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SECOND ORDER PARALLEL TENSORS ON (k,μ)-CONTACT METRIC MANIFOLDS

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Abstract

The object of the present paper is to study the symmetric and skew-symmetric properties of a second order parallel tensor in a (k,μ)-contact metric manifold.

1 Introduction

In 1926, H. Levy [8] proved that a second order symmetric parallel non-singular tensor on a space of constant curvature is a constant multiple of the metric tensor. In recent papers R. Sharma ([10], [11], [12]) generalized Levy's result and also studied a second order parallel tensor on Kaehler space of constant holomorphic sectional curvature as well as on contact manifolds . In 1996, U. C. De [6] studied second order parallel tensors on $P-$Sasakian manifolds. Recently L. Das [5] studied second order parallel tensors on α-Sasakian manifolds. In this study we consider second order parallel tensors on (k,μ)-contact metric manifolds.

The paper is organized as follows:
In Section 2, we give a brief account of contact metric and (k,μ)-contact metric manifolds. In section 3, it is shown that if a (k,μ)-contact metric manifold admits a second order symmetric parallel tensor then either the manifold is locally isometric to the Riemannian product $E^{n+1}(0)\times S^n(4)$, or the second order symmetric parallel tensor is a constant multiple of the associated metric tensor. As an application of this result we obtain that a Ricci symmetric

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($\nabla S = 0$) (k, μ)-contact metric manifold is either locally isometric to the Riemannian product $E^{n+1}(0) \times S^n(4)$, or an Einstein manifold. Further, it is shown that on a (k, μ)-contact metric manifold with $k^2 + (k-1)\mu^2 \neq 0$ there is no nonzero parallel 2-form.

2 Contact Metric Manifolds

A $(2n+1)$-dimensional manifold M is said to admit an almost contact structure if it admits a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η satisfying

$$(a)\ \phi^2 = -I + \eta \otimes \xi,\ \ (b)\ \eta(\xi) = 1,\ \ (c)\ \phi\xi = 0,\ \ (d)\ \eta \circ \phi = 0. \tag{1}$$

An almost contact metric structure is said to be normal if the induced almost complex structure J on the product manifold $M \times \mathbb{R}$ defined by

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$$

is integrable, where X is tangent to M, t is the coordinate of $\mathbb{R}$ and f is a smooth function on $M \times \mathbb{R}$. Let g be a compatible Riemannian metric with almost contact structure (ϕ, ξ, η), that is,

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2}$$

Then M becomes an almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g). From (1) it can be easily seen that

$$(a) g(X, \phi Y) = -g(\phi X, Y), (b) g(X, \xi) = \eta(X)$$

for all vector fields X, Y. An almost contact metric structure becomes a contact metric structure if

$$g(X, \phi Y) = d\eta(X, Y)$$

for all vector fields X, Y. The 1-form η is then a contact form and ξ is its characteristic vector field. We define a $(1,1)$ tensor field h by $h = \frac{1}{2}\pounds_\xi \phi$, where $\pounds$ denotes the Lie-differentiation. Then h is symmetric and satisfies $h\phi = -\phi h$. We have $Tr.h = Tr.\phi h = 0$ and $h\xi = 0$. Also,

$$\nabla_X \xi = -\phi X - \phi h X \tag{3}$$

holds in a contact metric manifold. A normal contact metric manifold is a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \phi)(Y) = g(X, Y)\xi - \eta(Y)X, \quad X, Y \in TM,$$

where ∇ is Levi-Civita connection of the Riemannian metric g. A contact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$ for which ξ is a Killing vector is said to be a K-contact manifold. A Sasakian manifold is K-contact but not conversely. However a 3-dimensional K-contact manifold is Sasakian [7]. It is well known that the tangent sphere bundle of a flat Riemannian manifold admits a contact metric structure satisfying $R(X,Y)\xi = 0$ [2]. On the other hand, on a Sasakian manifold the following holds:

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y.$$

As a generalization of both $R(X,Y)\xi = 0$ and the Sasakian case; D. Blair, T. Koufogiorgos and B. J. Papantoniou [4] considered the (k,μ)-nullity condition on a contact metric manifold and gave several reasons for studying it. The (k,μ)-nullity distribution $N(k,\mu)$ ([4], [9]) of a contact metric manifold M is defined by

$$\begin{aligned} N(k,\mu) \quad &: \quad p \longrightarrow N_p(k,\mu) = \\ &= \quad \{W \in T_pM : R(X,Y)W = (kI+\mu h)(g(Y,W)X - g(X,W)Y)\}, \end{aligned}$$

for all $X, Y \in TM$, where $(k,\mu) \in \mathbb{R}^2$. A contact metric manifold M^{2n+1} with $\xi \in N(k,\mu)$ is called a (k,μ)-contact metric manifold (see also [3]). In particular on a (k,μ)-contact metric manifold, we have

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY]. \tag{4}$$

On a (k,μ)-contact metric manifold $k \leq 1$. If $k = 1$, the structure is Sasakian ($h = 0$ and μ is indeterminant) and if $k < 1$, the (k,μ)-nullity condition determines the curvature of M^{2n+1} completely [4]. In fact, for a (k,μ)-contact metric manifold, the condition of being a Sasakian manifold, a K-contact manifold, $k = 1$ and $h = 0$ are all equivalent.
Also, if M is a contact metric manifold with $\xi \in N(k,\mu)$, we have the following relations [4]:

$$R(\xi,X)Y = k\{g(X,Y)\xi - \eta(Y)X\} + \mu\{g(hX,Y)\xi - \eta(Y)hX\}, \tag{5}$$

$$h^2 = (k-1)\phi^2, k \leq 1. \tag{6}$$

We now state some results which will be used later on.

Lemma 2.1. ([2]) *A contact metric manifold M with $R(X,Y)\xi = 0$ for all vector fields X, Y is locally isometric to the Riemannian product of a flat $(n+1)-$dimensional manifold and an n-dimensional manifold of positive curvature 4, that is, $E^{n+1} \times S^n(4)$.*

Lemma 2.2. [4] *Let M be a contact metric manifold with ξ belonging to the $(k,\mu)-$nullity distribution, then $k \leq 1$. If $k = 1$, then $h = 0$ and $M(\xi,\eta,\phi,g)$ is a Sasakian manifold. If $k < 1$, the contact metric structure is not Sasakian and M admits three mutually orthogonal integrable distributions, the eigen distributions of the tensor field h : $D(0)$, $D(\lambda)$ and $D(-\lambda)$, where 0, $\lambda = \sqrt{1-k}$ and $-\lambda$ are the (constant) eigenvalues of h.*

Lemma 2.3. [4] *Let M be a contact metric manifold with ξ belonging to the $(k,\mu)-$nullity distribution. If $k < 1$, then for any X orthogonal to ξ, the $\xi-$sectional curvature $K(X,\xi)$ is given by*

$$\begin{aligned} K(X,\xi) = k + \mu g(hX,X) &= k + \lambda\mu \quad if \quad X \in D(\lambda) \\ &= k - \lambda\mu \quad if \quad X \in D(-\lambda). \end{aligned}$$

3 Second order parallel tensor

Definition 3.1 A tensor α of second order is said to be a parallel tensor if $\nabla\alpha = 0$, where ∇ denotes the operator of the covariant differentiation with respect to the metric tensor g.

Let α be a $(0,2)$-symmetric tensor field on a (k,μ)-contact metric manifold M such that $\nabla\alpha = 0$. Then it follows that

$$\alpha(R(W,X)Y,Z) + \alpha(Y,R(W,X)Z) = 0, \tag{7}$$

for arbitrary vector fields $W,X,Y,Z \in T(M)$.

Substitution of $W = Y = Z = \xi$ in (7) gives us

$$\alpha(R(\xi,X)\xi,\xi) = 0,$$

since α is symmetric.

Now take a non-empty connected open subset U of M and restrict our considerations to this set.

As the manifold is a (k,μ)-contact metric manifold, using (5) in the above equation we get

$$k\{g(X,\xi)\alpha(\xi,\xi) - \alpha(X,\xi)\} - \mu\alpha(hX,\xi) = 0. \tag{8}$$

We now consider the following cases:

Case 1. $k = \mu = 0$,

Case 2. $k \neq 0, \mu = 0$,

Case 3. $k \neq 0, \mu \neq 0$.

For the Case 1, we have from (4) that $R(X,Y)\xi = 0$ for all X,Y and hence by Lemma 2.1, the manifold is locally isometric to the Riemannian product $E^{n+1}(0) \times S^n(4)$.

For the Case 2, it follows from (8) that

$$\alpha(X,\xi)-\alpha(\xi,\xi)g(X,\xi)=0. \tag{9}$$

Differentiating (9) covariantly along Y, we get

$$\begin{aligned} g(\nabla_Y X,\xi)\alpha(\xi,\xi) \quad &+ \quad g(X,\nabla_Y\xi)\alpha(\xi,\xi)+2g(X,\xi)\alpha(\nabla_Y\xi,\xi) \\ &- \quad \alpha(\nabla_Y X,\xi)-\alpha(X,\nabla_Y\xi)=0. \end{aligned} \tag{10}$$

Changing X by $\nabla_Y X$ in (9) we have

$$g(\nabla_Y X,\xi)\alpha(\xi,\xi)-\alpha(\nabla_Y X,\xi)=0. \tag{11}$$

From (10) and (11) it follows that

$$g(X,\nabla_Y\xi)\alpha(\xi,\xi)+2g(X,\xi)\alpha(\nabla_Y\xi,\xi)-\alpha(X,\nabla_Y\xi)=0. \tag{12}$$

Using (1), (3) and (9) we have from (12)

$$\alpha(X,\phi Y)-\alpha(X,h\phi Y)=\alpha(\xi,\xi)g(X,\phi Y)-\alpha(\xi,\xi)g(X,h\phi Y). \tag{13}$$

Replacing Y by ϕY in (13) and using (1) we get

$$\alpha(X,Y)-g(X,Y)\alpha(\xi,\xi)=\alpha(X,hY)-\alpha(\xi,\xi)g(X,hY). \tag{14}$$

Changing Y by hY in (14) and using (6) we have

$$\alpha(X,hY)-\alpha(\xi,\xi)g(X,hY)=-(k-1)\{\alpha(X,Y)-\alpha(\xi,\xi)g(X,Y)\}. \tag{15}$$

Using (14) in (15) we obtain

$$k(\alpha(X,Y)-\alpha(\xi,\xi)g(X,Y))=0,$$

Since $k\neq 0$,

$$\alpha(X,Y)-\alpha(\xi,\xi)g(X,Y)=0.$$

Hence, since α and g are parallel tensor fields, $\alpha(\xi,\xi)$ is constant on U. By the parallelity of α and g, it must be $\alpha(X,Y)=\alpha(\xi,\xi)g(X,Y)$ on whole of M.

Finally for the Case 3, changing X by hX in the equation (8) and using (6) we obtain

$$k\alpha(hX,\xi)=(k-1)\mu(\alpha(X,\xi)-g(X,\xi)\alpha(\xi,\xi)). \tag{16}$$

Using (16) in (8) we get

$$(k^2+(k-1)\mu^2)\{\alpha(X,\xi)-\alpha(\xi,\xi)g(X,\xi)\}=0. \tag{17}$$

Now $k^2 + (k-1)\mu^2 \neq 0$ means $\{k + \mu\sqrt{1-k}\}\{k - \mu\sqrt{1-k}\} \neq 0$ which implies $\{k + \mu\sqrt{1-k}\} \neq 0$ and $\{k - \mu\sqrt{1-k}\} \neq 0$. Also

$$TM = [\xi] \oplus [D(\lambda)] \oplus [D(-\lambda)],$$

where $D(\lambda)$(resp. $D(-\lambda)$) is the distribution defined by the vector fields $hX = \lambda X$ (resp. $hX = -\lambda X$), $\lambda = \sqrt{1-k}$ which follows from (6)). Hence the relation $k^2 + (k-1)\mu^2 \neq 0$ basically means that the sectional curvatures of plane sections containing ξ are non-vanishing, that is, $K(X, \xi) \neq 0$ for any vector field X perpendicular to ξ. Again from Lemma 2.3, it follows that $K(X, \xi) = 0$ if and only if

$$k + \lambda\mu = 0 \quad for \quad X \in D(\lambda)$$

$$k - \lambda\mu = 0 \quad for \quad X \in D(-\lambda),$$

where $\lambda = \sqrt{1-k}$. Then we have $k + \mu\sqrt{1-k} = 0$ and $k - \mu\sqrt{1-k} = 0$. These two relations gives us $k = \mu = 0$. But in this case we have assumed that $k \neq 0$ and $\mu \neq 0$. Consequently we must have $K(X, \xi) \neq 0$ for all X perpendicular to ξ in this case. Hence we must have $k^2 + (k-1)\mu^2 \neq 0$. Then (17) implies that the relation (9) holds and hence proceeding in the same way as in case 2, we can show that $\alpha(X, Y) = \alpha(\xi, \xi)g(X, Y)$ on whole of M.

Therefore considering all the cases we can state the following:

Theorem 3.1. *If a (k, μ)-contact metric manifold admits a second order symmetric parallel tensor then either the manifold is locally isometric to the Riemannian product $E^{n+1}(0) \times S^n(4)$ including the 3-dimensional case, or the second order symmetric parallel tensor is a constant multiple of the associated metric tensor.*

Application: We consider the Ricci symmetric $(k, \mu)-$contact metric manifold. Then $\nabla S = 0$. Hence from Theorem 3.1, we have the following:

Corollary 3.1. *A Ricci symmetric ($\nabla S = 0$) (k, μ)-contact metric manifold is either locally isometric to the Riemannian product $E^{n+1}(0) \times S^n(4)$, or an Einstein manifold.*

The above Corollary has been proved by Papantoniou in [9].

Next, let M be a (k, μ)-contact metric manifold admitting a second order skew-symmetric parallel tensor. Putting $Y = W = \xi$ in (7) and using (5), we obtain

$$\begin{aligned} k\{\eta(X)\alpha(\xi, Z) &- \alpha(X, Z) - \eta(Z)\alpha(\xi, X)\} \\ &= \mu\{\alpha(hX, Z) + \eta(Z)\alpha(\xi, hX)\}. \end{aligned} \tag{18}$$

Changing X by hX in (18) we have

$$\begin{aligned} k\{\alpha(hX,Z)+\eta(Z)\alpha(\xi,hX)\} &= (k-1)\mu\{\alpha(X,Z) \\ &+ \eta(Z)\alpha(\xi,X)-\eta(X)\alpha(\xi,Z)\}. \end{aligned} \tag{19}$$

Using (18) and (19) we obtain

$$(k^2+(k-1)\mu^2)\{\alpha(X,Z)-\eta(X)\alpha(\xi,Z)+\eta(Z)\alpha(\xi,X)\}=0. \tag{20}$$

Consider a non-empty open subset U of M such that $k^2+(k-1)\mu^2 \neq 0$ and $k \neq 0$ on U. Then

$$\alpha(X,Z)-\eta(X)\alpha(\xi,Z)+\eta(Z)\alpha(\xi,X)=0. \tag{21}$$

Now, let A be a $(1,1)$ tensor field which is metrically equivalent to α, that is, $\alpha(X,Y)=g(AX,Y)$. Then from (21) we have

$$g(AX,Z)=\eta(X)g(A\xi,Z)-\eta(Z)g(A\xi,X),$$

and thus

$$AX=\eta(X)A\xi-g(A\xi,X)\xi. \tag{22}$$

Since α is parallel, then A is parallel. Hence, using (1), (22) follows that

$$\nabla_X(A\xi)=A(\nabla_X\xi)=-A(\phi X)+A(h\phi X).$$

Using (1), we have

$$\nabla_{\phi X}(A\xi)=A(X)-\eta(X)A\xi-A(hX). \tag{23}$$

Using (22) in (23) we obtain

$$\nabla_{\phi X}(A\xi)=-A(hX)-g(A\xi,X)\xi. \tag{24}$$

Also from (22) we get

$$g(A\xi,\xi)=0. \tag{25}$$

Using (25), from (24) we have

$$g(\nabla_{\phi X}(A\xi),A\xi)=-g(A(hX),A\xi).$$

Thus,

$$g(\nabla_{\phi X}\xi,A^2\xi)=-g(hX,A^2\xi). \tag{26}$$

Now from (3) we get

$$\begin{aligned} \nabla_{\phi X}\xi &= -\phi^2X+h\phi^2X \\ &= X-hX-\eta(X)\xi. \end{aligned}$$

Using this in (26) we have

$$A^2\xi = -\|A\xi\|^2\xi. \tag{27}$$

Differentiating (27) covariantly along X, it follows that

$$\nabla_X(A^2\xi) = A^2(\nabla_X\xi) = A^2(-\phi X - \phi hX) = -\|A\xi\|^2(\nabla_X\xi).$$

Hence

$$-A^2(\phi X) - A^2(\phi hX) = \|A\xi\|^2\phi X + \|A\xi\|^2\phi hX. \tag{28}$$

Replacing X by ϕX and using (1) we obtain from (27)

$$A^2(X) - A^2(hX) = -\|A\xi\|^2 X + \|A\xi\|^2 hX. \tag{29}$$

Changing X by hX in (29) and using (1) and (29) we obtain

$$A^2(hX) + (k-1)A^2(X) = -\|A\xi\|^2 hX - (k-1)\|A\xi\|^2 X. \tag{30}$$

Using (29) from (30) we get

$$k\{A^2X + \|A\xi\|^2X\} = 0.$$

Now $k \neq 0$ implies $A^2X = -\|A\xi\|^2X$.

Now, if $\|A\xi\| \neq 0$, then $J = \frac{1}{\|A\xi\|}A$ is an almost complex structure on U. In fact, (J, g) is a Kaehler structure on U. The fundamental second order skew-symmetric parallel tensor is $g(JX, Y) = \kappa g(AX, Y) = \kappa\alpha(X, Y)$, with $\kappa = \frac{1}{\|A\xi\|} = constant$. But (21) means $\alpha(X, Y) = \eta(X)\alpha(\xi, Y) - \eta(Y)\alpha(\xi, X)$ and thus α is degenerate, which is a contradiction. Therefore $\|A\xi\| = 0$ and hence $\alpha = 0$ on U. Since α is parallel on U, $\alpha = 0$ on M.

Hence we can state the following:

Theorem 3.2. *On a (k, μ)-contact metric manifold with $k \neq 0$ there is no nonzero second order skew symmetric parallel tensor provided $k^2 + (k-1)\mu^2 \neq 0$.*

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