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HOCHSCHILD (CO)HOMOLOGY IN COMMUTATIVE ALGEBRA. A SURVEY

Cristodor Ionescu

Dedicated to Professor Mirela Ştefănescu on the occasion of her 60th birthday

1. Introduction

Originally introduced for associative algebras over commutative rings, Hochschild (co)homology theory first played a big role in commutative algebra by the famous theorem obtained by Hochschild, Kostant and Rosenberg. For some time, the interest in this theory was not so big, but was reinitialized around 1990, due in principal to the works of A. Rodicio and its collaborators. Further important research by L. Avramov, M. Vigué-Poirrier and others, made this theory a fascinating subject for commutative algebra. We want to point out in this survey some of the most important results, as well as some of the possible further directions of research in this field.

In the following by a ring we mean a commutative ring with unit. We shall freely use the definitions and results of [24], [9], [16] and [2].

2. Hochschild (co)homology. Definition and properties

We shall present in this section the definitions and some basic properties of Hochschild (co)homology in the case of commutative rings.

Let A be a commutative ring, B be an A-algebra, M a $B \otimes_A B$-module. To define Hochschild (co)homology, we need to define the Hochschild complex. For this let us denote

$$C_n(B,M) := M \otimes_A B^{\otimes n} = M \otimes_A \underbrace{B \otimes_A ... \otimes_A B}_{n-times}$$

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and by $b : C_n(B,M) \longrightarrow C_{n-1}(B,M)$ the Hochschild boundary

$$b(m, a_1, ..., a_n) := (ma_1, a_2, ..., a_n)+$$

$$+\sum_{i=1}^{n-1}(-1)^i(m, a_1, ..., a_i a_{i+1}, ..., a_n) + (-1)^n(a_n m, a_1, ..., a_{n-1}).$$

2.1 Remark: $C_\bullet(B,M) := (C_n(B,M), b)$ is a complex of B-modules(i.e. $b^2 = 0$).

In the case $M = B$ we have

$$C_\bullet(B) := C_\bullet(B,B) : ... \to B^{\otimes(n+1)} \xrightarrow{b} B^{\otimes n} \xrightarrow{b} ... \xrightarrow{b} B^{\otimes 2} \xrightarrow{b} B$$

The n-th Hochschild homology module $H_n(B|A,M)$ is the n-th homology module(remember that everything is commutative !) of this complex. In the case $M = B$, we denote $H_n(B|A) := H_n(B|A,B)$. We shall also denote $H_\bullet(B|A) := \bigoplus_{n\geq 0} H_n(B|A)$.

2.2 Remarks: i) $H_0(B|A) \simeq B$ and $H_1(B|A) \simeq \Omega^1_{B/A}$ (the module of Kähler differentials of B over A).

ii) If B is a flat A-algebra, there is an isomorphism of B-modules

$$H_n(B|A,M) \simeq Tor_n^{B\otimes_A B}(M,B)$$

iii) $H_\bullet(B|A)$ becomes a strictly anticommutative graded B-algebra. The product is induced by the "shuffle" product:

$$C_p(B) \otimes C_q(B) \longrightarrow C_{p+q}(B)$$

$$(\alpha \otimes a_1 \otimes ... \otimes a_p) \otimes (\beta \otimes a_{p+1} \otimes ... \otimes a_{p+q} \mapsto \sum_\sigma \epsilon_\sigma \alpha\beta \otimes a_{\sigma^{-1}(1)} \otimes ... \otimes a_{\sigma^{-1}(p+q)}$$

where the sum is taken over all permutations $\sigma \in S_{p+q}$ such that $\sigma(1) < \sigma(2) < ... < \sigma(p)$ and $\sigma(p+1) < ... < \sigma(p+q)$.

2.3 Remarks: i) The B-module isomorphism $\gamma_1 : \Omega^1_{B|A} \longrightarrow H_1(B|A)$ extends to a morphism of graded B-algebras $\gamma : \Omega^\bullet_{B/A} \longrightarrow H_\bullet(B|A)$, where $\Omega^\bullet_{B/A}$ is the exterior algebra of $\Omega^1_{B/A}$. In degree n one has

$\gamma(da_1 \wedge ... \wedge da_n) =$class of $(\sum_{\sigma\in S_n} \epsilon_\sigma 1 \otimes a_{\sigma^{-1}(1)} \otimes ... \otimes a_{\sigma^{-1}(n)})$

ii) If A contains $\mathbf{Q}$ and B is a flat A-algebra, γ has a left inverse. More generally, in this case,

$$H_n(B|A) \simeq H_n^{(1)}(B|A) \oplus ... \oplus H_n^{(n)}(B|A), \forall n \geq 0$$

where $H_n^{(p)}(B|A) = H_{n-p}(\bigwedge^p \mathbf{L}_{B|A})$ and $\mathbf{L}_{B|A}$ is the cotangent complex of B over A. Moreover $H_n^{(1)}(B|A) \simeq H_{n-1}(A,B,B)$ is the $(n-1)$-th André-Quillen homology module of B over A with coefficients in B and $H_n^{(n)}(B|A) \simeq \Omega^n_{B/A} = \bigwedge^n \Omega^1_{B/A}$.

Next we define Hochschild cohomology using the same complex. Namely we define $H^n(B|A,M) := H^n(\mathrm{Hom}_B(C_\bullet(B,B),M)$ as the Hochschild cohomology of

B with coefficients in M. As before, in the case $M = B$ we denote $H^n(B|A) = H^n(B|A, B)$, the Hochschild cohomology of B over A.

2.4 Remarks: i) $H^0(B|A) \simeq B$ and $H^1(B|A) \simeq Der_A(B, B)$ the B-module of A-derivations of B.

ii) If B is a projective A-algebra, $H^n(B|A) \simeq Ext_n^{B \otimes_A B}(B, B)$.

iii) If B is a projective A-algebra, there exists a graded homomorphism

$$\omega : H^\bullet(B|A) := \underset{n \geq 0}{\oplus} H^n(B|A) \longrightarrow Hom_B(\Omega^\bullet_{B/A}, B)$$

iv) If B is flat over A and A contains $\mathbf{Q}$, ω has a right inverse. In this case we have also

$$H^n(B|A) \simeq H^n_{(1)}(B|A) \oplus ... \oplus H^n_{(n)}(B|A),$$

where $H^n_{(p)}(B|A) = H^{n-p}(\mathrm{Hom}_B(\overset{p}{\bigwedge} \mathbf{L}_{B/A}, B))$, so that $H^n_{(1)}(B|A) = H^{n-1}(A, B, B)$ is the $(n-1)$-th André-Quillen cohomology module of B over A.

2.5 Remark: Hochschild cohomology can be also described as the cohomology of the complex $(\underset{n \in \mathbf{N}}{\oplus} Mult^n_A(B, M), d)$ defined as follows: $Mult^n_A(B, M)$ denotes the A-multilinear functions on the n-fold cartesian product of B (if $M = B$ we write simply $Mult^n_A(B, B) = Mult^n_A(B)$) and

$$d(f)(b_1, ..., b_{n+1}) = b_1 f(b_2, ..., b_{n+1}) +$$
$$+ \sum_{0<i<n+1} (-1)^i f(b_1, ..., b_i b_{i+1}, ..., b_{n+1}) + (-1)^{n+1} f(b_1, ..., b_n) b_{n+1}$$

Finally we present a generalization of the Hochschild (co)homology to topological algebras introduced by R. Hübl[13].

Let A be a commutative ring, (B, τ) a topological algebra, where τ is a linear topology on B, $\widehat{B} = \widehat{(B, \tau)}$ the completion of B with respect to the topology τ. Let us denote by $C_n(B|A, \tau) := \underbrace{B \widehat{\otimes} ... \widehat{\otimes} B}_{(n+1)-times}$ the $(n+1)$-fold complete tensor product of (B, τ) over A. Let also denote by b the map induced by b on $C_n(B|A, \tau)$. Then $(C_\bullet(B|A, \tau), b)$ is a complex of $\widehat{B}$-modules. The $\widehat{B}$-module $H_n(B|A, \tau) := H_n(C_\bullet(B|A, \tau), b)$ is called the n-th Hochschild homology module of the topological algebra (B, τ) over A. We also denote $H_\bullet(B|A, \tau) := \underset{n \geq 0}{\oplus} H_n(B|A, \tau)$.

2.6 Remarks: i) $H_0(B|A, \tau) \simeq \widehat{B}; H_1(B|A, \tau) \simeq \Omega^1_{(B/A, \tau)}$(see [16]).

ii) $H_\bullet(B|A, \tau)$ is a graded anticommutative $\widehat{B}$-algebra with the product induced by the shuffle product.

iii) If $A \longrightarrow B$ is flat, τ is the I-adic topolgy on B for some ideal I of B and $C_n(B|A)$ is noetherian for all $n \in \mathbf{N}$, then there is an isomorphism of graded $\widehat{B}$-modules $H_\bullet(B|A, \tau) \simeq Tor_\bullet^{B \widehat{\otimes} B}(\widehat{B}, \widehat{B})$.

iv) If τ is the discrete topolgy on B, then $H_\bullet(B|A, \tau) \simeq H_\bullet(B|A)$.

One can also introduce the Hochschild cohomology of a topological algebra(see [13]).

Let (B, τ) be a topological A-algebra and let $Mult^n_{A,c}(B) \subseteq Mult^n_A(B)$ be the $B\widehat{\otimes}_A B$-submodule of continuous multilinear forms for $n > 0$ and $Mult^0_{A,c}(B) = B$. Then $d(Mult^n_{A,c}(B)) \subseteq Mult^{n-1}_{A,c}(B)$ so we have a subcomplex:

$$(\bigoplus_{n\in\mathbf{N}} Mult^n_{A,c}(B), d) \subseteq (\bigoplus_{n\in\mathbf{N}} Mult^n_A(B), d)$$

We define $H^n(B|A,\tau) := H^n((\bigoplus_{n\in\mathbf{N}} Mult^n_{A,c}(B), d)$, the n-th Hochschild cohomology module of the topological algebra (B, τ) over A and

$$H^\bullet(B|A,\tau) := \bigoplus_{n\in\mathbf{N}} H^n(B|A,\tau).$$

2.7 Remarks: i) If τ is the discrete topology on B, then $H^n(B|A,\tau) \simeq H^n(B|A)$.
ii) $H^0(B|A,\tau) \simeq \widehat{B}$.
iii) $H^n(B|A,\tau)$ has a structure of a $\widehat{B}$-module.
iv) There exists a canonical cohomology product

$$H^p(B|A,\tau) \otimes H^q(B|A,\tau) \longrightarrow H^{p+q}(B|A,\tau)$$

which induces a structure of an associative graded $\widehat{B}$-algebra on $H^\bullet(B|A,\tau)$.

3. Hochschild homology in commutative algebra

The story begins with a famous result obtained 40 years ago and known as HKR-theorem:

3.1 Theorem(Hochschild, Kostant, Rosenberg[12]): *Let K be a perfect field, A a regular K-algebra essentially of finite type. Then*

$$\gamma : H_\bullet(A|K) \longrightarrow \Omega^\bullet_{A/K}$$

is an isomorphism.

Using methods of André-Quillen homology this result was brought to a nice general form by André:

3.2 Theorem [1]: *Let A be a noetherian ring, B a noetherian flat A-algebra. The following are equivalent:*

1) $u : A \longrightarrow B$ is a regular morphism;

2) $\Omega^1_{B/A}$ is a flat B-module and γ is an isomorphism.

In [25], A. Rodicio proposed a conjecture which represents a converse of the theorem of Hochschild, Kostant and Rosenberg in a special case:

3.3 Conjecture [25]: *Let K be field of characteristic zero, A a K-algebra essentially of finite type. If there is a natural number $n > 0$ such that $H_m(A|K) = 0, \forall m \geq n$, then A is smooth over K.*

3.4 Remark: In the above context, the smoothness of A is equivalent with the regularity of the structural morphism of A over K or with the regularity of the noetherian ring A.

The first result was given by J. Majadas and A. Rodicio[22] in the case A is locally a complete intersection, i.e. $A = R/J$, where R is a polynomial ring in a finite number of variables and J is an ideal locally generated by a regular sequence. The conjecture was solved by BACH[7] and generalized by L. Avramov and M. Vigué-Poirrier[5] to the case of any field. A further generalization was given by A. Rodicio[26], simplifying the proof of Avramov and Vigué-Poirrier:

3.5 Theorem [27]: *Let A be a ring and B a noetherian augmented A-algebra with augmentation ideal I. If there are a positive even integer i and a positive odd integer j such that $Tor_i^B(A,A) = Tor_j^B(A,A) = 0$, then I is locally generated by a regular sequence.*

As a corollary one obtains at once:

3.6 Corollary [27]: *Let $u : A \longrightarrow B$ be a flat morphism such that $B \otimes_A B$ is noetherian. The following are equivalent:*

1) B is smooth over A;

2) γ is an isomorphism;

3) $H_\bullet(B|A)$ is generated as a B-algebra by its elements of degree 1;

4) There exists $n > 0$ such that $H_i(B|A) = 0, \forall i > n$.

Further work by L. Avramov and S. Iyengar led to the folowing characterization of smoothness:

3.7 Theorem [3]: *Let A be a noetherian ring, B a flat A-algebra essentially of finite type. The following conditions are equivalent:*

1) B is a smooth A-algebra;

2) $\Omega^1_{B/A}$ is a finite projective B-module and γ is an isomorphism;

3) There exists $n > 0$ such that $H_m(B|A) = 0, \forall n \geq m$;

4) There exist an even positive integer i and an odd positive integer j such that $H_i(B|A) = H_j(B|A) = 0$;

5) $H_\bullet(B|A)$ is a finitely generated B-algebra.

3.8 Example: Condition iv) of the above theorem cannot be weakened to the vanishing of a single index i. Namely Larsen and Lindenstrauss[17] proved that if R is a Dedekind domain with perfect residue fields and with fraction field K, L a finite separable field extension of K and S the integral closure of R in L then:

$$H_n(S|R) = \begin{cases} S & \text{if} \quad n = 0 \\ \Omega^1_{S/R} & \text{if} \quad n \geq 1 \ \ odd \\ 0 & \text{if} \quad n \geq 2 \ \ even \end{cases}$$

One can see [28] for a simpler proof.

However, in special cases, it is enough that a single Hochschild homology module vanishes:

3.9 Theorem [26]: *Let K be a field, A a K-algebra essentially of finite type, locally a complete intersection. If there is a positive integer $i \in \mathbf{N}$ such that $H_i(A|K) = 0$, then A is a smooth K-algebra.*

The previous results give a characterization of regular rings which are algebras of finite type over a field of characteristic zero(or over a perfect field). The next ones are following this road, characterizing the algebras which are complete intersections.

3.10 Theorem [31]: *Let K be a field of characteristic zero and A a K-algebra of finite type, locally a complete intersection. Then* $H_n^{(p)}(A|K) = 0, \forall p < \frac{n}{2}, \forall n \geq 0$.

The reciprocal has a difficult proof.

3.11 Theorem [29]: *Let K be a field of characteristic zero and A a K-algebra of finite type. Suppose there is $N > 0$ such that $\forall n > N, \forall p < \frac{n}{2}, H_n^{(p)}(A|K) = 0$. Then A is locally a complete intersection.*

In [29] one can see how these results are applied in K-theory.

There are also some other connections between the vanishing of Hochschild homology and regular morphisms. First let us note:

3.12 Remark: Let $u : A \longrightarrow B$ be a morphism of rings. Then the following are clearly equivalent:

i) $fd_{B\otimes_A B}(B) < \infty$;

ii)$\exists n > 0$ such that $Tor_r^{B\otimes_A B}(M, B) = 0, \forall r > n$ and for any B-module M;

iii) (if moreover u is flat) $\exists n > 0$ such that $H_r(B|A, M) = 0, \forall r > n$ and for all A-modules M.

3.13 Theorem [23]; *Let $u : A \longrightarrow B$ be a flat morphism of noetherian rings. If $fd_{B\otimes_A B}(B) < \infty$, then u is a regular morphism.*

In special situations one can characterize the morphisms $u : A \longrightarrow B$ such that $fd_{B\otimes_A B}(B) < \infty$:

3.14 Theorem [23]: *Let $K \subseteq L$ be a field extension. The following are equivalent:*

i) $fd_{L\otimes_K L}(L) < \infty$;

ii) L is a separable extension of K and $trdeg(L|K) < \infty$.

3.15 Theorem [23]: *Let K be a perfect field and A a K-algebra which is a Dedekind domain. Let also $L = Q(A)$ be the field of fractions of A. The following are equivalent:*

i) $fd_{A\otimes_K A}(A) < \infty$;

ii) $trdeg(L|K) < \infty$.

4. Hochschild cohomology in commutative algebra

Hochschild cohomology was not so intensively studied with respect to its significance in commutative algebra. However, some results analogous with those presented in the previous section exist.

Firts of all, note that a result similar to 3.1 was already obtained by Hochschild, Kostant and Rosenberg.

4.1 Theorem [12]: *Let $u : A \longrightarrow B$ be a smooth norphism of noetherian rings. Then ω is an isomorphism.*

There are not so many converses as in the homological case. We mention them briefly.

4.2 Theorem [18]: *Let K be a field of characteristic zero, A a K-algebra essentially of finite type, locally a complete intersection. If there is an even natural number i such that $H^i(A|K) = 0$, then A is a smooth K-algebra.*

This result was generalized to arbitrary characteristic by J.A. Guccione and J.J Guccione.

4.3 Theorem [10]: *Let K be field, A a smooth K-algebra essentially of finite type, I an ideal of A that is locally generated by a regular sequence, B:=A/I. If there is an even i such that* $\omega_i : H^i(B|K) \longrightarrow Hom_B(\Omega^i_{B/K}, B)$ *is injective, then B is smooth over K.*

4.4 Corollary [10]: *If K,A and B are as above, if there is some even i such that* $H^i(B|K) = 0$ *then B is smooth over K.*

Another result was obtained in the case of Gorenstein algebras by A. Blanco and J. Majadas.

4.5 Theorem [6] *Let* $u : A \longrightarrow B$ *be a flat morphism such that* $B \otimes_A B$ *is noetherian. Suppose that:*

i) B is a Gorenstein ring of dimension d(not necessarily finite!);

ii) $H^n(B|A) = 0, \forall n \in \{m, ..., m+d+2\}$.

Then u is a regular morphism.

In special cases, as a corollary one can obtain stronger results:

4.6 Corollary [6]: *Let K be a field of characteristic zero, A a K-algebra essentially of finite type. Suppose that:*

i) A is a Cohen-Macaulay normal ring of dimension d;

ii) $\Omega^1_{A/K}$ *has constant rank r;*

iii) $H^n(A|K) = 0, \forall n \in \{r+1, ..., r+d\}$.

Then A is a regular ring.

5. The topological case

In the topological case a first analogue of the theorem of Hochschild, Kostant and Rosenberg was proved by Hübl:

5.1 Proposition [14]: *Let K be a perfect field, A a K-algebra, I an ideal of A,* τ *the I-adic topology on A. Suppose that* $\underbrace{A\widehat{\otimes}...\widehat{\otimes}A}_{n-times}$ *is noetherian for all* $n \in \mathbf{N}$*. If A is a formally smooth K-algebra, then* $H_\bullet(A|K,\tau) \simeq \Omega^\bullet_{(A/K,\tau)}$.

This result was generalized in [15]:

5.2 Theorem [15]: *Let* $u : A \longrightarrow B$ *be a flat morphism of noetherian rings, I an ideal of B,* τ *the I-adic topology on B. Suppose that* $\underbrace{B\widehat{\otimes}...\widehat{\otimes}B}_{n-times}$ *is noetherian for all* $n \in \mathbf{N}$*. If u is formally smooth, then* $H_\bullet(B|A,\tau) \simeq \Omega^\bullet_{(B/A,\tau)}$.

The converse was proved by Majadas([21]) and the final result can be stated in the following nice form:

5.3 Theorem: *Let* $u : A \longrightarrow B$ *be a flat morphism of noetherian rings, I an ideal of B,* τ *the I-adic topology on B. Assume that* $B\widehat{\otimes}B$ *is a noetherian ring. Then the following are equivalent:*

i) u is formally smooth;

ii) $\Omega^\bullet_{(B/A,\tau)} \simeq Tor^{B\widehat{\otimes}B}_\bullet(\widehat{B}, \widehat{B})$;

iii) There is an even i and an odd j such that

$$Tor_i^{B\widehat{\otimes}B}(\widehat{B},\widehat{B}) = Tor_j^{B\widehat{\otimes}B}(\widehat{B},\widehat{B}) = 0;$$

iv) $fd_{B\widehat{\otimes}B}(\widehat{B}) < \infty$.

6. Further possible research

From the previous sections it comes out at once that there are still open problems and several ideas of further research. In this section we shall point out some of these problems and directions.

6.1 Question [23]: *Characterize the regular morphisms $u : A \longrightarrow B$ such that* $fd_{B\otimes_A B}(B) < \infty$.

In view of 3.14 and 3.15, it is normal to consider the following:

6.2 Question [23]: *Let K be a field, A a noetherian local K-algebra. Suppose that A is geometrically regular over K and $trdeg(A|K) < \infty$. Does it follow that* $fd_{A\otimes_K A} < \infty$?.

Another difficult problem connected with the morphism γ is the following:

6.3 Question: *Let K be a field, A a K-algebra. Is γ injective?*

In connection with the example 3.8 and theorem 3.9 , it would be very interesting to study the following conjecture proposed in [26]:

6.4 Conjecture: *Let K be a field, A a K-algebra essentially of finite type. If there is some i such that $H_i(A|K) = 0$, A is smooth over K.*

In [19], A. Lago and A. Rodicio proposed the following question thinking of some generalization of 3.5 to the case when B is no more noetherian :

6.5 Question: *Let A be a noetherian ring, M an A-module, $B = Sym_A(M)$ the symmetric algebra of M. If $fd_B(A) < \infty$, is M a flat A-module?*

The theorems 3.6, 3.10, 3.11 give characterizations of algebras essentially of finite type over a field of characteristic zero(or over a perfect field) which are regular or complete intersections. Following this idea M. Vigué-Poirrier suggested:

6.6 Problem [30]: *Characterize, in terms of Hochschild (co)homology, algebraic properties of algebras over a field,e.g. Cohen-Macaulay, Gorenstein etc.*

Less studied than the homology, Hochschild cohomology offers clearly a big land of research. It is easily seen, from all the results presented, that almost all are obtained in the homology. So:

6.7 Problem: *To what extent the results known to be true for Hochschild homology have counterparts for the Hochschild cohomology?*

Hochschild homology of a topological algebra was used to characterize formally smooth morphisms(5.2, 5.3), generalizing the Hochschild-Kostant-Rosenberg theorem and its reciprocals.

6.8 Problem: *To what extent the results presented for the classical Hochschild homology have counterparts in the topological case?*

In our survey no results about Hochschild cohomology of a topological algebra was mentioned. This is because this kind of results is missing completely.

6.9 Problem: *Study the implications of Hochschild cohomology of a topological algebra in characterizing morphisms of commutative rings and classes of noetherian rings.*

Finally let us mention another idea of further research suggested by L. Avramov and S. Iyengar[4]. In its attempt to use operator algebras in differential geometry, A. Connes discovered cyclic homology[20], which is strongly related to Hochschild homology. It appears that, in the smooth case, cyclic homology is a generalization of de Rham cohomology of an algebra. The strong connection between the Hochschild homology and cyclic homology suggests:

6.10 Problem [4]: *Study the connections between the properties of an algebra and the properties of its cyclic homology.*

One can see for example [8] and [11] for some idea about using cyclic homology in commutative algebra.

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Institute of Mathematics "Simion Stoilow",
of the Romanian Academy,
P.O. Box 1-764
RO-70700 Bucharest,
Romania
e-mail: cristodor.ionescu@imar.ro