An. Şt. Univ. Ovidius Constanţa Vol. 18(1), 2010, 329–336

A THEOREM ON COMMON FIXED POINTS OF EXPANSION TYPE MAPPINGS IN CONE METRIC SPACES

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Abstract

In this paper, we present a common fixed point theorem for expansion type mappings in complete cone metric spaces. This result generalizes and extends the theorem of S. Z. Wang, B. Y. Li and Z. M. Gao and K. Iseki [8, Theorem 4] for a pair of mappings to cone metric spaces.

1 Introduction

In [3], Guang and Xian generalized the concept of a metric space, replacing the set of real numbers by an ordered Banach space and proved some fixed point theorems for mapping satisfying various contractive conditions. Recently, Rezapour and Hamlbarani [6] generalized some results of [3] by omitting the assumption of normality in the results. Also many authors proved some fixed point theorems for contractive type mappings in cone metric [1, 2, 4, 5, 7] spaces.

The main purpose of this paper is to present a common fixed point result for expansion type mappings in complete cone metric spaces.

2 Preliminaries

Throughout this paper, we denote by **N** the set of positive integers and by **R** the set of real numbers.

Key Words: Cone metric space; Complete cone metric space; Fixed point.
Mathematics Subject Classification: 47H10, 54H25
Received: May, 2009
Accepted: January, 2010

Definition 2.1 Let E be a real Banach space and P a subset of E. Then P is called a cone if:

(i) P is closed, nonempty and satisfies $P \neq \{0\}$,

(ii) $a, b \in \mathbf{R}$, $a, b \geq 0$, $x, y \in P$ implies $ax + by \in P$,

(iii) $x \in P$ and $-x \in P$ implies $x = 0$.

Given a cone $P \subseteq E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ if $x \leq y$ and $x \neq y$, and $x \ll y$ if $y - x \in intP$, where $intP$ is the interior of P. The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$.

The least positive number satisfying the above condition is then called the normal constant of P.

Lemma 2.1 ([9]) *Let E be a real Banach space with a cone P. Then:*

(i) *If $x \leq y$ and $0 \leq a \leq b$, then $ax \leq by$ for $x, y \in P$,*

(ii) *If $x \leq y$ and $u \leq v$, then $x + u \leq y + v$,*

(iii) *If $x_n \leq y_n$ for each $n \in \mathbf{N}$,* and $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$ *then $x \leq y$.*

Lemma 2.2 ([7]) *If P is a cone, $x \in P$, $\alpha \in \mathbf{R}$, $0 \leq \alpha < 1$, and $x \leq \alpha x$, then $x = 0$.*

In the following definition, we suppose that E is a real Banach space, P is a cone in E with $intP \neq \emptyset$ and $\leq$ is partial ordering with respect to P.

Definition 2.2 Let X be a non-empty set. Suppose the mapping $d : X \times X \to E$ satisfies:

(d1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,

(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(d3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space. This definition is more general than that of a metric space.

Example 2.1 Let $E = \mathbf{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\} \subset \mathbf{R}^2$, $X = \mathbf{R}^2$ and $d : X \times X \to E$ defined by

$$d(x, y) = d((x_1, x_2), (y_1, y_2)) = (\max\{|x_1 - y_1|, |x_2 - y_2|\}, \alpha \max\{|x_1 - y_1|, |x_2 - y_2|\}),$$

where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

3 Definitions and Lemmas

In this section we shall give some definitions and lemmas.

Definition 3.1([3]) Let (X, d) be a cone metric space. A sequence $\{x_n\}$ in X is said to be:

(a) A convergent sequence if for every $c \in E$ with $0 \ll c$, there is $N \in \mathbf{N}$ such that for all $n \geq N$, $d(x_n, x) \ll c$ for some fixed x in X. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x, n \to \infty$.

(b) A Cauchy sequence if for every $c \in E$ with $0 \ll c$, there is $N \in \mathbf{N}$ such that for all $n, m \geq N$, $d(x_n, x_m) \ll c$.

A cone metric space (X, d) is said to be complete if every Cauchy sequence is convergent in X.

The following lemma was recently proved in ([2]), by omitting the normality condition.

Lemma 3.1 *Let (X, d) be a cone metric space. If $\{x_n\}$ is a convergent sequence in X, then the limit of $\{x_n\}$ is unique.*

The proof of the following lemma is straighforward and is omitted.

Lemma 3.2 *Let (X, d) be a cone metric space, $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x and $\{x_{n_k}\}$ is any subsequence of $\{x_n\}$, then $\{x_{n_k}\}$ converges to x.*

Lemma 3.3 *Let (X, d) be a cone metric space, $\{x_n\}$ be a sequence in X and $x \in X$. If there exists a sequence $\{c_n\}$ in E with $c_n \to 0$ as $n \to \infty$ such that $d(x_n, x) \leq c_n$ for all $n \in \mathbf{N}$, then $\{x_n\}$ converges to x.*

Proof. For $c \in E$ with $0 \ll c$, choose $\varepsilon > 0$ such that $c + N_\varepsilon(0) \subseteq P$, where $N_\varepsilon(0) = \{y \in E : \|y\| < \varepsilon\}$. Since $c_n \to 0\,(n \to \infty)$, there exists a natural number N such that $\|c_n\| < \varepsilon$, for all $n \geq N$. Thus we have $c_n \in N_\varepsilon(0)$ and $-c_n \in N_\varepsilon(0)$, for all $n \geq N$. Hence $c - c_n \in c + N_\varepsilon(0)$ and so $c - c_n \in intP$, for all $n \geq N$. Thus, $c_n \ll c$ for all $n \geq N$. Then by hypothesis, $d(x_n, x) \ll c$, for all $n \geq N$. It follows that $\{x_n\}$ converges to x.

Lemma 3.4 ([7]) *Let (X, d) be a cone metric space and $\{x_n\}$ be a sequence in X. If there exists a sequence $\{a_n\}$ in $\mathbf{R}$ with $a_n > 0$ for all $n \in \mathbf{N}$ and*

$\sum a_n < \infty$, which satisfies $d(x_{n+1}, x_n) \leq a_n M$ for all $n \in \mathbf{N}$ and for some $M \in E$ with $M \geq 0$, then $\{x_n\}$ is a Cauchy sequence in (X, d).

Definition 3.2 Let E and F be reel Banach spaces and P and Q be cones on E and F, respectively. Let (X, d) and (Y, ρ) be cone metric spaces, where $d : X \times X \to E$ and $\rho : Y \times Y \to F$. A function $f : X \to Y$ is said to be continuous at $x_0 \in X$, if for every $c \in F$ with $0 \ll c$, there exists $b \in E$ with $0 \ll b$ such that, $\rho(f(x), f(x_0)) \ll c$ whenever $x \in X$ and $d(x, x_0) \ll b$.

If f is continuous at every point of X, then it is said to be continuous on X.

We now prove the following lemma.

Lemma 3.5 *Let (X, d) and (Y, ρ) be cone metric spaces as in Definition 3.2. A function $f : X \to Y$ is continuous at a point $x_0 \in X$ if and only if whenever a sequence $\{x_n\}$ in X converges to x_0, the sequence $\{f(x_n)\}$ converges to $f(x_0)$.*

Proof. Suppose that f is continuous at $x_0 \in X$ and let $\{x_n\}$ be a sequence in X converging to x_0. We shall show that $\{f(x_n)\}$ converges to $f(x_0)$. Since f is continuous at x_0, given $c \in F$ with $0 \ll c$ we can find $b \in E$ with $0 \ll b$ such that $x \in X$ and $d(x, x_0) \ll b$ implies $\rho(f(x), f(x_0)) \ll c$. Since the sequence $\{x_n\}$ converges x_0, there exists N such that $d(x_n, x_0) \ll b$ for all $n \geq N$. Therefore for all $n \geq N$, $\rho(f(x_n), f(x_0)) \ll c$. Thus $\lim_{n\to\infty} f(x_n) = f(x_0)$.

Now suppose that for every sequence $\{x_n\}$ in X converging to x_0, the sequence $\{f(x_n)\}$ converges to $f(x_0)$. We shall show that f is continuous at x_0. Suppose this is false. Then there exists $c \in F$ with $0 \ll c$ such that for every $b \in E$ with $0 \ll b$ there exists $x \in X$ such that $d(x, x_0) \ll b$ but $c - \rho(f(x), f(x_0)) \notin intQ$. For fixed $0 \ll b$, we have $0 \ll \frac{b}{n}$ for all $n \in \mathbf{N}$. Therefore we can find a sequence $\{x_n\}$ in X such that $d(x_n, x_0) \ll \frac{b}{n}$ but $c - \rho(f(x_n), f(x_0)) \notin intQ$ for $n = 1, 2, \ldots$. Since $\frac{b}{n} \to 0$ as $n \to \infty$, by Lemma 3.3, the sequence $\{x_n\}$ converges to x_0. But the sequence $\{f(x_n)\}$ does not converges to $f(x_0)$, because of $c - \rho(f(x_n), f(x_0)) \notin intQ$. This contradicts the assumption and the proof is complete.

4 Main result

In this section we prove a common fixed point theorem for expansion type mappings in complete cone metric spaces.

Theorem 4.1 *Let (X, d) be a complete cone metric space and P be a cone. Let f and g be surjective self-mappings of X satisfying the following inequalities*

$$d(gfx, fx) \geq ad(fx, x), \tag{1}$$

$$d(fgx, gx) \geq bd(gx, x) \tag{2}$$

for all x in X, where $a, b > 1$. If either f or g is continuous, then f and g have a common fixed point.

Proof. Let x_0 be an arbitrary point in X. Since f and g are surjective mappings, there exist points $x_1 \in f^{-1}(x_0)$ and $x_2 \in g^{-1}(x_1)$. Continuing in this way, we obtain the sequence $\{x_n\}$ with $x_{2n+1} \in f^{-1}(x_{2n})$ and $x_{2n+2} \in g^{-1}(x_{2n+1})$.

Note that if $x_n = x_{n+1}$ for some n, then x_n is a fixed point of f and g. Indeed, if $x_{2n} = x_{2n+1}$ for some $n \geq 0$, then x_{2n} is a fixed point of f. On the other hand, we have from equation (2) that

$$\begin{aligned} 0 = d(x_{2n}, x_{2n+1}) &= d(fx_{2n+1}, gx_{2n+2}) = d(fgx_{2n+2}, gx_{2n+2}) \\ &\geq bd(x_{2n+1}, x_{2n+2}) \end{aligned}$$

which implies $-bd(x_{2n+1}, x_{2n+2}) \in P$. Also we have $bd(x_{2n+1}, x_{2n+2}) \in P$. Hence $bd(x_{2n+1}, x_{2n+2}) = 0$ and so $x_{2n+1} = x_{2n+2}$, since $b > 1$. Thus, x_{2n} is a common fixed point of f and g. If $x_{2n+1} = x_{2n+2}$ for some $n \geq 0$, similarly, by using inequality (1) leads to x_{2n+1} is a common fixed point of f and g.

Now we suppose that $x_n \neq x_{n+1}$ for all n. Using inequality (1), we have

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(gx_{2n+2}, fx_{2n+3}) = d(gfx_{2n+3}, fx_{2n+3}) \\ &\geq ad(x_{2n+2}, x_{2n+3}). \end{aligned} \tag{3}$$

Similarly, using inequality (2) we have

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &= d(fx_{2n+1}, gx_{2n+2}) = d(fgx_{2n+2}, gx_{2n+2}) \\ &\geq bd(x_{2n+1}, x_{2n+2}). \end{aligned} \tag{4}$$

Suppose that $\alpha = \min\{a, b\}$. Then from inequalities (3) and (4) we have

$$d(x_{2n+2}, x_{2n+3}) \leq \alpha^{-1} d(x_{2n+1}, x_{2n+2})$$

and

$$d(x_{2n+1}, x_{2n+2}) \leq \alpha^{-1} d(x_{2n}, x_{2n+1}).$$

Thus, we obtain

$$d(x_{n+1}, x_{n+2}) \leq \alpha^{-1} d(x_n, x_{n+1})$$

for $n = 0, 1, 2, \ldots$ and it follows that

$$d(x_n, x_{n+1}) \leq \alpha^{-n} d(x_0, x_1).$$

for $n = 1, 2, 3, \ldots$. Since $\sum_{n=0}^{\infty} \alpha^{-n} < \infty$, it follows from Lemma 3.4 that $\{x_n\}$ is a Cauchy sequence in the complete cone metric space (X, d) and so has a limit z in X.

Now we consider that f is continuous. Since $x_{2n} = fx_{2n+1}$, it follows from Lemma 3.5 that

$$z = \lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} fx_{2n+1} = fz$$

and so z is a fixed point of f. Since g is surjective, there exists y such that $gy = z$. Thus, using inequality (2) we have

$$0 = d(fz, gy) = d(fgy, gy) \geq bd(gy, y) = bd(z, y)$$

which implies $-bd(z, y) \in P$. Also we have $bd(z, y) \in P$. Hence $bd(z, y) = 0$ and so $y = z$, since $b > 1$. Thus $z = gz$. We have therefore proved that z is a common fixed point of f and g.

Similarly, considering the continuity of g, it can be seen that f and g have a common fixed point and this completes the proof.

Putting $f = g$ and $k = \min\{a, b\}$ in Theorem 4.1, we get

Corollary 4.1. *Let (X, d) be a complete cone metric space and P be a cone. Let f be a surjective self-mapping of X satisfying the following inequality*

$$d(f^2x, fx) \geq kd(fx, x)$$

for all x in X, where $k > 1$. If f is continuous, then f has a fixed point.

Putting $E = \mathbf{R}$, $P = \{x \in \mathbf{R} : x \geq 0\} \subset \mathbf{R}$ and $d : X \times X \to \mathbf{R}$ in Corollary 4.1, then we obtain the following corollary.

Corollary 4.2 ([8],Theorem 4) *Let (X, d) be a complete metric space and let f be a surjective self-mapping of X satisfying the following inequality*

$$d(f^2x, fx) \geq kd(fx, x)$$

for all x in X, where $k > 1$. If f is continuous, then f has a fixed point.

We illustrate Theorem 4.1 by the following example.

Example 4.1 Let $E = \mathbf{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\} \subset \mathbf{R}^2$, $X = \mathbf{R}$ and the mapping $d : X \times X \to E$ defined by $d(x, y) = (\,|x - y|, |x - y|)$. Then (X, d) is a complete cone metric space.

Define the surjective self mappings $f, g : X \to X$ by

$$fx = 2x \text{ and } gx = 4x$$

for all x in X. Then we have

$$d(fx, gfx) = (|2x-8x|, |2x-8x|) = (6|x|, 6|x|) \geq (2|x-2x|, 2|x-2x|) = 2d(x, fx)$$

and

$$d(gx, fgx) = (|4x-8x|, |4x-8x|) = (4|x|, 4|x|) = (\frac{4}{3}|x-4x|, \frac{4}{3}|x-4x|) = \frac{4}{3}d(x, gx)$$

hold for all x in X. Thus, inequalities (1) and (2) are satisfied and also $x = 0$ is a common fixed point of f and g.

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