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BLOW-UP BOUNDARY SOLUTIONS FOR QUASILINEAR ANISOTROPIC EQUATIONS

Manuel Amzoiu

Abstract

This article refers to the study of the equation $\Delta_p\ u = m(x)f(u)$. Our aim is to find the conditions for f and m in which the equation has at least a positive solution and in which case the solution is large.

1 Introduction

In this paper we consider the following equation

$$\begin{cases} \Delta_p\ u = m(x)f(u) & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega, \end{cases} \tag{1}$$

where $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ is the Laplace operator and $\Omega \in R^N$ is a smooth domain(bounded or unbounded) with a compact boundary. Throughout this paper we assume that m is a non-negative function with $m \in C^{0,\alpha}(\bar{\Omega})$ if Ω is bounded, and $m \in C^{0,\alpha}_{loc}(\Omega)$ if Ω is unbounded. The non-decreasing non-linearity f fulfills

$$(f1)\ f \in C^1[0,\infty), f' \geq 0, f(0) = 0, f > 0 \text{ in } (0,\infty) \text{ and } \sup_{s\in(0,1]} \frac{f(s)^{\frac{1}{p-1}}}{s} < \infty,$$

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$(f2)$ $\int_1^\infty [F(t)]^{-\frac{1}{p}} dt < \infty$ where $F(t) = \int_0^t f(s)ds$,
$(f3)$ $\frac{f(x)}{(x+\beta)^{p-1}}$ is non-decreasing, for some $\beta \in R$.

A solution u to the problem (1) is called *large* (*explosive*, *blow − up*) if $u(x) \to \infty$ as $\text{dist}(x, \partial\Omega) \to 0$ (when Ω is bounded). In the case of $\Omega = R^N$ we call u an *entire large* (*explosive*) solution and the condition can be written $u(x) \to \infty$ as $|x| \to \infty$.

Remark 1.1. *The case $p = 2$ has been intensively studied for different forms of f. The results of this article extend the work of Cîrstea and Radulescu from [5] where most of the results, especially the uniqueness, are proved using the linearity of Δ. The case of Δ_p raises some problems mainly because it is not linear. We overcome this problems by using a special technique developed by Covei in [8].*

The paper is organized as follows: in Section 2, we present the main results as theorems and the proofs of theorems are given in Section 3.

2 The main results

Theorem 2.1. *Let Ω be a bounded domain. Assume that f satisfies the conditions $(f1), (f2), (f3)$, $m \in C^{0,\alpha}(\bar{\Omega})$ and $g : \partial\Omega \to (0, \infty)$ is a continuous function. Then the problem*

$$\begin{cases} \Delta_p \ u = m(x)f(u) & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \\ u \geq 0 & \text{in } \Omega \end{cases} \tag{2}$$

has a unique positive solution.

Theorem 2.2. *Consider Ω to be a bounded domain and m satisfies the next condition*
$(m1)$ for every $x_0 \in \Omega$ with $m(x_0) = 0$, there exists a domain Ω_0 which contain x_0 such that $\overline{\Omega_0} \subset \Omega$ and $m > 0$ on $\partial\Omega_0$.
Then the problem (1) has a positive large solution .

Theorem 2.3. *Let's assume that the problem (1) has at least one solution for $\Omega = R^N$. If m satisfies the modified condition*
(m1)' there exists a sequence of smooth bounded domains $(\Omega_n)_{n\geq 1}$ such that $\overline{\Omega_n} \subset \Omega_{n+1}$, $R^N = \bigcup_{n=1}^\infty \Omega_n$ and (m1) holds in Ω_n, for every $n \geq 1$,
then a maximal solution U of (1) exists.
If m satisfies the additional condition
$(m2)$ $\int_0^\infty r\Phi(r)dr < \infty$ where $\Phi(r) = \max_{|x|=r} m(x)$,
then U is an entire large solution.

Theorem 2.4. *If the problem (1) has at least a solution for a unbounded $\Omega \neq R^N$ and m satisfies (m1)', then there exists a maximal solution U for the problem (1). If m satisfies (m2), with $\Phi(r) = 0$ for $r \in [0, R]$ and $\Omega = R^N \setminus \overline{B(0, R)}$, then U is a large solution that blows-up at infinity.*

3 Proof of results

3.1 Proof of Theorem 2.1

For start it is easy to observe that the function $u^+(x) = n$ is a super-solution for the problem (2), when n is sufficiently large. In order to find a subsolution, we consider an auxiliary problem:

$$\Delta_p v = \Phi(r), v > 0 \text{ in } A(\underline{r}, \overline{r}) = \{x \in R^N, \underline{r} < |x| < \overline{r}\}, \tag{1}$$

where

$$\underline{r} = \inf\{\tau > 0; \partial B(0, \tau) \bigcap \overline{\Omega} \neq \emptyset\}, \overline{r} = \sup\{\tau > 0; \partial B(0, \tau) \bigcap \overline{\Omega} \neq \emptyset\},$$

$$\Phi(r) = \max_{|x|=r} m(x) \text{ for any } r \in [\underline{r}, \overline{r}].$$

The assumptions on f and g imply

$$g_0 = \min_{\partial\Omega} \ g > 0 \text{ and } \lim_{z \searrow 0} \int_z^{g_0} \frac{dt}{f(t)^{\frac{1}{p-1}}} = \infty.$$

Using these relations, we prove the existence of a positive number c such that

$$\max_{\partial\Omega} \ v = \int_c^{g_0} \frac{dt}{f(t)^{\frac{1}{p-1}}}. \tag{2}$$

Now we can define u_- such that

$$v(x) = \int_c^{u_-(x)} \frac{dt}{f(t)^{\frac{1}{p-1}}}, \quad \text{for all } x \in \Omega. \tag{3}$$

Next we are going to prove that u_- is a subsolution. First we observe that

$$u_- \in C^{1,\alpha}(\Omega) \text{ and } u_- \geq c \text{ in } \Omega.$$

The way that u_- is defined let us say that

$$\nabla v = \frac{1}{f(u_-)^{\frac{1}{p-1}}} \cdot \nabla u_-.$$

It means

$$\nabla v|\nabla v|^{p-2} = \frac{1}{f(u_-)} \cdot \nabla u_-|\nabla u_-|^{p-2}.$$

Using the formula

$$div(u\vec{v}) = \nabla u\vec{v} + udiv\vec{v}$$

we find that

$$\Delta_p v = div(\nabla v|\nabla v|^{p-2}) = div(\frac{1}{f(u_-)} \cdot \nabla u_-|\nabla u_-|^{p-2}) =$$

$$-\frac{f'(u_-)}{f^2(u_-)} \cdot |\nabla u_-|^2|\nabla u_-|^{p-2} + \frac{1}{f(u_-)} \cdot \Delta_p u_-$$

and the relation can be written

$$m(x) \leq \Delta_p v \leq \frac{1}{f(u_-)} \cdot \Delta_p u_-.$$

This implies that $\Delta_p u_- \geq m(x)f(u_-)$ and using $u_-(x) \leq g(x)$ it follows that u_- is subsolution. So far we have proved that the equation (1) has a sub- and supersolution which imply that the equation has at least a solution. To complete the proof of this theorem we have to show the uniqueness of the solution .

In order to prove its uniqueness, we consider that the equation (1) has two solutions u and v. It is sufficient to show that $u \leq v$ or, equivalently, $\ln(u(x)+\beta) \leq \ln(v(x)+\beta)$, for any $x \in \Omega$. We assume the contrary. So we have

$$\lim_{|x| \to \partial\Omega} (\ln(u(x)+\beta) - \ln(v(x)+\beta)) = 0$$

and we deduce that

$$\max(\ln(u(x)+\beta) - \ln(v(x)+\beta)) \text{ on } \Omega$$

exists and is positive. We denote this point x_0. At x_0 we have

$$\nabla(\ln(u(x)+\beta) - \ln(v(x)+\beta)) = 0,$$

so

$$\frac{1}{u(x_0)+\beta} \cdot \nabla u(x_0) = \frac{1}{v(x_0)+\beta} \cdot \nabla v(x_0),$$

which implies that

$$\frac{1}{(u(x_0)+\beta)^{p-2}} \cdot |\nabla u(x_0)|^{p-2} = \frac{1}{(v(x_0)+\beta)^{p-2}} \cdot |\nabla v(x_0)|^{p-2}. \tag{4}$$

The condition $(f3)$ yields to

$$\frac{f(u(x_0))}{(u(x_0)+\beta)^{p-1}} > \frac{f(v(x_0))}{(v(x_0)+\beta)^{p-1}}.$$

We observe $0 \geq \Delta(ln(u(x_0)+\beta) - ln(v(x_0)+\beta))$, which yields to

$$\frac{\Delta(u(x_0))}{u(x_0)+\beta} \leq \frac{\Delta v(x_0)}{v(x_0)+\beta}.$$

And by (4) it follows that

$$\frac{1}{(u(x_0)+\beta)^{p-1}} \cdot |\nabla u(x_0)|^{p-2}\Delta u(x_0) \leq \frac{1}{(v(x_0)+\beta)^{p-1}} \cdot |\nabla v(x_0)|^{p-2}\Delta v(x_0). \tag{5}$$

Since

$$|\nabla \ln(u(x_0)+\beta)|^{p-2} = \frac{1}{(u(x_0)+\beta)^{p-2}} \cdot |\nabla u(x_0)|^{p-2},$$

it results that

$$\nabla(|\nabla \ln(u(x_0)+\beta)|^{p-2}) = -(p-2)\frac{|\nabla u(x_0)|^{p-2}(u(x_0)+\beta)^{p-3}}{(u(x_0)+\beta)^{2(p-2)}} \cdot \nabla u(x_0) + \frac{\nabla(|\nabla u(x_0)|^{p-2})}{(u(x_0)+\beta)^{p-2}}.$$

We conclude that

$$\nabla(|\nabla \ln(u(x_0)+\beta)|^{p-2}) \cdot \nabla \ln(u(x_0)+\beta) = -(p-2)\frac{|\nabla u(x_0)|^{p-2}|\nabla u(x_0)|^2}{(u(x_0)+\beta)^{p}} + \frac{\nabla(|\nabla u(x_0)|^{p-2}) \cdot \nabla u(x_0)}{(u(x_0)+\beta)^{p-1}} \tag{6}$$

and

$$|\nabla \ln(u(x_0)+\beta)|^{p-2} \cdot \Delta \ln(u(x_0)+\beta) = \frac{|\nabla u(x_0)|^{p-2}\Delta u(x_0)}{(u(x_0)+\beta)^{p-1}} - \frac{|\nabla u(x_0)|^{p}}{(u(x_0)+\beta)^{p}}.$$

By (4), (5) and (6) we have

$$0 \geq \Delta_p \ln(u(x_0)+\beta) - \Delta_p \ln(v(x_0)+\beta)$$
$$= \frac{\Delta_p u(x_0)}{(u(x_0)+\beta)^{p-1}} - (p-1)\frac{|\nabla u(x_0)|^{p}}{(u(x_0)+\beta)^{p}} - \frac{\Delta_p v(x_0)}{(v(x_0)+\beta)^{p-1}} + (p-1)\frac{|\nabla v(x_0)|^{p}}{(v(x_0)+\beta)^{p}}$$
$$= \frac{\Delta_p u(x_0)}{(u(x_0)+\beta)^{p-1}} - \frac{\Delta_p v(x_0)}{(v(x_0)+\beta)^{p-1}} = m(x_0)\left(\frac{f(u(x_0))}{(u(x_0)+\beta)^{p-1}} - \frac{f(v(x_0))}{(v(x_0)+\beta)^{p-1}}\right) > 0$$

and that is a contradiction. Hence $u \leq v$. By symmetry, we also obtain $v \leq u$ and the proof of its uniqueness is now complete.

3.2 Proof of Theorem 2.2

To complete the proof of Theorem 2.2, we need the next auxiliary result

Lemma 3.1. *If the conditions (f1) and (f2) are fulfilled, then*

$$\int_1^\infty \frac{1}{f(t)^{\frac{1}{p-1}}} < \infty$$

Proof. Being a low risk of confusion, we will denote $B = B(0, R)$ for some fixed $R > 0$. By Theorem 2.1, we find that the problem

$$\begin{cases} \Delta_p\ u_n = f(u_n) & \text{in } B \\ u_n = n & \text{on } \partial B \\ u \geq 0 & \text{in } B \end{cases} \tag{7}$$

has a unique solution. The fact that f is non-decreasing implies, by the maximum principle, that $u_n(x)$ increases with n, when $x \in B$ is fix.

The first thing on our agenda is to try to prove that (u_n) is uniformly bounded in every compact subdomain of B. In order to achieve that, let $K \subset B$ be any compact set and $d := \text{ dist}(K, \partial B)$. Then

$$0 < d \leq \text{ dist}(x, \partial B), \text{ for any } x \in K. \tag{8}$$

By Proposition 1 in [1], there exists a continuous, non-increasing function $\mu : R_+ \to R_+$ such that

$$u_n(x) \leq \mu(\text{ dist}(x, \partial B)), \text{ for any } x \in K,$$

and, using (8), the first part of the proof follows. This allows us to define $u(x) := \lim_{n\to\infty} u_n(x)$. The next step is to show that u is a large solution to

$$\Delta_p u = f(u) \text{ in } B. \tag{9}$$

To complete this step we make a change of variables, putting $u(x) = u(r), r = |x|$ and the equation (9) becomes

$$(p-1)(u')^{p-2}u" + (u')^{p-1}\frac{N-1}{r} = f(u).$$

Multiplying this by r^{N-1} the equation can be rewritten

$$(r^{N-1}(u')^{p-1})' = f(u)r^{N-1}. \tag{10}$$

Integrating from 0 to r, we obtain

$$(u')^{p-1} = r^{1-N} \int_0^r f(u(s))s^{N-1},\ 0 < r < R.$$

Taking into account the fact that f is non-decreasing,

$$u' \leq [r^{1-N} f(u(r)) \int_0^r s^{N-1} ds]^{\frac{1}{p-1}} = (\frac{r}{N} f(u))^{\frac{1}{p-1}},\ 0 < r < R. \tag{11}$$

It results that u is a non-decreasing function and, in the same way, that u_n is non-decreasing on $(0, R)$. It remains to prove that $u(r) \to \infty$ as $r \nearrow R$. We achieve that arguing by contradiction, assuming that there exists $C > 0$ such that $u(r) < C$ for all $0 \leq r < R$. Let $N_1 \geq 2C$ be fix. Using the facts that u_{N_1} is monotone and $u_{N_1}(r) \to N_1$ we find $r_1 \in (0, R)$ such that $C \leq u_{N_1}(r)$, for $r \in [0, R)$. Hence

$$C \leq u_{N_1}(r) \leq u_{N_1+1}(r) \leq ... \leq u_n(r) \leq ....$$

Passing to the limit $n \to \infty$, it follows that $u(r) > C$, which is a contradiction. Integrating (11) on $(0, R)$ and taking $r \nearrow R$ we obtain

$$\int_{u(0)}^{\infty} \frac{1}{f(t)^{\frac{1}{p-1}}} \leq \frac{p-1}{pN^{\frac{1}{p-1}}} \cdot R^{\frac{p}{p-1}},$$

which completes the proof of our lemma.

Proof of theorem 2.2. Using Theorem (2.1), the boundary value problem

$$\begin{cases} \Delta_p\ v_n = m(x)f(v_n) & \text{in } \Omega \\ v_n = n & \text{on } \partial\Omega \\ v_n \geq 0 & \text{in } \Omega \end{cases} \tag{12}$$

has a unique positive solution, for any $n \geq 1$. We claim that
(a) for all $x_0 \in \Omega$ there exists an open set $\vartheta \subset\subset \Omega$ containing x_0 and $M_0 = m_0(x_0) > 0$ such that $v_n \leq M_0$ in ϑ, for any $n \geq 1$;
(b)$\lim_{x \to \partial\Omega} v(x) = \infty$, where $v(x) = \lim_{n\to\infty} v_n(x)$.

The first thing to be observed is that the sequence v_n is non-decreasing.Using again the Theorem (2.1), the problem

$$\begin{cases} \Delta_p\ \zeta = ||m||_\infty f(\zeta) & \text{in } \Omega \\ \zeta = 1 & \text{on } \partial\Omega \\ \zeta > 0 & \text{in } \Omega \end{cases} \tag{13}$$

has a unique solution. Then we obtain with the maximum principle

$$0 < \zeta \leq v_1 \leq v_2 \leq ... \leq v_n \leq ... \text{ in } \Omega.$$

We observe that (a) and (b) are sufficient for completing the proof. From (a) we obtain that the sequence (v_n) is uniformly bounded on every compact subset of Ω. Then, with the latest relation and (b), we prove that v is a solution.

To prove (a) we distinguish two cases:

Case $m(x_0) > 0$: By the continuity of m, there exists a ball $B = B(x_0, r) \subset \Omega$ such that

$$m_0 := \min_{x \in \overline{B}} m(x) > 0.$$

Let w be a positive solution to the problem

$$\begin{cases} \Delta_p \, w = m_0 f(w) & \text{in } \Omega \\ w(x) \to \infty & \text{as } x \to \partial\Omega. \end{cases} \tag{14}$$

By the maximum principle, it follows that $v_n \leq w$ in B. Furthermore, w is bounded in $\overline{B(x_0, \frac{r}{2})}$. We denote $M_0 = \sup_\vartheta w$, where $\vartheta = B(x_0, \frac{r}{2})$ and we obtain (a).

Case $m(x_0) = 0$: The boundedness of Ω and (m1) implies that there exists a domain $\vartheta \subset \Omega$, which contains x_0 such that $m > 0$ on $\partial\vartheta$. Then for any $x \in \partial\vartheta$ there exists a ball $B(x, r) \subset \Omega$ and a constant $M_x > 0$ such that $v_n \leq M_x$ on $B(x, \frac{r_x}{2})$, for any n. But $\partial\vartheta$ is compact and it can be covered with a finite number of balls, $B(x_i, \frac{r_{x_i}}{2})$, $i = 1, ..., k_0$. Taking $M_0 = \max(M_{x_1}, ..., M_{x_{k_0}})$ and applying the maximum principle we obtain $v_n \leq M_0$ and (a) follows.

We now consider the problem

$$\begin{cases} \Delta_p \, z = -m(x) & \text{in } \Omega \\ z = 0 & \text{on } \partial\Omega \\ z \geq 0 & \text{in } \Omega \end{cases} \tag{15}$$

that has a unique positive solution (by the maximum principle from [8]). To prove (b) it is sufficient to show

$$\int_{v(x)}^{\infty} \frac{dt}{f(t)^{\frac{1}{p-1}}} \leq z(x), \text{ for any } x \in \Omega. \tag{16}$$

By Lemma 3.1, the left side of (16) is well defined in Ω.

For an easier following of the prof of (16), we denote $\overline{u} = \int_{v_n(x)}^{\infty} f(t)^{-\frac{1}{p-1}} dt$ and $\overline{v} = z(x)$. We want to show that $\overline{u} \leq \overline{v}$ or, equivalently, $\ln(\overline{u}(x) + \beta) \leq \ln(\overline{v}(x) + \beta)$, for any $x \in \Omega$. We assume the contrary. So we have

$$\lim_{|x| \to \partial\Omega} (\ln(\overline{u}(x) + \beta) - \ln(\overline{v}(x) + \beta)) = 0$$

and we deduce that

$$\max(\ln(\overline{u}(x) + \beta) - \ln(\overline{v}(x) + \beta)) \text{ on } \partial\Omega$$

exists and is positive. Let us denote this point x_0. At x_0 we have

$$\nabla(\ln(\overline{u}(x) + \beta) - \ln(\overline{v}(x) + \beta)) = 0,$$

so

$$\frac{1}{\overline{u}(x_0) + \beta} \cdot \nabla\overline{u}(x_0) = \frac{1}{\overline{v}(x_0) + \beta} \cdot \nabla\overline{v}(x_0)$$

which implies

$$\frac{1}{(\overline{u}(x_0) + \beta)^{p-2}} \cdot |\nabla\overline{u}(x_0)|^{p-2} = \frac{1}{(\overline{v}(x_0) + \beta)^{p-2}} \cdot |\nabla\overline{v}(x_0)|^{p-2}. \tag{17}$$

The condition $(f3)$ yields to

$$\frac{f(\overline{u}(x_0))}{(\overline{u}(x_0) + \beta)^{p-1}} > \frac{f(\overline{v}(x_0))}{(\overline{v}(x_0) + \beta)^{p-1}}.$$

Following the same thinking as in the proof of Theorem 2.1 , and taking into account that

$$\Delta_p\overline{u} = div(\nabla\overline{v_n}|\nabla\overline{v_n}|^{p-2}) = div(-\frac{1}{f(v_n)} \cdot \nabla v_n|\nabla v_n|^{p-2}) =$$

$$\frac{f'(v_n)}{f^2(v_n)} \cdot |\nabla v_n|^2|\nabla v_n|^{p-2} - \frac{1}{f(v_n)} \cdot \Delta_p v_n,$$

we have

$$0 \geq \Delta_p ln(\overline{u}(x_0) + \beta) - \Delta_p ln(\overline{v}(x_0) + \beta) =$$

$$= \frac{\Delta_p\overline{u}(x_0)}{(\overline{u}(x_0) + \beta)^{p-1}} - (p-1)\frac{|\nabla\overline{u}(x_0)|^p}{(\overline{u}(x_0) + \beta)^p} - \frac{\Delta_p\overline{v}(x_0)}{(\overline{v}(x_0) + \beta)^{p-1}} + (p-1)\frac{|\nabla\overline{v}(x_0)|^p}{(\overline{v}(x_0) + \beta)^p} =$$

$$= \frac{\Delta_p\overline{u}(x_0)}{(\overline{u}(x_0) + \beta)^{p-1}} - \frac{\Delta_p\overline{v}(x_0)}{(\overline{v}(x_0) + \beta)^{p-1}} =$$

$$= \frac{\frac{f'(v_n(x_0))}{f^2(v_n(x_0))} \cdot |\nabla v_n(x_0)|^2 |\nabla v_n(x_0)|^{p-2} - \frac{1}{f(v_n(x_0))} \cdot \Delta_p v_n(x_0)}{(\overline{u}(x_0)+\beta)^{p-1}} - \frac{\Delta_p z(x_0)}{(\overline{v}(x_0)+\beta)^{p-1}} >$$

$$> \frac{\frac{f'(v_n(x_0))}{f^2(v_n(x_0))} \cdot |\nabla v_n(x_0)|^p - \frac{1}{f(v_n(x_0))} \cdot m(x_0) f(v_n(x_0)}{(\overline{u}(x_0)+\beta)^{p-1}} + \frac{m(x_0)}{(\overline{u}(x_0)+\beta)^{p-1}} =$$

$$= \frac{\frac{f'(v_n(x_0))}{f^2(v_n(x_0))} \cdot |\nabla v_n(x_0)|^p}{(\overline{u}(x_0)+\beta)^{p-1}} > 0$$

and that is a contradiction. Hence the assumption is false and the proof is now complete.

3.3 Proof of Theorem 2.3

Now we consider the following boundary value problem

$$\begin{cases} \Delta_p \; v_n = m(x) f(v_n) & \text{in } \Omega \\ v_n \to \infty & \text{as } x \to \partial\Omega \\ v_n > 0 & \text{in } \Omega \end{cases} . \tag{18}$$

Again, using Theorem 2.1, the above problem has a solution. Since $\overline{\Omega_n} \subset \Omega$ applying the maximum principle we obtain $v_n \geq v_{n+1}$ in Ω_n. Since $R^N = \bigcup_{n=1}^{\infty} \Omega_n$ and $\overline{\Omega_n} \subset \Omega$ it follows that there exists $n_0 = n_0(x_0)$ such that $x_0 \in \Omega_n$ for all $n \geq n_0$ and $x_0 \in R^N$. We can define $U(x_0) = \lim_{n\to\infty} v_n(x_0)$. The regularity of U as in [9] is $U \in C^{1,\alpha}_{loc}(R^N)$ and $\Delta_p U = m(x) f(U)$.
To prove that U is the maximal solution, let u be a arbitrary solution of (1). By the maximum principle, we obtain $v_n \geq u$ in Ω_n, for all $n \geq 1$. It follows that $U \geq u$ in R^N.
We prove now that if m satisfies (m2), then U blows-up at infinity. For that is sufficient to find $w \in C^2(R^N)$ such that $U \geq w$ and $w(x) \to \infty$ as $|x| \to \infty$.

By Theorem 2.1, we obtain that the problem

$$\begin{cases} \Delta_p \; z = \Phi(r), & r = |x| < \infty \\ z(r) \to 0 & \text{as } |x| \to \infty \end{cases} \tag{19}$$

has a unique positive solution.
We define a function w impliciently by

$$z(x) = \int_{w(x)}^{\infty} \frac{dt}{f(t)^{\frac{1}{p-1}}} . \tag{20}$$

At the beginning of the article the condition imposed to f yields to

$$\lim_{t\searrow 0}\frac{f(t)^{\frac{1}{p-1}}}{t}\le C \text{ for a constant } C$$

which gives us the possibility to choose $\delta>0$ such that

$$\frac{f(t)^{\frac{1}{p-1}}}{t}<C \text{ for all } 0<t<\delta.$$

We obtain

$$f(t)^{\frac{1}{p-1}}<C\cdot t$$

and

$$\frac{1}{f(t)^{\frac{1}{p-1}}}>\frac{1}{C}\cdot\frac{1}{t}.$$

This implies that for every $s\in(0,\delta)$ we have

$$\int_s^\delta\frac{dt}{f(t)^{\frac{1}{p-1}}}>\frac{1}{C}\int_s^\delta\frac{dt}{t}=\frac{1}{C}(\ln\delta-\ln s).$$

Passing to the limit it follows that $\lim_{s\searrow 0}\int_s^\delta f(t)^{-\frac{1}{p-1}}dt=\infty$ and we have the possibility to define w as in (20).

3.4 Proof of Theorem 2.4

The fact that $\Omega\neq R^N$ forces us to make some changes in the argument from theorem 2.3.
Let $(\Omega_n)_{n\ge 1}$ be a sequence of bounded domains given by (m1)'. For some n let v_n be a positive solution of (12). Set $U(x)=\lim_{n\to\infty}v_n(x)$. We find that U is a maximal solution to (2). When $\Omega=R^N\setminus\overline{B(0,R)}$, we suppose that (m2) is fulfilled with $\Phi(r)=0$ for $r\in[0,R]$. To prove that U is a maximal solution is enough to show that a positive function $w\in C(R^N\setminus\overline{B(0,R)})$ with $U\ge w$ in $R^N\setminus\overline{B(0,R)}$ and $w(x)\to\infty$ as $|x|\to\infty$ and as $|x|\searrow R$. As in Theorem 2.3, z is the positive solution to the problem

$$\begin{cases}\Delta_p\, z=\Phi(r) & \text{if } |x|=r>R\\ z(x)\to 0 & \text{as } x\to\infty\\ z(x)\to 0 & \text{as } |x|\searrow R.\end{cases}\tag{21}$$

The uniqueness of z follows from the maximum principle.

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University of Craiova,
Department of Mathematics,
Romania
e-mail: m_amzoiu@yahoo.com