but not to $\mathcal{L}_0\mathfrak{C}$.

From the point of view of lower semisimple classes we have $G_n \in \mathcal{M}_{[n]}\mathfrak{C}$ and n is minimal with this property, but $G_n \in \mathcal{M}_1\mathfrak{C}$, and not to $\mathcal{M}_0\mathfrak{C}$.

The following natural question, formulated for all groups by B.J. Gardner in [6] should be asked in this place:

Problem 3. Let $0 < n < \infty$. Does there exist a homomorphically closed class $\mathfrak{C}$ such that $\mathcal{L}_n(\mathfrak{C}) = \mathcal{L}(\mathfrak{C})$ and n is the smallest with this propery?

Example 4.3. Let $\mathfrak{D}$ be the class of all solvable groups. Our groups are finite. Hence, for the class $\mathfrak{C}$ of cyclic groups we have $\mathcal{L}\mathfrak{C} = \mathfrak{D} = \mathcal{L}_2\mathfrak{C}$, and for the class $\mathfrak{A}$ of abelian groups we have $\mathcal{L}\mathfrak{A} = \mathfrak{D} = \mathcal{L}_1\mathfrak{A}$.

Remark. It is known (see [5, §1.16]) that in the case of all groups $\mathcal{L}_3\mathfrak{C} \neq \mathcal{L}_2\mathfrak{C}$, where $\mathfrak{C}$ is the class of all cyclic groups.

Let us go back to our universal class $\mathfrak{F}$. Keeping in mind the above remark we can formulate the following question related to Problem 3:

Problem 4. Does there exist a class $\mathfrak{C}$ such that $\mathcal{L}_3\mathfrak{C} \neq \mathcal{L}_2\mathfrak{C}$?

As a partial solution of this problem and a generalization of Example 4.2 we have

Theorem 4.4. *If $\mathfrak{C}$ is a hereditary class, then $\mathcal{L}_2\mathfrak{C} = \mathcal{L}_3\mathfrak{C} = \mathcal{L}\mathfrak{C}$.*

Proof. By Lemmas 2.6 and 3.1 $\mathcal{L}\mathfrak{C}$ is a hereditary radical and every simple $\mathcal{L}\mathfrak{C}$-group belongs to $\mathcal{L}_0\mathfrak{C}$.

Now let $G \in \mathcal{L}\mathfrak{C}$ and let $1 \neq H$ be a nontrivial homomorphic image of G. If N is a minimal normal subgroup of H then, by assumption, N is an $\mathcal{L}\mathfrak{C}$-group. On the other hand, from Theorem 1.1 we know that N is a direct product of isomorphic simple group. Hence $N \in \mathcal{L}_1\mathfrak{C}$. This means that $G \in \mathcal{L}_2\mathfrak{C}$. The other needed inclusions are obvious. □

The example 4.3 shows that in the above theorem $\mathcal{L}\mathfrak{C}$ need not be equal to $\mathcal{L}_1\mathfrak{C}$.

As in ring case let us agree that a group G is *unequivocal* if $\mathfrak{R}(G) = G$ or $\mathfrak{R}(G) = 1$ for every radical $\mathfrak{R}$.

Theorem 4.5. *A group G is unequivocal if and only if all composition factors of G are isomorphic.*

Proof. Assume that G is a group such that all its composition factors are isomorphic to a simple group A, and suppose that $\mathfrak{R}$ is a radical class. If $A \in \mathfrak{R}$ then, by Lemma 2.6, $G \in \mathfrak{R}$, and $\mathfrak{R}(G) = G$. If $A \notin \mathfrak{R}$, then $A \in \mathcal{S}\mathfrak{R}$ and by Lemma 2.7, $G \in \mathcal{S}\mathfrak{R}$. Hence $\mathfrak{R}(G) = 1$ and G is unequivocal.

Assume now that $1 \neq G$ is an unequivocal group. Let A be a minimal subnormal subgroup of G. Then, by Theorem 1.1, A is a simple group and is a composition factor of G. Put $\mathfrak{R} = \mathcal{L}(A)$. Then, by Lemma 2.6, $\mathfrak{R}(G) \neq 1$. Hence, by assumption, $\mathfrak{R}(G) = G$. In this way, by the definition of $\mathfrak{R}$, it follows that all composition factors of G are isomorphic to A. □

More generally, one can ask for a given group G and a subgroup $H \leq G$, whether H is a radical in G. An answer to this question is not difficult. We have

Proposition 4.6. *A subgroup $H \leq G$ is a radical in G if and only if H is normal and there is no nontrivial homomorphism $\varphi : H \longrightarrow G/H$ such that $\varphi(H)$ is subnormal in G/H.*

Next question considered on the level of associative rings is to describe all classes being radical and semsimple. As an easy consequence of Theorem 3.4, Theorem 2.3, Lemma 2.6 and Lemma 2.7 we have

Theorem 4.7. *Let $\mathfrak{R}$ be a class of groups. The following conditions are equvalent:*

(i) $\mathfrak{R}$ *is a radical semisimple class;*

(ii) $\mathfrak{R}$ *is a hereditary radical class;*

(iii) $\mathfrak{R}$ *is a semisimple class closed under homomorphisms;*

(iv) *There exists a class* $\mathfrak{T}$ *of simple groups, such that* $\mathfrak{R} = \mathcal{L}\mathfrak{T}$*;*

(v) $\mathfrak{R} = \mathfrak{F}_{\mathfrak{T}}$ *for a class* $\mathfrak{T}$ *of simple groups;*

(vi) $\mathfrak{R} = \mathcal{H}\mathfrak{R} = \mathcal{Q}\mathfrak{R} = \mathcal{E}\mathfrak{R}$.

5 Wreath product

Let $F \leqslant G$ be groups. Then G will be called an *outer extension* of F if $F \lhd G$ and for every $x \in G \setminus F$ the automorphism of F induced by conjugation of G by x is an outer automorphism of F.

Lemma 5.1. *Let* $F \lhd G$ *be groups. Then* G *is an outer extension of* F *if and only if* $C_G(F) = Z(F)$.

Proof. $\Rightarrow$ Let $c \in C_G(F)$. Then the inner automorphism of G induced by c is trivial on F, thus it is inner. Hence, by assumption, $c \in F$. By the choice of c we then have $C_G(F) \subseteq Z(F)$. The converse inclusion is trivial.

$\Leftarrow$ Let G be not an outer extension of F and let $x \in G \setminus F$ be such that the inner automorphism σ_x induced by x is inner on F and is induced by $f \in F$. Then $f^{-1}x$ induces the identity on F. Thus $f^{-1}x \in C_G(F)$. On the other hand $f^{-1}x \notin F$. Hence $C_G(F) \neq Z(F)$. □

In the sequel we are going to use a type of outer extensions, namely regular wreath products. We recall a definition of it here.

The (regular) wreath product of groups A by B is defined as follows: Let $A^B = F$ be the group of all functions from B to A with natural pointwise multiplication. Then F is the direct product of $|B|$ isomorphic copies of A. Now let $b \in B$. If $f \in F$, define $\sigma_b(f) = f^b$ by

$$f^b(x) = f(xb^{-1}) \text{ for all } x \in B.$$

The set of automorphisms $\{\sigma_b : b \in B\}$ is a group isomorphic to B in a natural way. We shall identify these groups. *The wreath product* $W = A \wr B$ *of* A *by* B is the semidirect product of F by this group of automorphisms; that is, $W = BF$ with the relations

$$bfcg = bcf^c g \text{ for all } b, c \in B \text{ and } f, g \in F.$$

This semidirect product is an outer extension of F. We shall refer to F as the *base group* of W. Further information on the wreath product can be found in [3, 10]. We will use the following observation, a consequence of Lemma 5.1.

Proposition 5.2. *Let A and B be nontrivial groups and let $W = A \wr B$. If $1 \neq N \lhd W$, then N intersects F nontrivially.*

Proof. Let $1 \neq N \lhd W$. If $N \cap F = 1$, then with the help of the assumption $N \subseteq C_W(F)$. However, by definition, W is an outer extension of $F = A^B$. Hence, by Lemma 5.1 $\quad 1 \neq N \subseteq Z(F) \subseteq F$. □

As a consequence we can connect radicals with wreath products.

Lemma 5.3. *Let A, B be groups and $\mathfrak{R}$ be a radical class. Then*

(i) *If $A, B \in \mathfrak{R}$, then $A \wr B \in \mathfrak{R}$.*

(ii) *If $A, B \in \mathcal{S}\mathfrak{R}$, then $A \wr B \in \mathcal{S}\mathfrak{R}$.*

(iii) *If $A \in \mathfrak{R}$ and $B \in \mathcal{S}\mathfrak{R}$, then $\mathfrak{R}(A \wr B)$ is the base group of $A \wr B$.*

(iv) *If $\mathfrak{R}$ is hereditary, $1 \neq A \in \mathcal{S}\mathfrak{R}$ and $B \in \mathfrak{R}$, then $A \wr B \in \mathcal{S}\mathfrak{R}$.*

Proof. The first three parts follow by Theorem 2.3. The last part follows by Proposition 5.2. □

Using wreath product we can give the following result

Example 5.4. Let G be a nonabelian simple group and let $\mathfrak{C} = (G)$. If we take $W = G \wr G$, then one can see that $W \in \mathcal{L}_{[2]}\mathfrak{C} \subseteq \mathcal{L}_2\mathfrak{C}$, but has no normal subgroup isomorphic to G. Hence, $W \notin \mathcal{L}_{[1]}\mathfrak{C}$ and even $W \notin \mathcal{L}_1\mathfrak{C}$. In this way we see that in Theorem 4.4 we can not replace $\mathcal{L}_2\mathfrak{C}$ by $\mathcal{L}_1\mathfrak{C}$.

The group W also shows that in Lemma 2.6(ii), subnormal series can not be replaced by normal series and in (iii) of the same lemma subnormal subgroups can not be replaced by normal subgroups.

In general, for a radical class $\mathfrak{R}$, $\mathfrak{R}(A)$ does not always contain all $\mathfrak{R}$-subgroups of A. We now consider radical classes for which this is the case.

Let us agree (see [6]) that a radical class $\mathfrak{R}$ is *strict* if $B \leqslant \mathfrak{R}(A)$ whenever $B \in \mathfrak{R}$ and $B \leqslant A$. The following characterization of strict radicals is well known, and is an easy consequence of results from Section 2:

Proposition 5.5. *A radical class $\mathfrak{R}$ is strict if and only if its semisimple class $\mathcal{S}\mathfrak{R}$ is strongly hereditary. If in particular $\mathfrak{X}$ is a strongly hereditary class of groups, then its upper radical is strict.*

Example 5.6. The class $\mathfrak{P}$ of all perfect groups is a strict radical class since $\mathcal{S}\mathfrak{P} = \mathfrak{D}$ is the class of all solvable groups.

As on the level of all groups (see [5, 6]) we have

Theorem 5.7. *The only hereditary, strict radical classes are* $\{1\}$ *and the whole class* $\mathfrak{F}$.

Proof. Let $\mathfrak{R} \neq \{1\}$ be a strict hereditary radical class. Due to Jordan-Hölder theorem it is enough to prove that $\mathfrak{S} \subseteq \mathfrak{R}$.

Using the assumption and Theorem 3.4 we have a simple $\mathfrak{R}$-group, say B. If $\mathfrak{S} \nsubseteq \mathfrak{R}$, then let A be a simple $\mathfrak{R}$-semisimple group. Then $A \wr B \in \mathcal{S}\mathfrak{R}$ by Lemma 5.3. Since $\mathfrak{R}$ is strict then by Proposition 5.5 it follows that $B \in \mathcal{S}\mathfrak{R}$ and we have a contradiction with the choice of B. □

Remark. The above theorem was proved by B.J. Gardner in the universal class of all groups. He used a free product of groups in the proof (see [5]).

We already know from Section 2, that for any subset $\pi \subseteq \mathbb{P}$, the class $\mathfrak{F}_\pi$ is strongly hereditary and semisimple. Hence by Proposition 5.5, the upper radical of the class $\mathfrak{F}_\pi$ is strict with the semisimple class $\mathfrak{F}_\pi$.

Theorem 5.8. *For any subset* $\pi \subseteq \mathbb{P}$ *let* $\mathfrak{D}_\pi$ *be the class of all solvable* π*-groups. Then:*

(i) *The class* $\mathfrak{D}_\pi$ *is strongly hereditary and semisimple;*

(ii) *The upper radical of the class* $\mathfrak{D}_\pi$ *is strict;*

(iii) *If* $\mathfrak{R}$ *is a strict radical with the semisimple class* $\mathcal{S}\mathfrak{R} \subseteq \mathfrak{D}$ *then there exists a subset* $\pi \subseteq \mathbb{P}$ *such that* $\mathcal{S}\mathfrak{R} = \mathfrak{D}_\pi$.

Proof. The statements (i) and (ii) follow directly from the equality $\mathfrak{D}_\pi = \mathfrak{D}_\pi \cap \mathfrak{F}_\pi$ and Proposition 5.5.

(iii) Let $\mathfrak{R}$ be a strict radical with $\mathcal{S}\mathfrak{R} \subseteq \mathfrak{D}$ and π be the set of all primes p with $C_p \in \mathcal{S}\mathfrak{R}$. Then by Proposition 5.5 $\mathfrak{D}_\pi \subseteqq \mathcal{S}\mathfrak{R}$.

Now let $G \in \mathcal{S}\mathfrak{R}$, and $p \mid |G|$. By Cauchy's Theorem there exists a subgroup H of G such that $H \cong C_p$. Then $C_p \in \mathcal{S}\mathfrak{R}$, because $\mathcal{S}\mathfrak{R}$ is a strongly hereditary class. Thus $p \in \pi$, and by definition $G \in \mathfrak{D}_\pi$. Hence $\mathcal{S}\mathfrak{R} = \mathfrak{D}_\pi$ as required. □

As a consequence we obtain that there exists $2^{\aleph_0}$ strict radicals in the class $\mathfrak{F}$.

Proposition 5.9. *Let* $\mathfrak{R}$ *be a strict radical class.*

(i) *If* $C_2 \in \mathfrak{R}$, *then there exists a set of odd numbers* $\pi \subseteq \mathbb{P}$ *such that* $\mathcal{S}\mathfrak{R} = \mathfrak{D}_\pi$.

(ii) *If for some* $n \geqslant 1 \quad A_n \in \mathfrak{R}$, *then* $A_k \in \mathfrak{R}$ *for all* $k \geqslant n$.

(iii) *If* $A_n \in \mathcal{S}\mathfrak{R}$, *then* $A_k \in \mathcal{S}\mathfrak{R}$ *for all* $k \leqslant n$. *In particular* $\mathfrak{F}_\pi \subseteq \mathcal{S}\mathfrak{R}$ *with* $\pi = \{p \in \mathbb{P} \mid p \leqslant n\}$.

(iv) *If* $A_n \in \mathcal{S}\mathfrak{R}$ *for all* $n \geqslant 5$ *then* $\mathcal{S}\mathfrak{R} = \mathfrak{F}$.

Proof. (i) By famous Theorem of Feit and Thompson every group of odd order is solvable. Hence, by Proposition 5.5 $\mathcal{S}\mathfrak{R} \subseteq \mathfrak{D}$ and by Theorem 5.8(iii) there exists a subset $\pi \subseteq \mathbb{P}$ such that $\mathcal{S}\mathfrak{R} = \mathfrak{D}_\pi$.

(ii)-(iii) It is evident because $S_k \leqslant S_{k+1}$ for every $k \geqslant 1$ and $C_p \leqslant S_k$ for all $p \leqslant k$.

(iv) It is evident by Caylay's embedding Theorem. □

If either $\mathfrak{T} = (\mathfrak{S})$ or $\mathfrak{T} = \{1\}$ then, by the existence of composition series, the associated radical $\mathcal{L}\mathfrak{T}$ is trivial. This is not true in the class of all groups and in the class of associative rings.

It is known on the level of rings and of all groups, that every partition of simple objects gives at least two radicals related to this partition. Similar result, up to the exception mentioned above, is true for finite groups.

Theorem 5.10. *Let* $\mathfrak{T} \subset \mathfrak{S}$ *be a proper subclass of simple groups. Then the lower radical* $\mathcal{L}\mathfrak{T}$ *is different from the upper radical* $\mathcal{U}(\mathfrak{S} \setminus \mathfrak{T})$.

Proof. Let $A \in \mathfrak{T}$ and $B \in \mathfrak{S} \setminus \mathfrak{T}$. If $\mathfrak{R} = \mathcal{L}\mathfrak{T}$ then, by Lemma 5.3(iv), the group $B \wr A \in \mathcal{S}\mathfrak{R}$. Hence, $\mathfrak{R}(B \wr A) = 1$.

On the other hand, if $\mathfrak{U} = \mathcal{U}(\mathfrak{S} \setminus \mathfrak{T})$, then $\mathfrak{U}(B \wr A) = B \wr A$, because there is no homomorphism of $B \wr A$ onto a group from $\mathfrak{S} \setminus \mathfrak{T}$. □

As a consequence of earlier results we have

Theorem 5.11. *There exists a nonhereditary radical* $\mathfrak{R} \subseteq \mathfrak{D}$. *Every hereditary radical* $\mathfrak{R} \subseteq \mathfrak{D}$ *is strongly hereditary.*

Proof. Let $\mathfrak{C} = \{S_3, C_2\}$. Then the radical $\mathcal{L}\mathfrak{C}$ is not hereditary, because $C_3 \notin \mathcal{L}\mathfrak{C}$.

The second claim follows immediately from Theorem 3.5, because solvable simple groups are cyclic of prime order, hence they form a full class. □

One can easily observe, that the class $\mathfrak{N}$ of all nilpotent groups is neither radical nor semisimple, because it is not closed under extensions. However, as an easy consequence of the structure of nilpotent groups we have

Theorem 5.12. *The only nontrivial radicals* $\mathfrak{R} \subseteq \mathfrak{N}$ *are classes* $\mathfrak{F}_p$, *where* $p \in \mathbb{P}$.

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Jan Krempa
Institute of Mathematics
University of Warsaw
ul. Banacha 2
02-097 Warszawa
Poland
jkrempa@mimuw.edu.pl

Izabela Agata Malinowska
Institute of Mathematics
University of Białystok
ul. Akademicka 2
15-267 Białystok
Poland
izabelam@math.uwb.edu.pl