An. Şt. Univ. Ovidius Constanţa Vol. 18(1), 2010, 181–196

STRONG CONVERGENCE THEOREMS ON ITERATIVE METHODS FOR STRICTLY PSEUDO-CONTRACTIVE MAPPINGS

Yan Hao

Abstract

The aim of this work is to consider an iterative method for a family of λ-strict pseudo-contractions. Strong convergence theorems are established in a real 2-uniformly smooth Banach space

1. Introduction and Preliminaries

Throughout this paper, we assume that E is a real Banach space with the normalized duality mapping J from E into 2^{E^*} give by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2, \quad \|f\| = \|x\|\}, \quad \forall x \in E,$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. We assume that C is a nonempty subset of E and we use $F(T)$ to denote the fixed point set of the mapping T. $\rightarrow$ and $\rightharpoonup$ denote strong and weak convergence, respectively.

Among nonlinear mappings, the classes of non-expansive mappings and strict pseudo-contractions are two kinds of important nonlinear mappings. The studies on them have a very long history(see, e.g., [1-28]). Recall that T is called a λ-strict pseudo-contraction in the terminology of Browder and Petryshyn [3] if there exists a constant $\lambda \in (0, 1)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2 - \lambda \|(I - T)x - (I - T)y\|^2, \tag{1.1}$$

Key Words: non-expansive mapping; strict pseudo-contraction; iterative method; fixed point

Mathematics Subject Classification: 47H05, 47H09, 47J05

Received: April, 2009

Accepted: January, 2010

for all $x, y \in C$ and for some $j(x-y) \in J(x-y)$. If I denotes the identity operator, we can see that (1.1) is equivalent to the following inequality

$$\langle (I-T)x - (I-T)y, j(x-y)\rangle \geq \lambda \|(I-T)x - (I-T)y\|^2, \tag{1.2}$$

for all $x, y \in C$ and for some $j(x-y) \in J(x-y)$. The class of strict pseudo-contractions was first introduced in Hilbert spaces by Browder and Petryshyn [3]. Let C be a nonempty subset of a real Hilbert space H, and $T : C \to C$ be a mapping. In light of [3], T is said to be k-strict pseudo-contractive, if there exists a $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 + k\|(I-T)x - (I-T)y\|^2, \tag{1.3}$$

for all $x, y \in C$. It is well-known that (1.3) is equivalent to

$$\langle Tx - Ty, x - y\rangle \leq \|x-y\|^2 - \frac{1-k}{2}\|(I-T)x - (I-T)y\|^2, \tag{1.4}$$

for all $x, y \in C$. Note that the class of strict pseudo-contractions strictly includes the class of non-expansive mappings which are mappings T on C such that

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.5}$$

Recall that a self mapping $f : C \to C$ is a contraction, if there exists a constant $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|, \quad \forall x, y \in C.$$

We use Π_C to denote the collection of all contractions on C.

Remark 1.1. It is shown in [12] that the strict pseudo-contractions are L-Lipschitz (i.e., $\|Tx - Ty\| \leq L\|x - y\|$ for all $x, y \in C$ and for some $L > 0$).

Remark 1.2. When $\lambda = 0$, T is said to be pseudo-contractive, and it is said to be strongly pseudo-contractive, if there exists a positive constant $a \in (0, 1)$ such that $T - aI$ is pseudo-contractive. We remark that the class of strongly pseudo-contractive mappings is independent of the class of λ-strict pseudo-contractions. This can be seen from some examples (see, Chidume and Mutangadurea [5] and Zhou [28]).

A Banach space E is called uniformly convex if for each $\epsilon > 0$ there is a $\delta > 0$ such that for $x, y \in E$ with $\|x\|, \|y\| \leq 1$ and $\|x - y\| \geq \epsilon, \|x + y\| \leq 2(1 - \delta)$ holds. The modulus of convexity of E is defined by

$$\delta_E(\epsilon) = \inf\{1 - \|\frac{1}{2}(x + y)\| : \|x\|, \|y\| \leq 1, \|x - y\| \geq \epsilon\},$$

for all $\epsilon \in [0,1]$. E is said to be uniformly convex if $\delta_E(0) = 0$ and $\delta(\epsilon) > 0$ for all $0 < \epsilon \leq 2$. A Hilbert space H is 2-uniformly convex, while L_p is $\max\{p,2\}$-uniformly convex for every $p > 1$.

Let $S(E) = \{x \in E : \|x\| = 1\}$. Then the norm of E is said to be Gâteaux differentiable if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{1.6}$$

exists for each $x, y \in S(E)$. In this case, E is called smooth. The norm of E is said to be uniformly Gâteaux differentiable if for each $y \in S(E)$, the limit (1.6) is attained uniformly for $x \in S(E)$. The norm of E is called Fréchet differentiable, if for each $x \in S(E)$, the limit (1.6) is attained uniformly for $y \in S(E)$. The norm of E is called uniformly Fréchet differentiable, if the limit (1.6) is attained uniformly for $x, y \in S(E)$. It is well-known that the (uniform) Fréchet differentiability of the norm of E implies the (uniform) Gâteaux differentiability of the norm of E.

The modulus of smoothness of E is the function $\rho_E : [0,\infty) \to [0,\infty)$ defined by

$$\rho_E(\tau) = \{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| \leq 1,\ \|y\| \leq \tau\}, \quad \forall \tau \geq 0.$$

The Banach space E is uniformly smooth if and only if $\lim_{\tau\to\infty} \frac{\rho_E(\tau)}{\tau} = 0$ for all $\tau > 0$. Let $q > 1$. The Banach space E is said to be q-uniformly smooth (or to have the modulus of smoothness of power type $q > 1$) if there exists a constant $c > 0$ such that $\rho_E(\tau) \leq c\tau^q$. It is known that, if E is q-uniformly smooth then $q \leq 2$ and E is uniformly smooth and hence the norm of E is uniformly Fréchet differentiable, in particular, the norm of E is Fréchet differentiable. There are typical examples of both uniformly convex and uniformly smooth Banach space L^p, where $p > 1$. More precisely, L^p is min $\{p,2\}$-uniformly smooth for every $p > 1$.

One classical way to study non-expansive mappings is to use contractions to approximate a non-expansive mapping ([2], [17]). More precisely, take $t \in (0,1)$ and define a contraction $T_t : K \to K$ by

$$T_t x = tu + (1-t)Tx, \quad x \in K, \tag{1.7}$$

where $u \in K$ is a fixed point. Banach's Contraction Mapping Principle guarantees that T_t has a unique fixed point x_t in K. It is unclear, in general, what is the behavior of x_t as $t \to 0$, even if T has a fixed point. However, in the case of T having a fixed point, Browder [1] proved that, if X is a Hilbert space, then x_t converges strongly to a fixed point of T that is nearest to u. Reich [17] extended Browder's result to the setting of Banach spaces and proved that, if

X is a uniformly smooth Banach space, then x_t converges strongly to a fixed point of T and the limit defines the (unique) sunny non-expansive retraction from C onto $F(T)$.

Very recently, Xu [24] studied the following iterative process by so-called viscosity approximation which first introduced by Moudafi [11].

$$x_0 = x \in C, \quad x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \quad n \geq 0 \tag{1.8}$$

and proved the following theorem in a Banach space.

Theorem X. *Let E be a uniformly smooth Banach space, C be a closed convex subset of E, $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$, and $f : C \to C$ be a contraction. Let $\{x_n\}$ be generated by (1.8). Then, under the hypotheses*

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) *either* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n\to\infty}(\alpha_{n+1}/\alpha_n) = 1$,

$\{x_n\}$ converges strongly to a fixed point of T, which is the unique solution of some variational inequality.

Recall that the normal Mann's iterative process was introduced by Mann [10] in 1953. Since then, the construction of fixed points for non-expansive mappings and strict pseudo-contractions via the normal Mann's iterative process has been extensively investigated by many authors.

The normal Mann's iterative process generates a sequence $\{x_n\}$ in the following manner:

$$\forall x_1 \in C, \quad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad \forall n \geq 1, \tag{1.9}$$

where the sequence $\{\alpha_n\}_{n=1}^{\infty}$ is in the interval (0,1).

If T is a non-expansive mapping with a fixed point and the control sequence $\{\alpha_n\}$ is chosen so that $\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by normal Mann's iterative process (1.9) converges weakly to a fixed point of T (this is also valid in a uniformly convex Banach space with the Fréchet differentiable norm [18]). In 1967, Browder and Petryshyn [3] established the first convergence result for strictly pseudo-contractive self-mappings in real Hilbert spaces. They proved weak and strong convergence theorems by using algorithm (1.9) with a constant control sequence $\{\alpha_n\} = \alpha$ for all n. Afterward, Rhoades [20] generalized in part the corresponding results in [3] in the sense that a variable control sequence $\{\alpha_n\}$ was taken into consideration.

Attempts to modify the Mann iteration method (1.9) for non-expansive mappings and strict pseudo-contractions so that strong convergence is guaranteed have recently been made; see, e.g., [6,8,9,14,16,26,28] and the references therein.

Kim and Xu [8] introduced the following iteration process.

$$\begin{cases} x_0 = x \in C \ \ \textit{arbitrarily chosen}, \\ y_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)y_n, \end{cases} \tag{1.10}$$

where T is a non-expansive mapping of K into itself, $u \in C$ is a given point. They proved the sequence $\{x_n\}$ defined by (1.10) converges strongly to a fixed point of T provided the control sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy appropriate conditions.

More recently, Yao et al. [26] improved the results of Kim and Xu [8] by using the so-called viscosity approximation methods. To be more precisely, they introduced the following iterative scheme

$$\begin{cases} x_0 = x \in C \ \ \textit{arbitrarily chosen}, \\ y_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)y_n, \end{cases} \tag{1.11}$$

where f is a contraction on f. They obtained a strong convergence theorem for a non-expansive mapping in a Banach space.

Very recently, Zhou [27] modified the normal Mann's iterative process (1.9) for non-self strict pseudo-contractions to have strong convergence in Hilbert spaces. To be more precisely, he proved the following results.

Theorem Z. Let C be a closed convex subset of a real Hilbert space H and let $T : C \to H$ be a k-strictly pseudo-contractive nonself-mapping such that $F(T) \neq \emptyset$. Given $u \in C$ and sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in (0,1), the following control conditions are satisfied:

(i) $\beta_n \to 0$;

(ii) $k \le \alpha_n \le b < 1$ for all $n \ge 1$;

(iii) $\sum_{n=1}^{\infty} \beta_n = \infty$;

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$; or $\frac{\beta_n}{\beta_{n+1}} \to 1$ as $n \to \infty 1$.

Let a sequence $\{x_n\}$ be generated in the following manner:

$$\begin{cases} x_1 = x \in C, \\ y_n = P_C[\alpha_n x_n + (1-\beta_n)Tx_n], \\ x_{n+1} = \beta_n u + (1-\beta_n)y_n, \quad n \geq 1, \end{cases}$$

where P_C is a projection from H onto C. Then, $\{x_n\}$ converges strongly to a fixed point z of T, where $z = P_{F(T)}u$.

In this paper, motivated by Cho et al. [6], Kim and Xu [8], Qin et al. [14], Xu [24], Yao et al. [26] and Zhou [27,28], we prove strong convergence theorems for a finite family of λ-strict pseudo-contractions in Banach spaces. Our results improve and extend the corresponding ones announced by many others.

In order to prove our main results, we need the following definitions and lemmas.

Lemma 1.1. (Xu [25]). *Let E be a real 2-uniformly smooth Banach space with the best smooth constant K. Then the following inequality holds:*

$$\|x+y\|^2 \leq \|x\| + 2\langle y, Jx\rangle + 2\|Ky\|^2,$$

for all $x, y \in E$.

Lemma 1.2 (Zhou [28]). *Let C be a nonempty subset of a real 2-uniformly smooth Banach spaces and $T : C \to C$ be a λ-strict pseudo-contraction. For $\alpha \in (0,1)$, we define $T_\alpha x = (1-\alpha)x + \alpha Tx$. Then as $\alpha \in (0, \frac{\lambda}{K_2})$, $T_\alpha : C \to C$ is non-expansive such that $F(T_\alpha) = F(T)$.*

Lemma 1.3 (Zhou [28]). *Let E be a smooth Banach space and C be a nonempty convex subset of E . Given an integer $r \geq 1$, assume that for each $i \in \Lambda$, $T_i : C \to C$ is a λ_i-strict pseudo-contraction for some $0 \leq \lambda < 1$. Assume that $\mu_{i=1}^r$ is a positive sequence such that $\sum_{i=1}^r \mu_i = 1$. Then $\sum_{i=1}^r \mu_i T_i : C \to C$ is a λ-strict pseudo-contraction with $\lambda = \min\{\lambda_i : 1 \leq i \leq r\}$.*

Lemma 1.4 (Zhou [28]). *Let E be a smooth Banach space and C be a nonempty convex subset of E . Given an integer $r \geq 1$, assume that $\{T_i\}_{i=1}^r : C \to C$ is a finite family of λ_i-strict pseudo-contractions for some $0 \leq \lambda < 1$ such that $F = \cap_{i=1}^r F(T_i) \neq \emptyset$. Assume that $\{\mu_i\}_{i=1}^r$ is a positive sequence such that $\sum_{i=1}^r \mu_i = 1$. Then $F(\sum_{i=1}^r \mu_i T_i) = F$.*

Lemma 1.5 (Xu [24]). *Let E be a uniformly smooth Banach space, C be a closed convex subset of E, $T : C \to C$ be a non-expansive mapping with $F(T) \neq \emptyset$, and $f \in \Pi_C$. Then $\{x_t\}$, defined by*

$$x_t = tf(x_t) + (1-t)Tx_t,$$

converges strongly to a point in $F(T)$. *If we define* $Q : \Pi_C \to F(T)$ *by*

$$Q(f) := \lim_{t\to 0} x_t, \quad f \in \Pi_C. \tag{1.12}$$

Then $Q(f)$ *solves solves the variational inequality*

$$\langle (I-f)Q(f), J(Q(f)-p)\rangle \le 0, \quad f \in \Pi_C, p \in F(T).$$

Lemma 1.6. *In a Banach space* E*, there holds the inequality*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad x, y \in E,$$

where $j(x+y) \in J(x+y)$.

Lemma 1.7 (Xu [22]). *Assume that* $\{\alpha_n\}$ *is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \le (1-\gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ *is a sequence in (0,1) and* $\{\delta_n\}$ *is a sequence such that*

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n/\gamma_n \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

3. Main Results

Lemma 2.1. *Let* C *be a nonempty closed convex subset of a real* 2*-uniformly Banach space* E *and let* $T : C \to C$ *be a* λ*-strict pseudo-contraction such that* $F(T) \ne \emptyset$. *Given* $f \in \Pi_C$ *and* $x_0 \in C$ *and sequences* $\{\alpha_n\}$ *and* $\{\beta_n\}$*, the following control conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \beta_n < \frac{\lambda}{K^2}$ *for all* $n \ge 0$.

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then $\{x_n\}$ *generated by*

$$\begin{cases} y_n = \beta_n T x_n + (1-\beta_n)x_n \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)y_n, \quad n \ge 1, \end{cases}$$

converges strongly to a fixed point x^* of T, which solves the following variational inequality

$$\langle f(x^*) - x^*, J(p - x^*)\rangle \leq 0, \quad \forall p \in F(T).$$

Proof. From Lemma 1.2, we have $y_n = T_{\beta_n} x_n$, $F(T_{\beta_n}) = F(T)$ and T_{β_n} is non-expansive for every n. First, we show $\{x_n\}$ is bounded. Indeed, taking a point $p \in F(T)$, we have

$$\|y_n - p\| \leq \|x_n - p\|.$$

It follows that

$$\begin{aligned}\|x_{n+1} - p\| &= \|\alpha_n(f(x_n) - p) + (1 - \alpha_n)(y_n - p)\| \leq \\ &\leq \alpha_n\|f(x_n) - f(p) + f(p) - p\| + (1 - \alpha_n)\|x_n - p\| \leq \\ &\leq [1 - \alpha_n(1 - \alpha)]\|x_n - p\| + \alpha_n\|f(p) - p\|.\end{aligned}$$

By simple induction, we have

$$\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{\|p - f(p)\|}{1 - \alpha}\}, \quad n \geq 1,$$

which gives that the sequence $\{x_n\}$ is bounded. On the other hand, we have

$$\begin{aligned}\|y_{n+1} - y_n\| &= \|T_{\beta_{n+1}} x_{n+1} - T_{\beta_n} x_n\| = \\ &= \|T_{\beta_{n+1}} x_{n+1} - T_{\beta_{n+1}} x_n + T_{\beta_{n+1}} x_n - T_{\beta_n} x_n\| \leq \\ &\leq \|x_{n+1} - x_n\| + \|T_{\beta_{n+1}} x_n - T_{\beta_n} x_n\| \leq \\ &\leq \|x_{n+1} - x_n\| + M_1|\beta_{n+1} - \beta_n|,\end{aligned} \tag{2.1}$$

where M_1 is an appropriate constant such that $M_1 \geq \sup\{\|Tx_n - x_n\|\}$. Observing that

$$\begin{aligned}x_{n+2} - x_{n+1} = &(1 - \alpha_{n+1})(y_{n+1} - y_n) - (\alpha_{n+1} - \alpha_n)y_n \\ &+ \alpha_{n+1}(f(x_{n+1}) - f(x_n)) + f(x_n)(\alpha_{n+1} - \alpha_n),\end{aligned}$$

we have

$$\begin{aligned}\|x_{n+2} - x_{n+1}\| \leq &(1 - \alpha_{n+1})\|y_{n+1} - y_n\| + |\alpha_{n+1} - \alpha_n|M_2 \\ &+ \alpha_{n+1}\alpha\|x_{n+1} - x_n\|,\end{aligned} \tag{2.2}$$

where M_2 is a appropriate constant such that $M_2 \geq \sup\{\|y_n\| + \|f(x_n)\|\}$. Substituting (2.1) into (2.2), we have

$$\|x_{n+2} - x_{n+1}\| \leq [1 - \alpha_{n+1}(1 - \alpha)]\|x_{n+1} - x_n\| + M_3(|\beta_{n+1} - \beta_n| + |\alpha_{n+1} - \alpha_n|), \tag{2.3}$$

M_3 is a appropriate constant such that $M_3 \geq \max\{M_1, M_2\}$. Noticing conditions (i), (ii) and (iv) and applying Lemma 1.6 to (2.3), we have

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{2.4}$$

Notice that

$$\begin{aligned}\|x_n - y_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \leq \\ &\leq \|x_n - x_{n+1}\| + \alpha_n\|f(x_n) - y_n\|.\end{aligned}$$

It follows, from condition (i) and (2.4), that $\lim_{n\to\infty} \|x_n - y_n\| = 0$. That is,

$$\lim_{n\to\infty} \|x_n - T_{\beta_n}x_n\| = 0. \tag{2.5}$$

Next, we claim that

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, J(x_n - x^*)\rangle \leq 0, \tag{2.6}$$

where $x^* = \lim_{t\to 0} x_t$ with x_t being the fixed point of the contraction

$$x \mapsto tf(x) + (1-t)T_{\beta_n}x.$$

Then x_t solves the fixed point equation $x_t = t\gamma f(x_t) + (1-t)T_{\beta_n}x_t$. Thus we have

$$\|x_t - x_n\| = \|(1-t)(T_{\beta_n}x_t - x_n) + t(f(x_t) - x_n)\|.$$

It follows from Lemma 1.6 that

$$\begin{aligned}\|x_t - x_n\|^2 &= \|(1-t)(T_{\beta_n}x_t - x_n) + t(f(x_t) - x_n)\|^2 \leq \\ &\leq (1-t)^2\|T_{\beta_n}x_t - x_n\|^2 + 2t\langle f(x_t) - x_n, J(x_t - x_n)\rangle \leq \\ &\leq (1 - 2t + t)^2)\|x_t - x_n\|^2 + f_n(t) \leq \\ &+ 2t\langle f(x_t) - x_t, J(x_t - x_n)\rangle + 2t\langle x_t - x_n, J(x_t - x_n)\rangle,\end{aligned} \tag{2.7}$$

where

$$f_n(t) = (2\|x_t - x_n\| + \|x_n - T_{\beta_n}x_n\|)\|x_n - T_{\beta_n}x_n\| \to 0, \quad \text{as } n \to 0. \tag{2.8}$$

It follows from (2.7) that

$$2t\langle x_t - f(x_t), x_t - x_n\rangle \leq t^2\|x_t - x_n\|^2 + f_n(t).$$

That is,

$$\langle x_t - f(x_t), J(x_t - x_n)\rangle \leq \frac{t}{2}\|x_t - x_n\|^2 + \frac{1}{2t}f_n(t). \tag{2.9}$$

Let $n \to \infty$ in (2.9) and note that (2.8) yields

$$\limsup_{n\to\infty}\langle x_t - f(x_t), J(x_t - x_n)\rangle \le \frac{t}{2}M_4, \tag{2.10}$$

where $M_4 > 0$ is a constant such that $M_4 \ge \|x_t - x_n\|^2$ for all $t \in (0,1)$ and $n \ge 1$. Taking $t \to 0$ from (2.10), we have

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle x_t - f(x_t), J(x_t - x_n)\rangle \le 0.$$

Since E is uniformly smooth, $J : E \to E^*$ is uniformly continuous on any bounded sets of E, which ensures that the order of $\limsup_{t\to 0}$ and $\limsup_{n\to\infty}$ is exchangeable, and hence (2.6) holds. Now from Lemma 1.6, we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|(1-\alpha_n)(y_n - x^*) + \alpha_n(f(x_n) - x^*)\|^2 \le \\ &\le \|(1-\alpha_n)(y_n - x^*)\|^2 + 2\alpha_n\langle f(x_n) - x^*, J(x_{n+1} - x^*)\rangle \le \\ &\le (1-\alpha_n)^2\|x_n - x^*\|^2 + \alpha_n\alpha(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) + \\ &\quad + 2\alpha_n\langle f(x^*) - x^*, J(x_{n+1} - x^*)\rangle,\end{aligned}$$

which implies that

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 + \\ &\le \frac{(1-\alpha_n)^2 + \alpha_n\alpha}{1-\alpha_n\alpha}\|x_n - x^*\|^2 + \frac{2\alpha_n}{1-\alpha_n\alpha}\langle f(x^*) - x^*, J(x_{n+1} - x^*)\rangle \le \\ &\le [1 - \frac{2\alpha_n(1-\alpha)}{1-\alpha_n\alpha}]\|x_n - x^*\|^2 + \\ &\quad + \frac{2\alpha_n(1-\alpha)}{1-\alpha_n\alpha}[\frac{1}{1-\alpha}\langle f(x^*) - x^*, J(x_{n+1} - x^*)\rangle + \frac{\alpha_n}{2(1-\alpha)}M_5],\end{aligned} \tag{2.11}$$

where M_5 is an appropriate constant such that $M_5 \ge \sup_{n\ge 1}\{\|x_n - x^*\|^2\}$. Put $j_n = \frac{2\alpha_n(1-\alpha)}{1-\alpha_n\alpha}$ and

$$t_n = \frac{1}{1-\alpha}\langle f(x^*) - x^*, J(x_{n+1} - x^*)\rangle + \frac{\alpha_n}{2(1-\alpha)}M_5.$$

That is,

$$\|x_{n+1} - q\|^2 \le (1 - j_n)\|x_n - q\| + j_n t_n. \tag{2.12}$$

It follows, from conditions (i), (ii) and (2.6), that $\lim_{n\to\infty} j_n = 0, \sum_{n=1}^{\infty} j_n = \infty$ and $\limsup_{n\to\infty} t_n \le 0$. Apply Lemma 1.7 to (2.12) to conclude $x_n \to q$. This completes the proof.

Remark 2.1. Lemma 2.1 improves Yao et al. [26] from non-expansive mappings to λ-strict pseudo-contractions.

Theorem 2.2. *Let C be a nonempty closed convex subset of a real 2-uniformly Banach space E and let $\{T\}_{i=1}^{r} : C \to C$ be a λ_i-strict pseudo-contraction such that $F = \cap_{i=1}^{r} F(T_i) \neq \emptyset$. Let $\{\mu_i\} \subset (0,1)$ be r real numbers with $\sum_{i=1}^{r} \mu_i = 1$ Given $f \in \Pi_C$ and $x_0 \in C$ and sequences $\{\alpha_n\}$ and $\{\beta_n\}$, we assume the following control conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \beta_n < \frac{\lambda}{K^2}$ *for all* $n \geq 0$.

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then $\{x_n\}$ generated by

$$\begin{cases} y_n = \beta_n \sum_{i=1}^{r} \mu_i T_i x_n + (1-\beta_n) x_n \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) y_n, \quad n \geq 1, \end{cases}$$

converges strongly to a common fixed point x^* of $\{T_i\}_{i=1}^{r}$, which solves the following variational inequality

$$\langle f(x^*) - x^*, J(p - x^*) \rangle \leq 0, \quad \forall p \in F(T).$$

Proof. Define $Tx = \sum_{i=1}^{r} T_i x$. By Lemma 1.3 and Lemma 1.4, we have $T : C \to C$ is λ-strict pseudo-contraction with $\lambda = \min\{\lambda_i : 1 \leq i \leq r\}$ and $F(T) = F(\sum_{i=1}^{r} T_i) = \cap_{i=1}^{r} F(T_i) = F$. From Lemma 2.1, we can conclude the desired conclusions easily.

Remark 2.2. Lemma 2.1 and Theorem 2.2 are applicable to l^p and L^p for all $p \geq 2$.

Applications

As applications of Lemma 2.1 and Theorem 2.2, we have the following results.

If $f(x) = u \in C$ for all $x \in C$ in Lemma 2.1 and Theorem 2.2, respectively, we have the following theorems.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real 2-uniformly Banach space E and let $T : C \to C$ be a λ-strict pseudo-contraction such that $F(T) \neq \emptyset$. Given $u, x_0 \in C$ and sequences $\{\alpha_n\}$ and $\{\beta_n\}$, such that the following control conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \beta_n < \frac{\lambda}{K^2}$ *for all* $n \geq 0$.

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,

then $\{x_n\}$ *generated by*

$$\begin{cases} y_n = \beta_n T x_n + (1-\beta_n)x_n \\ x_{n+1} = \alpha_n u + (1-\alpha_n)y_n, \quad n \geq 1, \end{cases}$$

converges strongly to a fixed point x^* *of* T, *where* $x^* = Q_F u$ *and* $Qu : C \to F$ *is the unique sunny nonexpansive retraction from* C *onto* F.

Remark 3.2. Theorem 3.1 improves Theorem 3.2 of Zhou [27] from Hilbert spaces to Banach spaces and Theorem 1 of Kim and Xu [8] from non-expansive mappings to λ-strict pseudo-contraction.

Theorem 3.3. Let C be a nonempty closed convex subset of a real 2-uniformly Banach space E and let $\{T\}_{i=1}^{r} : C \to C$ be a λ_i-strict pseudo-contraction such that $F = \cap_{i=1}^{r} F(T_i) \neq \emptyset$. Let $\{\mu_i\} \subset (0,1)$ be r real numbers with $\sum_{i=1}^{r} \mu_i = 1$ Given $u, x_0 \in C$ and sequences $\{\alpha_n\}$ and $\{\beta_n\}$, such that the following control conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \beta_n < \frac{\lambda}{K^2}$ for all $n \geq 0$.

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,

then $\{x_n\}$ generated by

$$\begin{cases} y_n = \beta_n \sum_{i=1}^{r} \mu_i T_i x_n + (1-\beta_n)x_n \\ x_{n+1} = \alpha_n u + (1-\alpha_n)y_n, \quad n \geq 1, \end{cases}$$

converges strongly to a common fixed point x^* of $\{T_i\}_{i=1}^{r}$, where $x^* = Q_F u$ and $Qu : C \to F$ is the unique sunny non-expansive retraction from C onto F.

Remark 3.4. Theorem 3.3 improves Theorem 1 of Kim and Xu [8] in the following senses:

(i) from non-expansive mappings to λ-strict pseudo-contractions;

(ii) from a single mapping to a finite family of mappings;

(iii) relaxing the restriction on parameters.

Acknowledgments

This project is supported by the National Natural Science Foundation of China (no. 10901140).

References

[1] F.E. Browder, Fixed point theorems for noncompact mappings in Hilbert spaces, Proc. Natl. Acad. Sci. USA, 53 (1965), 1272-1276.

[2] F.E. Browder, Convergence of approximants to fixed points of nonexpansive nonlinear mappings in Banach spaces, Arch. Ration. Mech. Anal., 24 (1967), 82-90.

[3] F.E. Browder and W.V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert space, J. Math. Anal. Appl., 20 (1967), 197-228.

[4] R.E. Bruck, Nonexpansive projections on subsets of Banach spaces, Pacific J. Math., 47 (1973), 341-355.

[5] C.E Chidume and S.A. Mutangadurea, An example on Mann iteration method for Lipschitz pseudo-contractions, Proc. Amer. Math. Soc., 129 (2001), 2359-2363.

[6] Y.J. Cho, S.M. Kang and X. Qin, Approximation of common fixed points of an infinite family of nonexpansive mappings in Banach spaces, Comput. Math. Appl., 56 (2008), 2058-2064.

[7] K. Deimling, Zeros of accretive operators, Manuscripta Math., 13 (1974), 365-374.

[8] T.H. Kim and H.K. Xu, Strong convergence of modified Mann iterations, Nonlinear Anal., 61 (2005), 51-60.

[9] G. Marino and H.K. Xu, Weak and strong convergence theorems for k-strict pseudo-contractions in Hilbert spaces, J. Math. Anal. Appl., 329 (2007), 336-349.

[10] W.R. Mann, Mean value methods in iteration, Proc. Amer. Math. Soc., 4 (1953), 506-510.

[11] A. Moudafi, Viscosity approximation methods for fixed points problems J. Math. Anal. Appl., 241 (2000), 46-55.

[12] M.O. Osilike and A. Udomene, Demiclosedness principle and convergence theorems for strictly pseudocontractive mappings of Browder-Petryshyn type, J. Math. Anal. Appl., 256 (2001), 431-445.

[13] X. Qin and Y. Su, Approximation of a zero point of accretive operator in Banach spaces, J. Math. Anal. Appl., 329 (2007), 415-424.

[14] X. Qin, M. Shang and S.M. Kang, Strong convergence theorems of modified Mann iterative process for strict pseudo-contractions in Hilbert spaces, Nonlinear Anal., 70 (2009), 1257-1264.

[15] X. Qin, Y.J. Cho and S.M. Kang, Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces, J. Comput. Appl. Math., 225 (2009), 20-30.

[16] X. Qin and Y. Su, Strong convergence theorems for relatively nonexpansive mappings in a Banach space, Nonlinear Anal., 67 (2007), 1958-1965.

[17] S. Reich, Strong convergence theorems for resolvents of accretive operators in Banach spaces, J. Math. Anal. Appl., 75 (1980), 287-292.

[18] S. Reich, Weak convergence theorems for nonexpansive mappings in Banach spaces, J. Math. Anal. Appl., 67 (1979), 274-276.

[19] S. Reich, Some problems and results in fixed point theory, Contemp. Math., 21 (1983), 179-187.

[20] B.E. Rhoades, Fixed point iterations using infinite matrices, Trans. Amer. Math. Soc., 196 (1974), 162-176.

[21] R. Wittmann, Approximation of fixed points of nonexpansive mappings, Arch. Math., 58 (1992), 486-491.

[22] H.K. Xu, An iterative approach to quadratic optimization, J. Optim. Theory Appl., 116 (2003), 659-678.

[23] H.K. Xu, Iterative algorithms for nonlinear operators, J. London Math. Soc., 66 (2002), 240-256.

[24] H.K. Xu, Viscosity approximation methods for nonexpansive mappings, J. Math. Anal. Appl., 298 (2004), 279-291.

[25] H.K. Xu, Inequalities in Banach spaces with applications, Nonlinear Anal., 16 (1991), 1127-1138.

[26] Y. Yao, R. Chen and J.C. Yao, Strong convergence and certain control conditions for modified Mann iteration, Nonlinear Anal., 68 (2008), 1687-1693.

[27] H. Zhou, Convergence theorems of fixed points for k-strict pseudo-contractions in Hilbert space, Nonliner Anal., 69 (2008), 456-462.

[28] H. Zhou, Convergence theorems for λ-strict pseudo-contration in 2-uniformly smooth Banach spaces, Nonlinear Anal., 69 (2008), 3160-3173.

Zhejiang Ocean University
School of Mathematics, Physics and Information Science
Zhoushan 316004, China
Email: zjhaoyan@yahoo.cn