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SOME EQUATIONS IN ALGEBRAS OBTAINED BY THE CAYLEY-DICKSON PROCESS

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Dedicated to Professor Mirela Ştefănescu on the occasion of her 60th birthday

Abstract

In this paper we try to solve three fundamental equations $ax = xb$, $ax = \bar{x}b$ and $x^2 = a$, in a division algebra, A over K, obtained with the Cayley-Dickson process (see [Br; 67]), in the case when $K$is an arbitrary field of characteristic $\neq 2$.

§1. INTRODUCTION

Unless otherwise indicated, K denotes a commutative field with characteristic $\neq 2$ and Adenotes a non-associative algebra over K.

Definition 1.1. The algebra A is called **alternative** if $x^2y = x(xy)$ and $yx^2 = (yx)x$, $\forall x, y \in A$.

Let A be an alternative algebra and $x, y, z \in A$. We define the **associator** of elements x, y, z by the equality: $(x, y, z) := (xy)z - x(yz)$ .This is linear in each argument and satisfies the identities:

i) $(x, y, z) = -(y, x, z) = -(x, z, y) = (z, x, y)$;

ii) $(x, x, y) = 0$;

ii) $(x, y, a) = 0, a \in K$.

Definition 1.2. An algebra A is called **power-associative**, if each element of A generates an associative subalgebra.

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In a power-associative algebra,the power a^n $(n \geq 1)$ of an element a is defined in a unique way and we have : $(a^n)^m = a^{nm}, a^n a^m = a^{n+m}$.

Definition 1.3. An algebra A is called **a composition algebra** if there exists a quadratic form $n : A \rightarrow K$ such that $n(xy) = n(x)n(y)$, for any $x, y \in A$ and the bilinear associated form $f : A \times A \rightarrow K$, $f(x,y) = \frac{1}{2}(n(x+y) - n(x) - n(y))$is non-degenerate.The quadratic form n is also called **the norm** on A.

A composition algebra with unity is also called a **Hurwitz** algebra**.** The non-zero finite-dimensional composition algebras over fields with characteristic different from 2 can have only the dimensions $1, 2, 4$ or 8.[El, Pe-I; 99]

Definition 1.4. An algebra A is called **flexible** if $x(yx) = (xy)x$, for all $x, y \in A$.

Definition 1.5. The vector space morfism $\phi : A \rightarrow A$ is called **an involution of the algebra** A if $\phi(\phi(x)) = x$ and $\phi(xy) = \phi(x)\phi(y)$, for all $x, y \in A$.

Let A be an arbitrary finite-dimensional algebra with unity 1.We consider the involution of the algebra $A, \phi : A \rightarrow A, \phi(a) = \overline{a}$,where $a+\overline{a}$ and $a\overline{a} \in K \cdot 1$, for all $a \in A$. Let $\alpha \in K$ be a fixed non-zero element.On the vector space $A \oplus A$ we define the following operation of multiplication

$$(a_1, a_2)(b_1, b_2) = \left(a_1 b_1 - \alpha \overline{b_2} a_2, a_2 \overline{b_1} + b_2 a_1\right).$$

The resulting algebra is denoted by (A, α) and is called the **algebra derived from the algebra** A by the **Cayley-Dickson process.** We can easily prove that A is isomorphic with a subalgebra of algebra the (A, α) and $\dim(A, \alpha) = 2 \dim A$. We denote $v = (0, 1)$ and we get $v^2 = -\alpha \cdot 1$,where $\mathbf{1} = (0, 1)$, therefore $(A, \alpha) = A \oplus Av$.

Let $x = a_1 + a_2 v \in (A, \alpha)$, and denote $\overline{x} = \bar{a}_1 - a_2 v$.Then $x + \overline{x} = a_1 + \overline{a_1} \in K \cdot 1$, $x\overline{x} = a_1\overline{a_1} + \alpha a_2 \overline{a_2} \in K \cdot 1$, therefore the mapping

$\psi : (A, \alpha) \rightarrow (A, \alpha)$, $\psi(x) = \bar{x}$,is an involution of the algebra (A, α) extending the given involution ϕ.

For $x \in A$ $t(x) = x + \overline{x} \in K$ and $n(x) = x\overline{x} \in K$ are called the *trace* and the *norm* of the element $x \in A$.

If $z \in (A, \alpha)$, $z = x + yv$, then $z + \overline{z} = t(z) \cdot 1$ and $z\overline{z} = \overline{z}z = n(z) \cdot 1$, where $t(z) = t(x)$ and $n(z) = n(x) + \alpha n(y)$. Therefore $(z + \overline{z})z = z^2 + \overline{z}z = z^2 + n(z)$ and $z^2 - t(z)z + n(z) = 0, \forall z \in (A, \alpha)$ that is each algebra which is obtained by the Cayley-Dickson process is a **quadratic algebra.** In [Sc; 54], it appears that such an algebras is power- associative flexible and satisfies the identities: $t(xy) = t(yx), t((xy)z) = t(x(yz)), \forall x, y, z \in (A, \alpha)$.

The algebra (A, α) is a Hurwitz algebra if and only if it is alternative and (A, α) is alternative if and only if A is an associative algebra.[Ko, Sh; 95].

Proposition 1.6. *Let (A, α) be an algebra obtained by the Cayley-Dickson process.*

i) If A is an alternative algebra, then $(xy)\overline{x} = x(y\overline{x}) = xy\overline{x}$, $\forall x, y \in (A, \alpha)$.

ii) If $n(x) \neq 0$, then there exists $x^{-1} = \frac{\overline{x}}{n(x)}$, for all $x \in (A, \alpha)$. If (A, α) is an alternative algebra, then $(xy)x^{-1} = x(yx^{-1}) = xyx^{-1}$, for all $x, y \in (A, \alpha)$.

Proof. The following identities are true : $(x, y, x) = 0$ and $(x, y, \pi) = 0, \pi \in K$. Then $(x, y, \overline{x}) + (x, y, x) = (x, y, t(x)) = 0$, therefore $(x, y, \overline{x}) = 0$.□

The Cayley-Dickson process can be applied to each Hurwitz algebra. If $A = K$, this process leads to the following Hurwitz algebras over K :

1) The field K of characteristic $\neq 2$.

2) $\mathbb{C}(\alpha) = (K, \alpha), \alpha \neq 0$. If the polynomial $X^2 + \alpha$ is irreducible over K , then $\mathbb{C}(\alpha)$ is a field. Otherwise $\mathbb{C}(\alpha) = K \oplus K$.

3) $\mathbb{H}(\alpha, \beta) = (\mathbb{C}(\alpha), \beta), \beta \neq 0$, the algebra of the generalized quaternions,which is associative but it is not commutative.

4) $\mathbb{O}(\alpha, \beta, \gamma) = (\mathbb{H}(\alpha, \beta), \gamma), \gamma \neq 0$,the algebra of the generalized octonions (also a Cayley-Dickson algebra).The algebra $\mathbb{O}(\alpha, \beta, \gamma)$ is non-associative, therefore the process of obtaining Hurwitz algebras ends here. [Ko,Sh;95]

Definition 1.7. Let A be an arbitrary algebra over the field K. It is a **division algebra** if $A \neq 0$ and the equations $ax=b$, $ya=b$, for every $a, b \in A$, $a \neq 0$, have unique solutions in A.

Proposition 1.8.[Ko, Sh; 95] *Let A be a Hurwitz algebra.The following statements are equivalent:*

i) There exists $x \in A, x \neq 0$ such that $n(x) = 0$.

ii) There exists $x, y \in A, x \neq 0, y \neq 0$,such that $xy = 0$;

iii) A contains a non-trivial idempotent (i.e.an element e, $e \neq 0, 1$ such that $e^2 = e$).□

Definition 1.9. Any Hurwitz algebra which satisfies one of the above equivalent conditions is called a **split Hurwitz algebra.**

Every Hurwitz algebra is either a division algebra or a split algebra.

If K is an algebrically closed field, then we obtain only split algebras.

We have obtained by the Cayley-Dickson process some algebras with dimension bigger than 8 which are non-alternative and non-associative but are quadratic and flexible algebras. Every one of these algebras is central simple (i.e. $A_F = F \otimes_K A$ is a simple algebra, for every extension F of K and for every dimension).

Remark. 1.10. *For every algebra A obtained by the Cayley-Dickson process we has the relation: $2f(x,1) = t(x)$, $\forall x \in A$, where f is the bilinear form associated with the norm n.*

Proposition 1.11. *In each algebra obtained by the Cayley-Dickson process the following relation is satisfied: $xy + \bar{y}\bar{x} = 2f(x,\bar{y})1$, where f is the bilinear form associated with the norm n.*

Proof . As $\bar{x} = 2f(x,1) - x$, we have:
$xy + \bar{y}\bar{x} = xy + (2f(y,1) - y)(2f(x,1) - x) = xy + 4f(y,1)f(x,1) -$
$-2f(y,1)x - 2f(x,1)y + yx$.
$2f(x,\bar{y}) = 2f(x, 2f(y,1) \cdot 1 - y) = 2f(x, 2f(y,1) \cdot 1) - 2f(x,y) =$
$= 4f(x,1)f(y,1) - 2f(x,y) = 4f(x,1)f(y,1) - n(x+y) + n(x) +$
$+n(y) = 4f(x,1)f(y,1) - (x+y)(\bar{x}+\bar{y}) + x\bar{x} + y\bar{y} =$
$= 4f(x,1)f(y,1) - x\bar{x} - x\bar{y} - \bar{x}y - y\bar{y} + x\bar{x} + y\bar{y} =$
$= 4f(x,1)f(y,1) - x\bar{y} - \bar{x}y =$
$= 4f(x,1)f(y,1) - x(2f(y,1) - y) - y(2f(1,x) - x) =$
$= 4f(x,1)f(y,1) - 2f(y,1)x - 2f(x,1)y + xy + yx$ and we get the required equality.□

Proposition 1.12. *Let A be a composition division algebra, $f: A \times A \to K$, $n: A \to K$ be the bilinear form and respectively the norm of A. Then, for $v, w \in A \backslash \{0\}$, we have $f^2(v,w) = f(v,v)f(w,w)$, if and only if $v = rw$, $r \in K$.*

Proof. If $v = rw, r \in K$, then the equality is true.

Conversely, if $f^2(v,w) = f(v,v)f(w,w)$, for $v \neq 0, w \neq 0$, we have $f^2(v,w) \neq 0$. We suppose that $r \in K$ with $v = rw$ does not exist. Then, for non-zero elements $a, b \in K$, we have $av + bw \neq 0$. Indeed, if $av + bw = 0$ then $v = -\frac{b}{a}w$, with $-\frac{b}{a} \in K$, which is false. We get that $f(av + bw, av + bw) \neq 0$ and we have $a^2 f(v,v) + b^2 f(w,w) + 2ab f(v,w) \neq 0$. For $a = f(w,w)$, we obtain $f(w,w)f(v,v) + b^2 + 2bf(v,w) \neq 0$ and, for $b = -f(v,w)$, we have $f(w,w)f(v,v) + f^2(v,w) - 2f^2(v,w) \neq 0$, therefore $f(w,w)f(v,v) \neq f^2(v,w)$, which is false. Hence $av + bw = 0$ implies $v = rw$.□

Theorem 1.13.(Artin).[Ko, Sh; 95] *In each alternative algebra A, any two elements generate an associative subalgebra.*

Corollary 1.14.[Ko, Sh; 95] *Each alternative algebra is a power-associative algebra.*

Proposition 1.15. *Let A be a unitary division power-associative algebra (with finite or infinite dimension). Then every subalgebra of A is a unitary algebra.*

Proof. Let B be a subalgebra of the algebra A and $b \in B, b \neq 0$. We denote by $\mathcal{B}(b)$ the subalgebra of B generated by b, which is an associative algebra (A is power-associative). Since A is a division algebra, $\mathcal{B}(b)$ is a unitary algebra, then B is unitary. □

Proposition 1.16. *Let A be a unitary division power-associative algebra (with finite or infinite dimension). Then $\mathcal{A}(a,b) = \mathcal{A}(a-\pi, b-\theta)$, with $\pi, \theta \in K$, where by $\mathcal{A}(a,b)$ we denote the subalgebra generated by the elements $a, b \in A$.*

Proof. From *Proposition 1.15*, we have $1 \in \mathcal{A}(a-\pi, b-\theta)$, so $\pi, \theta \in \mathcal{A}(a-\pi, b-\theta)$. We obtain $a = (a-\pi)+\pi \in \mathcal{A}(a-\pi, b-\theta)$ and $b = (b-\theta)+\theta \in \mathcal{A}(a-\pi, b-\theta)$. Therefore $\mathcal{A}(a,b) \subset \mathcal{A}(a-\pi, b-\theta)$. Since $1 \in \mathcal{A}(a,b)$, we have $a-\pi, b-\theta, \pi, \theta \in \mathcal{A}(a,b)$, so we have the required equality.□

§ 2. Equations in the generalized quaternion algebras

Consider the generalized quaternion algebra, $\mathbb{H}(\alpha, \beta)$, with dimension 4, and the basis $\{1, e_1, e_2, e_3\}$, its multiplication operation is listed in the following table:

$\cdot$	1	e_1	e_2	e_3
1	1	e_1	e_2	e_3
e_1	e_1	$-\alpha$	e_3	$-\alpha e_2$
e_2	e_2	$-e_3$	$-\beta$	βe_1
e_3	e_3	αe_2	$-\beta e_1$	$-\alpha\beta$

Remark 2.1 The algebra $\mathbb{H}(\alpha, \beta)$ is either a division algebra or a split algebra, in this case being isomorphic to algebra $\mathcal{M}_2(K)$. In the following, we will show how to distinguish these two cases.

Let $x = a+be_1+ce_2+de_3 \in \mathbb{H}(\alpha,\beta)$. The element $\overline{x} = a-be_1-ce_2-de_3$ is called the conjugate of the element x. The *norm* and the *trace* of the element x are the elements of K: $n(x) = x\overline{x} = a^2+\alpha b^2+\beta c^2+\alpha\beta d^2, t(x) = x+\overline{x} = 2a$.

The algebra $\mathbb{H}(\alpha,\beta)$ is a division algebra if and only if for any $x \in \mathbb{H}(\alpha,\beta), x \neq 0$, implies $n(x) \neq 0$, therefore if and only if the equation $a^2+\alpha b^2+\beta c^2+\alpha\beta d^2 = 0$ has only the trivial solution. We write this equation under some equivalent forms: $\left(a^2+\alpha b^2\right) = -\beta c^2 - \alpha\beta d^2 = -\beta\left(c^2+\alpha d^2\right)$or $\beta = -\frac{n(a+be_1)}{n(c+de_1)} = -n\left(\frac{a+be_1}{c+de_1}\right) = -n(\varepsilon+\delta e_1) = -\varepsilon^2 - \alpha\delta^2$,where $\varepsilon+\delta e_1 = \frac{a+be_1}{c+de_1}$ or else $n(z) = -\beta$, where $z = \varepsilon+\delta e_1 \in \mathbb{C}(\alpha)$.

Therefore $\mathbb{H}(\alpha,\beta)$ is a division algebra if and only if $\mathbb{C}(\alpha)$ is a quadratic separable extension of the field K and the equation $n(z) = -\beta$ does not have non-zero solutions in $\mathbb{C}(\alpha)$. Otherwise $\mathbb{H}(\alpha,\beta)$ is a split algebra. Since, if $\mathbb{C}(\alpha)$ is a quadratic separable extension of the field K ,for $x \in \mathbb{H}(\alpha,\beta), x = a_1+a_2v$, with $a_1, a_2 \in \mathbb{C}(\alpha)$, $v^2 = -\beta, x \neq 0$ and $n(x) = 0$, then $a_2 \neq 0$. Indeed, if $n(x) = 0$ and $a_2 = 0$ we get $n(x) = n(a_1) = a^2+\alpha b^2, a_1 = a+bv, v^2 = -\alpha$, therefore the polynomial $X^2+\alpha$ has a solution in K,false.□

In the following, we consider that $\mathbb{H}(\alpha,\beta)$ is a division generalized quaternion algebra.

Definition 2.2. The linear applications $\overline{\lambda}, \overline{\rho} : \mathbb{H}(\alpha,\beta) \to End_K(\mathbb{H}(\alpha,\beta))$, given by

$\overline{\lambda}(a) : \mathbb{H}(\alpha,\beta) \to \mathbb{H}(\alpha,\beta), \overline{\lambda}(a)(x) = ax, a \in \mathbb{H}(\alpha,\beta)$and

$\overline{\rho}(a) : \mathbb{H}(\alpha,\beta) \to \mathbb{H}(\alpha,\beta), \overline{\rho}(a)(x) = xa, a \in \mathbb{H}(\alpha,\beta)$,are called **the left representation** and **the right representation** of the algebra $\mathbb{H}(\alpha,\beta)$.

We know that every associative finite-dimensional algebra A over an arbitrary field K is isomorphic with a subalgebra of the algebra $\mathcal{M}_n(K)$, with $n = \dim_K A$. So we could find a faithful representation for the algebra A in the algebra $\mathcal{M}_n(K)$. For the generalized quaternion algebra $\mathbb{H}(\alpha,\beta)$, the mapping:

$$\lambda : \mathbb{H}(\alpha,\beta). \to \mathcal{M}_4(K), \lambda(a) = \begin{pmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha\beta a_3 \\ a_1 & a_0 & -\beta a_3 & \beta a_2 \\ a_2 & \alpha a_3 & a_0 & -\alpha a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{pmatrix},$$

where $a = a_0+a_1e_1+a_2e_2+a_3e_3 \in \mathbb{H}(\alpha,\beta)$ is an isomorphism between $\mathbb{H}(\alpha,\beta)$ and the algebra of the matrices of the above form.

Obviously $\overline{\lambda}(a)(1) = a, \overline{\lambda}(a)(e_1) = ae_1, \overline{\lambda}(a)(e_2) = ae_2, \overline{\lambda}(a)(e_3) = ae_3$, represents the first, the second, the third and the fourth columns of the matrix $\lambda(a)$.

Definition 2.3. $\lambda(a)$ is called **the left matriceal representation** for the element $a \in \mathbb{H}(\alpha, \beta)$.

In the same manner, we introduce **the right matriceal representation** for the element $a \in \mathbb{H}(\alpha, \beta)$:

$$\rho : \mathbb{H}(\alpha, \beta) \rightarrow \mathcal{M}_4(K), \rho(a) = \begin{pmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha\beta a_3 \\ a_1 & a_0 & \beta a_3 & -\beta a_2 \\ a_2 & -\alpha a_3 & a_0 & \alpha a_1 \\ a_3 & a_2 & -a_1 & a_0 \end{pmatrix}, \text{where}$$

$a = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 \in \mathbb{H}(\alpha, \beta)$ and $\overline{\rho}(1) = a, \overline{\rho}(a)(e_1) = e_1 a, \overline{\rho}(a)(e_2) = e_2 a, \overline{\rho}(a)(e_3) = e_3 a$ represent the first, the second, the third and the fourth columns of the matrix $\rho(a)$.

Proposition 2.4.([Ti; 00], Lemma 1.2.)*Let* $x, y \in \mathbb{H}(\alpha, \beta)$ *and* $r \in K$. *Then the following statements are true:*

i) $x = y \Longleftrightarrow \lambda(x) = \lambda(y)$.

ii) $x = y \Longleftrightarrow \rho(x) = \rho(y)$.

iii) $\lambda(x+y) = \lambda(x) + \lambda(y), \lambda(xy) = \lambda(x)\lambda(y), \lambda(rx) = r\lambda(x)$, $\lambda(1) = I_4$, $r \in K$.

iv) $\rho(x+y) = \rho(x) + \rho(y), \rho(xy) = \rho(x)\rho(y), \rho(rx) = r\rho(x)$, $\rho(1) = I_4$, $r \in K$.

v) $\lambda(x^{-1}) = (\lambda(x))^{-1}, \rho(x^{-1}) = (\rho(x))^{-1}$, *for* $x \neq 0$; □

The following three propositions can be proved by straightforward calculations.

Proposition 2.5. *For all* $x \in \mathbb{H}(\alpha, \beta) \det(\lambda(x)) = \det(\rho(x)) = (n(x))^2$.□

Proposition 2.6. *Let* $x = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 \in \mathbb{H}(\alpha, \beta)$. *The following statedments are true:*

i) $x = \frac{1}{4} M_4 \lambda(x) M_4^*, x = \frac{1}{4} M_4^{*t} \rho^t(x) M_4^t$, *where* $M_4 = (1, e_1, e_2, e_3)$, $M_4^* = (1, -\alpha^{-1} e_1, -\beta^{-1} e_2, -\alpha^{-1}\beta^{-1} e_3)^t$.

ii) $\lambda(x) = D_1 \rho^t(x) D_2, \lambda(\overline{x}) = C_1 \lambda^t(x) C_2, \rho(x) = D_1 \lambda^t(x) D_2$, $\rho(\overline{x}) = C_1 \rho^t(x) C_2$, *where* $C_1, C_2, D_1, D_2 \in \mathcal{M}_4(K)$ and $C_1 = diag\{1, \alpha^{-1}, \beta^{-1}, \alpha^{-1}\beta^{-1}\}$, $C_2 = diag\{1, \alpha, \beta, \alpha\beta\}$, $D_1 = diag\{1, -\alpha^{-1}, -\beta^{-1}, -\alpha^{-1}\beta^{-1}\}$, $D_2 = diag\{1, -\alpha, -\beta, -\alpha\beta\}$.

iii) The matrices $C_1, C_2, D_1, D_2 \in \mathcal{M}_4(K)$ *satisfy the relations:* $C_1 C_2 = D_1 D_2 = I_4, D_1 M_1 = C_1, D_2 M_1 = C_2, C_1 M_1 = D_1$, $C_2 M_1 = D_2$, where $M_1 \in \mathcal{M}_4(K), M_1 = diag\{1, -1, -1, -1\}$.□

Proposition 2.7. **(**[Ti,00]**;** Lemma 1.3.)*Let* $x = a_0+a_1e_1+a_2e_2+a_3e_3 \in \mathbb{H}(\alpha,\beta)$. *Let* $\overrightarrow{x} = (a_0, a_1, a_3, a_3)^t \in \mathcal{M}_{1\times 4}(K)$, *be the vector representation of the element* x. *Then for every* $a, b, x \in \mathbb{H}(\alpha,\beta)$ *we have the relations:*

i) $\overrightarrow{ax} = \lambda(a)\overrightarrow{x}$.

ii) $\overrightarrow{xb} = \rho(b)\overrightarrow{x}$.

iii) $\overrightarrow{axb} = \lambda(a)\rho(b)\overrightarrow{x} = \rho(b)\lambda(a)\overrightarrow{x}$.

iv)$\rho(b)\lambda(a) = \lambda(a)\rho(b)$.□

Proposition 2.8. *Let* $a, b \in \mathbb{H}(\alpha,\beta), a \neq 0, b \neq 0$. *Then the linear equation*

$$ax = xb \tag{2.1.}$$

has non-zero solutions $x \in \mathbb{H}(\alpha,\beta)$, *if and only if*

$$t(a) = t(b) \text{ and } n(a - a_0) = n(b - b_0), \tag{2.2.}$$

where $a = a_0 + a_1e_1 + a_2e_2 + a_3e_3, b = b_0 + b_1e_1 + b_2e_2 + b_3e_3$.

Proof. We suppose that the equation (2.1.) has non-zero solutions $x \in \mathbb{H}(\alpha,\beta)$. Then we have $n(ax) = n(xb) \Rightarrow n(a)n(x) = n(x)n(b)$, therefore $n(a) = n(b)$. Since $a = xbx^{-1}$, $t(a) = t\left(xbx^{-1}\right) = t\left(x^{-1}xb\right) = t(b)$. We obtain that $a_0 = b_0$, and from $n(a) = n(b)$ we have $\alpha a_1^2 + \beta a_2^2 + \alpha\beta a_3^2 = \alpha b_1^2 + \beta b_2^2 + \alpha\beta b_3^2$, so $n(a - a_0) = n(b - b_0)$.

Conversely, by considering the vector representation, the equation (2.1.) becomes $\overrightarrow{ax} = \overrightarrow{xb}$, that is

$$(\lambda(a) - \rho(b))\overrightarrow{x} = \overrightarrow{0}. \tag{2.3.}$$

Equation (2.1.) has non-zero solutions if and only if the equation (2.3.) has a non-zero solution, that is, if and only if $\det(\lambda(a) - \rho(b)) = 0$. We compute this determinant: $\det(\lambda(a)\text{-}\rho(b)) =$

$$= \left[(a_0\text{-}b_0)^2 + n(a\text{-}a_0) + n(b\text{-}b_0)\right]^2 \text{-}\ 4n(a\text{-}a_0)\,n(b\text{-}b_0).$$

If $a_0 = b_0$ and $n(a - a_0) = n(b - b_0)$, then $\det(\lambda(a) - \rho(b)) = 0$, therefore the equation (2.1.) has a non-zero solution.□

Proposition 2.9. *With the notations of* Proposition 2.8., *if* $t(a) = t(b)$ *and* $n(a - a_0) = n(b - b_0)$, *then the matrix* $\lambda(a) - \rho(b)$ *has the rank two.*

Proof.

$$\lambda(a) - \rho(b) = \begin{pmatrix} a_0 - b_0 & -\alpha a_1 + \alpha b_1 & -\beta a_2 + \beta b_2 & -\alpha\beta a_3 + \alpha\beta b_3 \\ a_1 - b_1 & a_0 - b_0 & -\beta a_3 - \beta b_3 & \beta a_2 + \beta b_2 \\ a_2 - b_2 & \alpha a_3 + \alpha b_3 & a_0 - b_0 & -\alpha a_1 - \alpha b_1 \\ a_3 - b_3 & -a_2 - b_2 & a_1 + b_1 & a_0 - b_0 \end{pmatrix}$$

Case $a \neq b$.

We suppose $a_1 \neq b_1$. If $a_0 - b_0 = 0$, then $d_1 = \begin{vmatrix} 0 & -\alpha a_1 + \alpha b_1 \\ a_1 - b_1 & 0 \end{vmatrix} = \alpha (a_1 - b_1)^2 \neq 0$, and all the minors of order 3 are zero.

Therefore *rank* $(\lambda (a) - \rho (b)) = 2$ and the subspace of the solutions is of dimension two.

Case $a = b$.

$\lambda (a) - \rho (b) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -2\beta a_3 & 2\beta a_2 \\ 0 & 2\alpha a_3 & 0 & -2\alpha a_1 \\ 0 & -2a_2 & 2a_1 & 0 \end{pmatrix}$, and it results also

rank $(\lambda (a) - \rho (b)) = 2.\square$

Remark 2.10. By *Proposition 1.16.*, if $A=\mathbb{H}(\alpha, \beta)$, we have that $\mathcal{A}(a,b)=\mathcal{A}(a-a_0, b-b_0)=\mathcal{A}(a,\bar{b})=\mathcal{A}(\bar{a},\bar{b})=\mathcal{A}(\bar{a},b)$, where $a = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3, b = b_0 + b_1 e_1 + b_2 e_2 + b_3 e_3 \in A$, and $\mathcal{A}(a,b)$ represents the subalgebra generated by a and b.

Remark 2.11. Let $a, b \in \mathbb{H}(\alpha, \beta)$, as above, with $t(a) = t(b) = 0$. Then, by *Proposition 1.11.*, it results that $ab+ba=-2\alpha a_1 b_1 - 2\beta a_2 b_2 - 2\alpha\beta a_3 b_3 \in K$.

Remark 2.12. By *Proposition 1.12.* if $\mathbb{H}(\alpha, \beta)$ is a division algebra and $a, b \in \mathbb{H}(\alpha, \beta)$ with $t(a) = t(b) = 0$, then the equality

$$n(a)\, n(b) = \frac{1}{4}(ab + ba)^2 \qquad (2.4.)$$

is true if and only if $a = rb, r \in K$. If $n(a) = n(b)$, then $r = 1$ or $r = -1$.

Proof. From Proposition 2.11., $n(ab)=(\alpha a_1 b_1+\beta a_2 b_2+\alpha\beta a_3 b_3)^2$. Because $n(a,b) = \frac{1}{2}(n(a+b) + n(a) + n(b))$, then $n^2(a,b) = \frac{1}{4}(ab+ba)$,
$n(a,a)=n(a), n(b,b)=n(b)$ and by Proposition 1.12. we obtain
$n(a)\,n(b) = \frac{1}{4}(ab+ba)^2$ is true if and only if $a = rb, r \in K$. If $n(a) = n(b)$, then from the equality (2.4.) it results the equality
$(n(a) + \alpha a_1 b_1 + \beta a_2 b_2 + \alpha\beta a_3 b_3)(n(a) - \alpha a_1 b_1 - \beta a_2 b_2 - \alpha\beta a_3 b_3) = 0$. In the last relation we replace $(n(a) + rn(a))(n(a) - rn(a)) = 0$, and we get $n(a)^2 (1+r)(1-r) = 0$. Then either $r = -1$ or $r = 1.\square$

Proposition 2.13.

i) If $a = a_0 + a_1e_1 + a_2e_2 + a_3e_3, b = b_0 + b_1e_1 + b_2e_2 + b_3e_3 \in \mathbb{H}(\alpha, \beta)$ *with* $b \neq \bar{a}$, $a, b \notin K$ *then the solutions of the equation* (2.1.) *,with* $t(a) = t(b)$ *and* $n(a - a_0) = n(b - b_0)$ *, are found in* $\mathcal{A}(a, b)$ *and have the form :*

$$x = \lambda_1(a - a_0 + b - b_0) + \lambda_2 (n(a - a_0) - (a - a_0)(b - b_0)), \qquad (2.5.)$$

where $\lambda_1, \lambda_2 \in K$ *are arbitrary.*

ii) If $b = \bar{a}$, *then the general solution of the equation* (2.1.) *is* $x = x_1e_1 + x_2e_2 + x_3e_3$, *where* $x_1, x_2, x_3 \in K$ *and they satisfy the equality :*$\alpha a_1x_1 + \beta a_2x_2 + \alpha\beta a_3x_3\alpha\beta = 0$.

Proof. *i)* Let $x_1 = a - a_0 + b - b_0, x_2 = n(a - a_0) - (a - a_0)(b - b_0)$. If $b \neq \bar{a}$ then $x_2 \notin K$. We have $ax_1 - x_1b = a(a - a_0) + a(b - b_0) - (a - a_0)b -$
$- (b - b_0)b$,and we write $a = a_0 + v, b = b_0 + w$, with $t(v) = t(w) = 0$,
$ax_1 - x_1b = (a_0 + v)v + (a_0 + v)w - v(b_0 + w) - w(b_0 + w) =$
$= a_0v + v^2 + a_0w + vw - vb_0 - vw - wb_0 - w^2 = 0$, since by the hypothesis $n(v) = n(w)$, $v^2 = -n(v) = -n(w) = w^2$. Therefore x_1 is a solution.

Analogously, we have $ax_2 - x_2b = 0$, therefore x_2 is a solution. Obviously, $x_1, x_2 \in \mathcal{A}(a - a_0, b - b_0) = \mathcal{A}(a, b)$. We also note that x_1, x_2 are linearly independent.

If $\theta_1x_1 + \theta_2x_2 = 0$,with $\theta_1,\theta_2 \in K$, then $\theta_1v+\theta_1w+\theta_2n(v)$-$\theta_2vw$=0,which gives

$\theta_2(n(v) + \alpha a_1b_1 + \beta a_2b_2 + \alpha\beta a_3b_3)$=0,$\theta_1(a_1 + b_1) - \theta_2\beta(a_2b_3 - a_3b_2)$=0
$\theta_1(a_2 + b_2) - \theta_2\alpha(a_3b_1 - a_1b_3)$=0,$\theta_1(a_3 + b_3)$ $-\theta_2(a_1b_2 - a_2b_1)$=0.
Since $b \neq \bar{a}$, from Proposition 2.12., we have $\theta_2 = 0$ and
$\theta_1(a_1 + b_1) = 0, \theta_1(a_2 + b_2) = 0, \theta_1(a_3 + b_3) = 0$,therefore $\theta_1 = 0$.

If the subspace of the solutions of the equation (2.1.) has the dimension two, it results that each solution of this equation is of the form $\lambda_1x_1+\lambda_2x_2, \lambda_1, \lambda_2 \in K$.

We note that $\lambda_1x_1 + \lambda_2x_2 \in \mathcal{A}(v, w) = \mathcal{A}(a, b)$.

ii) Since $b = \bar{a}$, it results $b = a_0 - a_1e_1 - a_2e_2 - a_3e_3$, therefore $v = -w$. Then, if x is a solution of the equation, we have $ax = x\bar{a}$, therefore $(a_0 + v)(x_0 + y) = (x_0 + y)(a_0 - v)$ from where we get $2x_0v + vy + yv = 0$,where $x = x_0 + y$,with $x_0 \in K, y = x_1e_1 + x_2e_2 + x_3e_3, t(y) = 0$.

As $vy + yv \in K$, the last equality is equivalent with $x_0 = 0$ and $vy + yv = 0$, that is $x_0 = 0$ and $\alpha a_1x_1 + \beta a_2x_2 + \alpha\beta a_3x_3 = 0$.□

Remark 2.14. If $a_0 = b_0$ and $n(v) = n(w)$, the equation (2.1.) has the general solution under the form:

$$x = aq - q\bar{b}, \text{with } q \in \mathcal{A}(a, b), \qquad (2.6.)$$

or, equivalently $x = vq + qw$.

Proof. Indeed, suppose that $z \in \mathcal{A}(a,b)$ is an arbitrary solution of the equation (2.1.). It results $az = zb$, therefore $vz = wz$. Let $q = \frac{-vz}{2n(v)} = -\frac{zw}{2n(w)}$. We have $x = vq + qw = -\frac{v^2z}{2n(v)} - \frac{zw^2}{2n(w)} = \frac{z}{2} + \frac{z}{2} = z$, which proves that each solution of the equation (2.1.) can be written in the form (2.6.).□

Proposition 2.15. *Let* $a = a_0 + a_1e_1 + a_2e_2 + a_3e_3$, $b = b_0 + b_1e_1 + b_2e_2 + b_3e_3 \in \mathbb{H}(\alpha,\beta)$.
i)([Ti; 99] ,Theorem 2.3.) *The equation*

$$ax = \bar{x}b \tag{2.7.}$$

has non-zero solutions if and only if $n(a) = n(b)$. *In this case, if* $a + \bar{b} \neq 0$, *then* (2.7.) *has a solution of the form* $x = \lambda(\bar{a} + b), \lambda \in K$.

ii) If $a + \bar{b} = 0$, *then the general solution of the equation* (2.7.) *can be written in the form* $x = x_0 + x_1e_1 + x_2e_2 + x_3e_3$, *where* $a_0x_0 - \alpha a_1x_1 - \beta a_2x_2 - \alpha\beta a_3x_3 = 0$.

Proof. We suppose that (2.7.) has a non-zero solution $x \in \mathbb{H}(\alpha,\beta)$. Then we have $ax = \bar{x}b \Rightarrow n(ax) = n(\bar{x}b) \Rightarrow n(a)n(x) = n(x)n(b) \Rightarrow n(a) = n(b)$.

Conversely, suppose that $n(a) = n(b)$. We take $y = \bar{a}+b$ and we obtain $ay - \bar{y}a = a(\bar{a} + b) - (a + \bar{b})b = a\bar{a} + ab - ab - \bar{b}b = n(a) - n(b) = 0$.

If $a + \bar{b} = 0$, we have $b = -\bar{a}$ and the equation (2.7.) becomes $ax + \overline{ax} = 0$, that is $t(ax) = 0$. But $t(ax) = a_0x_0 - \alpha a_1x_1 - \beta a_2x_2 - \alpha\beta a_3x_3$.□

Proposition 2.16. *Let* $a \in \mathbb{H}(\alpha,\beta), a \notin K$. *If there exists* $r \in K$ *such that* $n(a) = r^2$, *then* $a = \bar{q}rq^{-1}$, *where* $q = r + \bar{a}$, *and* $q^{-1} = \frac{\bar{q}}{n(q)}$

Proof. By hypothesis, we have $a(r + \bar{a}) = ar + a\bar{a} = ar + n(a) = ar + r^2 = (a + r)r$. Since $\bar{q} = r + a$, it results $\bar{q}r = aq$.□

Proposition 2.17. *Let* $a \in \mathbb{H}(\alpha,\beta)$ *with* $a \notin K$. *If there exists* $r, s \in K$ *with the properties* $n(a) = r^4$ *and* $n(r^2 + \bar{a}) = s^2$, *then the quadratic equation*

$$x^2 = a \tag{2.8.}$$

has two solutions of the form $x = \pm\frac{r(r^2+a)}{n(r^2+\bar{a})}$.

Proof. By *Proposition 2.16.,* it results that a is of the form $a = \bar{q}r^2q^{-1}$, where $q=r^2+\bar{a}$. Because $q^{-1}=\frac{\bar{q}}{n(q)}$, we obtain $a=r^2\bar{q}q^{-1}=r^2\bar{q}\frac{\bar{q}}{n(q)}=r^2\frac{\bar{q}^2}{s^2}=\left(\frac{r}{s}\bar{q}\right)^2$,therefore $x_1=\frac{r}{s}\bar{q}, x_2=-\frac{r}{s}\bar{q}$ are solutions.□

Corollary 2.18. *Let $a,b,c\in\mathbb{H}(\alpha,\beta)$ so that ab and $b^2-c\notin K$. If ab and b^2-c satisfy the conditions of Proposition 2.17. then the equations $xax=b$ and*

$x^2+bx+xb+c=0$ have solutions.

Proof. $xax=b\Longleftrightarrow(ax)^2=ab$ and $x^2+bx+xb+c=0\Longleftrightarrow$

$\Leftrightarrow(x+b)^2=b^2-c$.□

Corollary 2.19. *If $b,c\in\mathbb{H}(\alpha,\beta)\setminus\{K\}$ satisfy the conditions $bc=cb$ and there exists $r\in K$, $r\neq0$ so that $n\left(\frac{b^2}{4}-c\right)=r^4$, and $n\left(r^2+\frac{\bar{b}^2}{4}-\bar{c}\right)=s^2$,*

$s\neq0$ then the equation

$$x^2+bx+c=0, \tag{2.9.}$$

has solutions in $\mathbb{H}(\alpha,\beta)$.

Proof. Let $x_0\in\mathbb{H}(\alpha,\beta)$ be a solution of the equation (2.9.). Because $x_0^2=t(x_0)x_0-n(x_0)$ and $x_0^2+bx_0+c=0$, it results that

$t(x_0)x_0-n(x_0)+bx_0+c=0$ hence $(t(x_0)+b)x_0=c+n(x_0)$. If $t(x_0)+b\neq0, t(x_0), n(x_0)\in K, 1\in\mathcal{A}(b,c)$, then $t(x_0)+b$ and $c+n(x_0)\in\mathcal{A}(b,c)$.

Therefore $x_0\in\mathcal{A}(b,c)$. Because $bc=cb$, it results that $\mathcal{A}(b,c)$ is commutative, therefore x_0 commutes with every element of $\mathcal{A}(b,c)$. Then the equation (2.9.) can be written under the form $\left(x+\frac{b}{2}\right)^2-\frac{b^2}{4}+c=0$ and by the *Proposition 2.17.* such an x_0 exists.□

§ 3. Equations in the generalized octonions algebra

Let $\mathbb{O}(\alpha,\beta,\gamma)$ be the generalized octonions algebra, with the basis

$\{1,f_1,f_2,f_3,f_4,f_5,f_6,f_7\}$, where $f_1=e_1,\ f_2=e_2,\ f_3=e_3, f_5=e_1f_4,\ f_6=e_2f_4, f_7=e_3f_4$. Its multiplication table is the following :

$\cdot$	1	f_1	f_2	f_3	f_4	f_5	f_6	f_7
1	1	f_1	f_2	f_3	f_4	f_5	f_6	f_7
f_1	f_1	$-\alpha$	f_3	$-\alpha f_2$	f_5	$-\alpha f_4$	$-f_7$	αf_6
f_2	f_2	$-f_3$	$-\beta$	βf_1	f_6	f_7	$-\beta f_4$	$-\beta f_5$
f_3	f_3	αf_2	$-\beta f_1$	$-\alpha\beta$	f_7	$-\alpha f_6$	βf_5	$-\alpha\beta f_4$
f_4	f_4	$-f_5$	$-f_6$	$-f_7$	$-\gamma$	γf_1	γf_2	γf_3
f_5	f_5	αf_4	$-f_7$	αf_6	$-\gamma f_1$	$-\alpha\gamma$	$-\gamma f_3$	$\alpha\gamma f_2$
f_6	f_6	f_7	βf_4	$-\beta f_5$	$-\gamma f_2$	γf_3	$-\beta\gamma$	$-\beta\gamma f_1$
f_7	f_7	$-\alpha f_6$	βf_5	$\alpha\beta f_4$	$-\gamma f_3$	$-\alpha\gamma f_2$	$\beta\gamma f_1$	$-\alpha\beta\gamma$

Remark 3.1. The algebra $\mathbb{O}(\alpha,\beta,\gamma)$ is a division algebra or a split algebra. As in the quaternion algebra case, we aim to find out conditions for getting a division algebra.

If $x = a_0 + a_1 f_1 + a_2 f_2 + a_3 f_3 + a_4 f_4 + a_5 f_5 + a_6 f_6 + a_7 f_7$, then $\bar{x} = a_0 - a_1 f_1 - a_2 f_2 - a_3 f_3 - a_4 f_4 - a_5 f_5 - a_6 f_6 - a_7 f_7$ is the *conjugate* of x and $n(x) = x\bar{x} = a_0^2 + \alpha a_1^2 + \beta a_2^2 + \alpha\beta a_3^2 + \gamma a_4^2 + \alpha\gamma a_5^2 + \beta\gamma a_6^2 + \alpha\beta\gamma a_7^2 \in K$ is the *norm* of x, while $t(x) = x + \bar{x} \in K$ is the *trace* of the element x.

If there exists $x \in \mathbb{O}(\alpha,\beta,\gamma), x \neq 0$, such that $n(x) = 0$, then $\mathbb{O}(\alpha,\beta,\gamma)$ is not a division algebra, and if $n(x) \neq 0, \forall x \in \mathbb{O}(\alpha,\beta,\gamma), x \neq 0$, then $\mathbb{O}(\alpha,\beta,\gamma)$ is a division algebra . Therefore, $\mathbb{O}(\alpha,\beta,\gamma)$ is a division algebra if and only if the equation $a_0^2 + \alpha a_1^2 + \beta a_2^2 + \alpha\beta a_3^2 + \gamma a_4^2 + \alpha\gamma a_5^2 + \beta\gamma a_6^2 + \alpha\beta\gamma a_7^2 = 0$ has only the trivial solution. This is equivalent with equation $a_0^2 + \alpha a_1^2 + \beta a_2^2 + \alpha\beta a_3^2 = -\gamma(a_4^2 + \alpha a_5^2 + \beta a_6^2 + \alpha\beta\gamma a_7^2)$ or $\gamma = -\frac{n(a_0 + a_1 f_1 + a_2 f_2 + a_3 f_3)}{n(a_4 f_4 + a_5 f_5 + a_6 f_6 + a_7 f_7)} = -n(b_0 + b_1 f_1 + b_2 f_2 + b_3 f_3) = -b_0^2 - \alpha b_1^2 - \beta b_2^2 - \alpha\beta b_3^2$, where $b_0 + b_1 f_1 + b_2 f_2 + b_3 f_3 = \frac{a_0 + a_1 f_1 + a_2 f_2 + a_3 f_3}{a_4 f_4 + a_5 f_5 + a_6 f_6 + a_7 f_7}$.

$\mathbb{O}(\alpha,\beta,\gamma)$ is a division algebra if and only if $\mathbb{H}(\alpha,\beta)$ is a division algebra and the equation $n(x) = -\gamma$ does not have solutions in $\mathbb{H}(\alpha,\beta)$.□

Based upon the matrix representation of the generalized quaternions , we introduce the matrix representation in the case of generalized octonions.

Let $a' = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3, a'' = a_4 + a_5 e_1 + a_6 e_2 + a_7 e_3 \in \mathbb{H}(\alpha,\beta)$ and $a = a' + a'' v \in \mathbb{O}(\alpha,\beta,\gamma)$. Then the matrix :

$$\Lambda(a) = \begin{pmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha\beta a_3 & -\gamma a_4 & -\alpha\gamma a_5 & -\beta\gamma a_6 & -\alpha\beta\gamma a_7 \\ a_1 & a_0 & -\beta a_3 & \beta a_2 & -\gamma a_5 & \gamma a_4 & \beta\gamma a_7 & -\beta\gamma a_6 \\ a_2 & \alpha a_3 & a_0 & -\alpha a_1 & -\gamma a_6 & -\alpha\gamma a_7 & \gamma a_4 & \alpha\gamma a_5 \\ a_3 & -a_2 & a_1 & a_0 & -\gamma a_7 & \gamma a_6 & -\gamma a_5 & \gamma a_4 \\ a_4 & \alpha a_5 & \beta a_6 & \alpha\beta a_7 & a_0 & -\alpha a_1 & -\beta a_2 & -\alpha\beta a_3 \\ a_5 & -a_4 & \beta a_7 & -\beta a_6 & a_1 & a_0 & \beta a_3 & -\beta a_2 \\ a_6 & -\alpha a_7 & -a_4 & \alpha a_5 & a_2 & -\alpha a_3 & a_0 & \alpha a_1 \\ a_7 & a_6 & -a_5 & -a_4 & a_3 & a_2 & -a_1 & a_0 \end{pmatrix}$$

is called **the left matriceal representation for the element** $a \in$

$\mathbb{O}(\alpha,\beta,\gamma)$.

Using the matrix representations for quaternions, we can write the left matrix representation:

$$\Lambda(a)=\begin{pmatrix} \lambda\left(a^{'}\right) & \text{-}\gamma\rho\left(a^{''}\right)M_1 \\ \lambda\left(a^{''}\right)M_1 & \rho\left(a^{'}\right) \end{pmatrix}, \text{where } M_1=diag\{1,\text{-}1,\text{-}1,\text{-}1\}\in\mathcal{M}_4(K).$$

Analogously, we define **the right matrix representation**:

$$\Delta(a)=\begin{pmatrix}
a_0 & \text{-}\alpha a_1 & \text{-}\beta a_2 & \text{-}\alpha\beta a_3 & \text{-}\gamma a_4 & \text{-}\alpha\gamma a_5 & \text{-}\beta\gamma a_6 & \text{-}\alpha\beta\gamma a_7 \\
a_1 & a_0 & \beta a_3 & \text{-}\beta a_2 & \gamma a_5 & \text{-}\gamma a_4 & \text{-}\beta\gamma a_7 & \beta\gamma a_6 \\
a_2 & \text{-}\alpha a_3 & a_0 & \alpha a_1 & \gamma a_6 & \alpha\gamma a_7 & \text{-}\gamma a_4 & \text{-}\alpha\gamma a_5 \\
a_3 & a_2 & \text{-}a_1 & a_0 & \gamma a_7 & \text{-}\gamma a_6 & \gamma a_5 & \text{-}\gamma a_4 \\
a_4 & \text{-}\alpha a_5 & \text{-}\beta a_6 & \text{-}\alpha\beta a_7 & a_0 & \alpha a_1 & \beta a_2 & \alpha\beta a_3 \\
a_5 & a_4 & \text{-}\beta a_7 & \beta a_6 & \text{-}a_1 & a_0 & \text{-}\beta a_3 & \beta a_2 \\
a_6 & \alpha a_7 & a_4 & \text{-}\alpha a_5 & \text{-}a_2 & \alpha a_3 & a_0 & \text{-}\alpha a_1 \\
a_7 & \text{-}a_6 & a_5 & a_4 & \text{-}a_3 & \text{-}a_2 & a_1 & a_0
\end{pmatrix}.$$

This matrix has as its columns, the coefficients in K of the elements $a,\ f_1a,\ f_2a,\ f_3a,\ f_4a,\ f_5a,\ f_6a,\ f_7a.$ Using the matrix representations of quaternions, we can also write that : $\Delta(a)=\begin{pmatrix} \rho\left(a^{'}\right) & -\gamma\lambda\left(\bar{a}^{''}\right) \\ \lambda\left(a^{''}\right) & \rho\left(\bar{a}^{'}\right) \end{pmatrix}=$

$=A_1\Lambda^t(a)A_2$, where $A_1,A_2\in\mathcal{M}_8(K)$ are matrices of the form:

$$A_1=\begin{pmatrix} -\gamma D_1 & 0 \\ 0 & C_1 \end{pmatrix}, A_2=\begin{pmatrix} -\gamma^{-1}D_2 & 0 \\ 0 & C_2 \end{pmatrix}, D_1,D_2,C_1,C_2\in\mathcal{M}_4(K)$$

being the matrices in *Proposition 2.6.*, and $A_1A_2=A_2A_1=I_8$. Indeed, we have

$$\Lambda^t(a)=\begin{pmatrix} \lambda^t\left(a^{'}\right) & M_1^t\lambda^t\left(a^{''}\right) \\ -\gamma M_1^t\rho^t\left(a^{''}\right) & \rho^t\left(a^{'}\right) \end{pmatrix}, \text{and}$$

$$A_1\Lambda^t(a)A_2=\begin{pmatrix} \text{-}\gamma D_1 & 0 \\ 0 & C_1 \end{pmatrix}\begin{pmatrix} \lambda^t\left(a^{'}\right) & M_1^t\lambda^t\left(a^{''}\right) \\ \text{-}\gamma M_1^t\rho^t\left(a^{''}\right) & \rho^t\left(a^{'}\right) \end{pmatrix}\begin{pmatrix} \text{-}\gamma^{\text{-}1}D_2 & 0 \\ 0 & C_2 \end{pmatrix}=$$

$$=\begin{pmatrix} -\gamma D_1\lambda^t\left(a^{'}\right) & -\gamma D_1M_1^t\lambda^t\left(a^{''}\right) \\ -\gamma C_1M_1^t\rho^t\left(a^{''}\right) & C_1\rho^t\left(a^{'}\right) \end{pmatrix}\begin{pmatrix} \text{-}\gamma^{-1}D_2 & 0 \\ 0 & C_2 \end{pmatrix}=$$

$$=\begin{pmatrix} D_1\lambda^t\left(a^{'}\right)D_2 & -\gamma D_1M_1^t\lambda^t\left(a^{''}\right)C_2 \\ \gamma C_1M_1^t\rho^t\left(a^{''}\right)D_2\gamma^{-1} & C_1\rho^t\left(a^{'}\right)C_2 \end{pmatrix}.$$

But, by *Proposition 2.6.*, it results that $D_1\lambda^t\left(a^{'}\right)D_2=\rho\left(a^{'}\right)$,

$C_1\rho^t\left(a^{'}\right)C_2=\rho\left(\bar{a}^{'}\right)$; We has $-\gamma D_1M_1^t\lambda^t\left(a^{''}\right)C_2=$

$$=-\gamma\text{diag}\{1,\alpha^{-1},\beta^{-1},\alpha^{-1}\beta^{-1}\}\begin{pmatrix} a_4 & \alpha a_5 & \beta a_6 & \alpha\beta a_7 \\ -\alpha a_5 & \alpha a_4 & \alpha\beta a_7 & -\alpha\beta a_6 \\ -\beta a_6 & -\alpha\beta a_7 & \beta a_4 & \alpha\beta a_5 \\ -\alpha\beta a_7 & \alpha\beta a_6 & -\alpha\beta a_5 & \alpha\beta a_4 \end{pmatrix}=$$

$$=\begin{pmatrix} -\gamma a_4 & -\alpha\gamma a_5 & -\beta\gamma a_6 & -\alpha\beta\gamma a_7 \\ \gamma a_5 & -\gamma a_4 & -\beta\gamma a_4 & \beta\gamma a_6 \\ \gamma a_6 & \alpha\gamma a_7 & -\gamma a_4 & -\alpha\gamma a_5 \\ \gamma a_7 & \gamma a_6 & \gamma a_5 & -\gamma a_4 \end{pmatrix}=-\gamma\lambda\left(\bar{a}^{''}\right) \text{ and } C_1M_1^t\rho^t\left(a^{''}\right)D_2=$$

$$=\text{diag}\{1,\alpha^{-1},\beta^{-1},\alpha^{-1}\beta^{-1}\}\begin{pmatrix} a_4 & -\alpha a_5 & -\beta a_6 & -\alpha\beta a_7 \\ -\alpha a_5 & -\alpha a_4 & \alpha\beta a_7 & -\alpha\beta a_6 \\ -\beta a_6 & -\alpha\beta a_7 & -\beta a_4 & \alpha\beta a_5 \\ -\alpha\beta a_7 & \alpha\beta a_6 & -\alpha\beta a_5 & -\alpha\beta a_4 \end{pmatrix}=\lambda\left(a^{''}\right).$$

Proposition 3.2. ([Ti; 00],Theorem 2.1. and Theorem 2.3.) *Let* $x=x_0+x_1f_1+x_2f_2+x_3f_3+x_4f_4+x_5f_5+x_6f_6+x_7f_7$ *and* $a=a_0+a_1f_1+a_2f_2+a_3f_3+a_4f_4+a_5f_5+a_6f_6+a_7f_7\in\mathbb{O}(\alpha,\beta,\gamma)$. *Denote* $\vec{x}=(x_0,x_1,x_2,x_3,x_4,x_5,x_6,x_7)^t$, *the vector representation for the element* x. *Then* $\overrightarrow{ax}=\Lambda(a)\vec{x}$ and $\overrightarrow{xa}=\Delta(a)\vec{x}$.

Proof. We take a and x under the form $a=a^{'}+a^{''}v$, $x=x^{'}+x^{''}v$ with $a^{'},a^{''},x^{'},x^{''}\in\mathbb{H}(\alpha,\beta)$. Then $ax=\left(a^{'}x^{'}-\gamma\bar{x}^{''}a^{''}\right)+$

$+\left(x^{''}a^{'}+a^{''}\bar{x}^{'}\right)v$ and we can write $\overrightarrow{ax}=\begin{pmatrix} \overrightarrow{a^{'}x^{'}-\gamma\bar{x}^{''}a^{''}} \\ \overrightarrow{x^{''}a^{'}+a^{''}\bar{x}^{'}} \end{pmatrix}=$

$$=\begin{pmatrix} \overrightarrow{a^{'}x^{'}}-\gamma\overrightarrow{\bar{x}^{''}a^{''}} \\ \overrightarrow{x^{''}a^{'}}+\overrightarrow{a^{''}\bar{x}^{'}} \end{pmatrix}=\begin{pmatrix} \lambda\left(a^{'}\right)\vec{x}^{'}-\gamma\rho\left(a^{''}\right)\overrightarrow{\bar{x}^{''}} \\ \rho\left(a^{'}\right)\vec{x}^{''}+\lambda\left(a^{''}\right)\overrightarrow{\bar{x}^{'}} \end{pmatrix}.$$

Given $\bar{x}^{''}=M_1x^{''}$, it results

$$\overrightarrow{ax}=\begin{pmatrix} \lambda\left(a^{'}\right)\vec{x}^{'}-\gamma\rho\left(a^{''}\right)M_1\overrightarrow{x^{''}} \\ \rho\left(a^{'}\right)\vec{x}^{''}+\lambda\left(a^{''}\right)M_1\overrightarrow{x^{'}} \end{pmatrix}=\begin{pmatrix} \lambda\left(a^{'}\right) & -\gamma\rho\left(a^{''}\right)M_1 \\ \lambda\left(a^{''}\right)M_1 & \rho\left(a^{'}\right) \end{pmatrix}\begin{pmatrix} \vec{x}^{'} \\ \vec{x}^{''} \end{pmatrix}=$$

$\Lambda(a)\vec{x}$.

Analogously, $\overrightarrow{xa}=\Delta(a)\,\vec{x}$.□

Proposition 3.3.([Ti; 00], Theorem 2.6.) *Let $x, y \in \mathbb{O}(\alpha,\beta,\gamma)$ and $m \in K$. Then the following relations are true:*
i) $x = y \Longleftrightarrow \Lambda(x) = \Lambda(y)$.
ii) $x = y \Longleftrightarrow \Delta(x) = \Delta(y)$.
iii) $\Lambda(x+y) = \Lambda(x) + \Lambda(y)$.
iv) $\Lambda(mx) = m\Lambda(x)$.
v) $\Delta(x+y) = \Delta(x) + \Delta(y)$.
vii) $\Delta(mx) = m\Delta(x)$.
viii) $\Lambda\left(x^{-1}\right) = \Lambda^{-1}(x)$.
ix) $\Delta\left(x^{-1}\right) = \Delta^{-1}(x)$.□
Since $\mathbb{O}(\alpha,\beta,\gamma)$ is a non-associative algebra, the equalities
$\Lambda(xy) = \Lambda(x)\Lambda(y)$, $\Delta(xy) = \Delta(x)\Delta(y)$ do not generally apply.

Proposition 3.4. *Let $x, y \in \mathbb{O}(\alpha,\beta,\gamma)$. Then, by using the notations in Proposition 2.6., we have:*
i) $\Lambda(\bar{x}) = E_1\Lambda^t(x)E_2$, *where* $E_1=\begin{pmatrix} \gamma C_1 & 0 \\ 0 & C_1 \end{pmatrix}, E_2=\begin{pmatrix} \gamma^{-1}C_2 & 0 \\ 0 & C_2 \end{pmatrix}$.
ii) $\Delta(\bar{x}) = F_1\Delta^t(x)F_2$, *where* $F_1=\begin{pmatrix} -\gamma C_1 & 0 \\ 0 & C_1 \end{pmatrix}, F_2=\begin{pmatrix} -\gamma^{-1}C_2 & 0 \\ 0 & C_2 \end{pmatrix}$.
iii) $E_1E_2 = F_1F_2 = A_1A_2 = I_8, E_1^t = E_1, E_2^t = E_2, F_1^t = F_1,$
$F_2^t = F_2, A_1^t = A_1, A_2^t = A_2$.
iv) $\Lambda(x) = A_1\Delta^t(x)A_2$, *where* $A_1=\begin{pmatrix} -\gamma D_1 & 0 \\ 0 & C_1 \end{pmatrix}, A_2=\begin{pmatrix} -\gamma^{-1}D_2 & 0 \\ 0 & C_2 \end{pmatrix}$.

Proof. *iv)* As $\Delta(x) = A_1\Lambda^t(x)A_2$, we multiplicate this last relation to the left and to the right with A_2 and with A_1, obtaining $A_2\Delta(x)A_1 = \Lambda^t(x)$, therefore $\Lambda(x) = A_1\Delta^t(x)A_2$. The other relations can be proved by calculations.□

Proposition 3.5. *Let $x \in \mathbb{O}(\alpha,\beta,\gamma)$. Then :*
i) $x = \frac{1}{8}H_1\Lambda(x)H_2$, *where* $H_1 = (1, f_1, f_2, f_3, f_4, f_5, f_6, f_7)$ *and*
$H_2=\left(1,-\alpha^{-1}f_1,-\beta^{-1}f_2,-\alpha^{-1}\beta^{-1}f_3,-\gamma^{-1}f_4,-\alpha^{-1}\gamma^{-1}f_5,-\beta^{-1}\gamma^{-1}f_6,-\alpha^{-1}\beta^{-1}\gamma^{-1}f_7\right)^t$;
ii) $x = \frac{1}{8}H_2^t\Delta^t(x)H_1^t$.

Proof. *i)* By calculation.
ii) $\Delta^t(x) = A_2\Lambda(x)A_1$ and the rest is proved by calculations.□

Propozition 3.6. *Let $x \in \mathbb{O}(\alpha,\beta,\gamma)$ with $x = x' + x''v$, where $x', x'' \in \mathbb{H}(\alpha,\beta)$. Then* $\det(\Lambda(x)) = \det(\Delta(x)) = (n(x))^4$.

Proof. We know that $\Delta(x) = A_1\Lambda^t(x)A_2$. Then $\det(\Delta(x))=$
$=\det(A_1\Lambda^t(x)A_2)=\det A_1 \det\Lambda^t(x)\det A_2 = \det\Lambda^t(x) = \det\Lambda(x)$. But
$\det\Delta(x) = \begin{vmatrix} \rho\left(x'\right) & -\gamma\lambda\left(\bar{x}''\right) \\ \lambda\left(x''\right) & \rho\left(\bar{x}'\right) \end{vmatrix} = \det\left(\rho\left(x'\right)\rho\left(\bar{x}'\right)+\gamma\lambda\left(\bar{x}''\right)\lambda\left(x''\right)\right) =$
$=\det\left(\rho\left(x'\bar{x}'\right)+\gamma\lambda\left(x''\bar{x}''\right)\right) = \det\left(n\left(x'\right)I_4+\gamma n\left(x''\right)I_4\right) =$
$=\left(n\left(x'\right)I_4+\gamma n\left(x''\right)I_4\right)^4 = (n(x))^4$.□
Let $a,b,\in\mathbb{O}(\alpha,\beta,\gamma)$. In the next, we consider the equation

$$ax = xb \qquad (3.1.)$$

in $\mathbb{O}(\alpha,\beta,\gamma)$. By using the vector representation, the equation is equivalent to:

$$[\Lambda(a)-\Delta(b)]\vec{x} = \vec{0}. \qquad (3.2.)$$

Proposition 3.7. *Let $a,b\in\mathbb{O}(\alpha,\beta,\gamma)$ with*
$a = a_0 + a_1f_1 + a_2f_2 + a_3f_3 + a_4f_4 + a_5f_5 + a_6f_6 + a_7f_7$
$b = b_0 + b_1f_1 + b_2f_2 + b_3f_3 + b_4f_4 + b_5f_5 + b_6f_6 + b_7f_7$. *Then, the linear equation $ax = xb$ has non-zero solutions if and only if :*

$$a_0 = b_0 \text{ and } n(a-a_0) = n(b-b_0). \qquad (3.3.)$$

Proof. We suppose that the equation $ax = xb$ has non-zero solutions, $x\in\mathbb{O}(\alpha,\beta,\gamma)$. It results that $n(ax) = n(xb)$, hence $n(a)n(x) =$
$= n(x)n(b)$,therefore $n(a) = n(b)$. As $a = xbx^{-1}$, it results
$t(a) = t\left(xbx^{-1}\right) = t\left(x^{-1}xb\right) = t(b)$,therefore $a_0 = b_0$ and from $n(a) = n(b)$, we obtain $n(a-a_0) = n(b-b_0)$.

Conversely, considering the vector representation, the equation (3.1.) has non-zero solutions if and only if the equation (3.2) has non-zero solutions, therefore if and only if $\det(\Lambda(a)-\Delta(b)) = 0$.We calculate this determinant. If $a_0 = b_0$, then the matrix $\Lambda(a)-\Delta(b)$ is of the form (MN), where the bloks M and N are the following matrices of type 8×4 :

$$M = \begin{pmatrix} 0 & -\alpha(a_1-b_1) & -\beta(a_2-b_2) & -\alpha\beta(a_3-b_3) \\ a_1-b_1 & 0 & -\beta(a_3+b_3) & \beta(a_2+b_2) \\ a_2-b_2 & \alpha(a_3+b_3) & 0 & -\alpha(a_1+b_1) \\ a_3-b_3 & -(a_2+b_2) & a_1+b_1 & 0 \\ a_4-b_4 & \alpha(a_5+b_5) & \beta(a_6+b_6) & \alpha\beta(a_7+b_7) \\ a_5-b_5 & -(a_4+b_4) & \beta(a_7+b_7) & -\beta(a_6+b_6) \\ a_6-b_6 & -\alpha(a_7+b_7) & -(a_4+b_4) & \alpha(a_5+b_5) \\ a_7-b_7 & a_6+b_6 & -(a_5+b_5) & -(a_4+b_4) \end{pmatrix},$$

$$N=\begin{pmatrix} -\gamma(a_4-b_4) & -\alpha\gamma(a_5-b_5) & -\beta\gamma(a_6-b_6) & -\alpha\beta\gamma(a_7-b_7)\\ -\gamma(a_5+b_5) & \gamma(a_4+b_4) & \beta\gamma(a_7+b_7) & -\beta\gamma(a_6+b_6)\\ -\gamma(a_6+b_6) & -\alpha\gamma(a_7+b_7) & \gamma(a_4+b_4) & \alpha\gamma(a_5+b_5)\\ -\gamma(a_7+b_7) & \gamma(a_6+b_6) & -\gamma(a_5+b_5) & \gamma(a_4+b_4)\\ 0 & -\alpha(a_1+b_1) & -\beta(a_2+b_2) & -\alpha\beta(a_3+b_3)\\ a_1+b_1 & 0 & \beta(a_3+b_3) & -\beta(a_2+b_2)\\ a_2+b_2 & -\alpha(a_3+b_3) & 0 & \alpha(a_1+b_1)\\ a_3+b_3 & a_2+b_2 & -(a_1+b_1) & 0 \end{pmatrix}.$$

Multiplying first the rows $2,3,5,6,7,8$ of the matrix $\Lambda(a)-\Delta(b)$ with $\alpha,\beta,\gamma,\alpha\gamma,\beta\gamma,\alpha\beta\gamma,$ and then the rowes $2,3,4,5,6,7,8$ with $a_1+b_1,\ a_2+b_2,\ a_3+b_3, a_4+b_4, a_5+b_5, a_6+b_6, a_7+b_7$ and adding them to the first row and then, multiplying the columns $2,3,4,5,6,7,8$ with $a_1+b_1, a_2+b_2,\ a_3+b_3, a_4+b_4, a_5+b_5, a_6+b_6, a_7+b_7$ and adding them to the column 7,we get a matrix B_1 with $\det B_1{=}\alpha^{-3}\beta^{-3}\gamma^{-3}\left(n\left(a\text{-}a_0\right)\text{-}n\left(b\text{-}b_0\right)\right)\left(a_7{+}b_7\right)^{-1}\det B_2$,

where $B_2\in\mathcal{M}_7(K)$.

Using the same tricks for B_2,we get, in the end, $\det(\Lambda(a)-\Delta(b)) = \alpha\beta\gamma\left(n\left(a-a_0\right)-n\left(b-b_0\right)\right)^2 n^2\left(a-a_0+b-b_0\right)$ and then $\det(\Lambda(a)\text{-}\Delta(b)){=}0$,

if $n(a\text{-}a_0)\text{-}n(b\text{-}b_0){=}0$. If $a_1+b_1=0,$ then we multiplicate with a_1 instead of a_1+b_1. Analogously, for $a_7+b_7=0$ and we obtain the same result.□

Corollary 3.8. *In the same hypothesis as in the Proposition 3.7., the matrix* $\Lambda(a)-\Delta(b)$ *has the rank* 6.

Proof. From the proof of the last proposition, it results that the matrix $\Lambda(a)-\Delta(b)$ is similar to the matrix

$$B_4=\begin{pmatrix} \dfrac{-n(a-a_0)+n(b-b_0)}{a_1+b_1} & E_2 & \dfrac{-n(a-a_0)+n(b-b_0)}{\alpha\beta\gamma(a_1+b_1)}\\ \dfrac{n(a-a_0)-n(b-b_0)}{a_7+b_7} & 0 & 0\\ E_1 & B_3 & 0 \end{pmatrix},$$

where $E_1\in\mathcal{M}_{6\times1}(K), E_2\in\mathcal{M}_{1\times6}(K), B_3\in\mathcal{M}_6(K)$, and if $n(a-a_0)=$ $=n(b-b_0)$, then $rank(\Lambda(a)-\Delta(b))=rankB_3=6.$□

Remark 3.9. Let $a,b\in\mathbb{O}(\alpha,\beta,\gamma)$ with

$a=a_0+a_1f_1+a_2f_2+a_3f_3+a_4f_4+a_5f_5+a_6f_6+a_7f_7$

$b=b_0+b_1f_1+b_2f_2+b_3f_3+b_4f_4+b_5f_5+b_6f_6+b_7f_7$, with $t(a)=t(b)$, then, from Propositions 1.11. and 1.12., it results that the relation

$$n(a)\,n(b)=\frac{1}{4}(ab+ba)^2 \tag{3.4.}$$

is true if and only if $a=rb,\ r\in K$. If $n(a)=n(b)$ then we have $r=1$ or $r=-1$. Indeed, the relation (3.4.) is equivalent to

$(n(a))^2 = (\alpha a_1 b_1 + \beta a_2 b_2 + \alpha\beta a_3 b_3 + \gamma a_4 b_4 + \alpha\gamma a_5 b_5 + \beta\gamma a_6 b_6 + \alpha\beta\gamma a_7 b_7)^2$
and, if $a = rb$, we obtain $(n(a) - rn(a))(n(a) + rn(a)) = 0$, therefore either $r = 1$ or $r = -1$.□

Proposition 3.10. *Let $a, b \in \mathbb{O}(\alpha, \beta, \gamma), a, b \notin K$ with $\bar{a} \neq b, t(a) = t(b)$,*
$n(a - a_0) = n(b - b_0)$. Then the solutions of the equation $ax = xb$ can be found in $\mathcal{A}(a, b)$ and are:
i) $x = \lambda_1 (a - a_0 + b - b_0) + \lambda_2 [n(a - a_0) - (a - a_0)(b - b_0)]$, where $\lambda_1, \lambda_2 \in K$, if $a \neq b$;
ii) The general solution of the equation $ax = xb$ can be expressed and by the form: $x = (a - a_0) q + q (b - b_0)$, where $q \in \mathcal{A}(a, b)$ is arbitrary;
iii) If $\bar{a} = b$, then the general solution for the equation (3.1.) is : $x = x_1 f_1 + x_2 f_2 + x_3 f_3 + x_4 f_4 + x_5 f_5 + x_6 f_6 + x_7 f_7$, where $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ satisfy the equality
$\alpha a_1 x_1 + \beta a_2 x_2 + \alpha\beta a_3 x_3 + \gamma a_4 x_4 + \alpha\gamma a_5 x_5 + \beta\gamma a_6 x_6 + \alpha\beta\gamma a_7 x_7 = 0$.

Proof. *i)* Let us given $x_1 = a - a_0 + b - b_0, x_2 = n(a\text{-}a_0)\text{-}(a\text{-}a_0)(b\text{-}b_0)$. If $b \neq \bar{a}$ it results $x_1 \neq 0$ and $x_2 \notin K$. Then
$ax_1 - x_1 b = a(a - a_0) + b(b - b_0) - (a - a_0) b - (b - b_0) b$. We write
$a = a_0 + v, b = b_0 + w$ with $t(v) = t(w) = 0$. Then $ax_1 - x_1 b =$
$= (a_0 + v) v + (a_0 + v) w - v (b_0 + w) - w (b_0 + w) = 0$, since
$n(v) = n(w), v^2 = -n(v), w^2 = -n(w)$. Therefore x_1 is a solution. Analougosly $ax_2 - x_2 b = 0$ and x_2 is a solution. It is obvious that $x_1, x_2 \in \mathcal{A}(a - a_0, b - b_0) = \mathcal{A}(a, b)$. We observe that x_1, x_2 are linear independent. Indeed, if $\theta_1 x_1 + \theta_2 x_2 = 0, \theta_1, \theta_2 \in K$, it results that $\theta_1 v + \theta_1 w + \theta_2 n(v) - \theta_2 vw = 0$. We have in turn:
$\theta_2 (n(v) + \alpha a_1 b_1 + \beta a_2 b_2 + \alpha\beta a_3 b_3 + \gamma a_4 b_4 + \alpha\gamma a_5 b_5 + \beta\gamma a_6 b_6 + \alpha\beta\gamma a_7 b_7) = 0$,
$\theta_1 (a_1 + b_1) - \theta_2 [\beta (a_2 b_3 - a_3 b_2) + \gamma (a_4 b_5 - a_5 b_4) + \beta\gamma (a_7 b_6 - a_6 b_7)] = 0$,
$\theta_1 (a_2 + b_2) - \theta_2 [\alpha (a_3 b_1 - a_1 b_3) + \gamma (a_4 b_6 - a_6 b_4) + \alpha\gamma (a_5 b_7 - a_7 b_5)] = 0$,
$\theta_1 (a_3 + b_3) - \theta_2 [(a_1 b_2 - a_2 b_1) + \gamma (a_4 b_7 - a_7 b_4) + \gamma (a_6 b_5 - a_5 b_6)] = 0$,
$\theta_1 (a_4 + b_4) - \theta_2 [\alpha (a_5 b_1 - a_1 b_5) + \beta (a_6 b_2 - a_2 b_6) + \alpha\beta (a_7 b_3 - a_3 b_7)] = 0$,
$\theta_1 (a_5 + b_5) - \theta_2 [(a_1 b_4 - a_4 b_1) + \beta (a_7 b_2 - a_2 b_7) + \beta (a_3 b_6 - a_6 b_3)] = 0$,
$\theta_1 (a_6 + b_6) - \theta_2 [\alpha (a_1 b_7 - a_7 b_1) + (a_2 b_4 - a_4 b_2) + \alpha (a_5 b_3 - a_3 b_5)] = 0$,
$\theta_1 (a_7 + b_7) - \theta_2 [(a_2 b_5 - a_5 b_2) + (a_6 b_1 - a_1 b_6) + (a_3 b_4 - a_4 b_3)] = 0$. Since $a \neq b$, from Remark 3.9. it results that $\theta_2 = 0$,therefore $\theta_1 (a_1 + b_1) = 0, ...,$
$\theta_1 (a_7 + b_7) = 0$, and from the fact that $b \neq \bar{a}$, it results $\theta_1 = 0$. As the solution subspace of the equation (3.1.) is of dimension two, it results that every solution of this equation has the form $\lambda_1 x_1 + \lambda_2 x_2$, with $\lambda_1, \lambda_2 \in K$, and $\lambda_1 x_1 + \lambda_2 x_2 \in \mathcal{A}(a - a_0, b - b_0) = \mathcal{A}(a, b)$.

ii) We prove that every element of the form $(a-a_0)\,q+q\,(b-b_0)$ is a solution for the equation (3.1.) : $ax-xb=(a_0+v)\,(vq+qw)-(vq+qw)\,(b_0+w)=$ $=a_0vq+a_0qw\,+v^2q+vqw-vqb_0-vqw-qwb_0-qw^2=0$. We suppose that z is a solution for the equation (3.1.) . It results that $az=zb$, therefore $vz=zw$. Take $q=-\frac{vz}{2n(v)}=-\frac{zw}{2n(v)},q\in\mathcal{A}\,(a,b)$. We have $x=vq+qw=$ $-\frac{v^2z}{2n(v)}-\frac{zw^2}{2n(v)}=\frac{z}{2}+\frac{z}{2}=z$, which gives that every solution can be written in the given form. Obviously, $z\in\mathcal{A}\,(a,b)$ for $a\neq b$. If $a=b$, let z be a solution for the equation $ax=xa$. Obviously $z\in\mathcal{A}\,(a)$ and for $q=\frac{-vz}{2n(v)}$, we obtain that every other solution, x, of the equation is of the form $x=$ $-\frac{v^2z}{2n(v)}-\frac{v^2z}{2n(v)}=z\in\mathcal{A}\,(a)$.

iii) If $b=\bar{a}$, it results $v=-w$. Then, if x is a solution for the equation (3.1.) , we obtain that $(a_0+v)\,(x_0+y)=(x_0+y)\,(a_0-v)$, hence $a_0x_0+a_0y+vx_0+vy=x_0a-x_0v+ya_0-yv$, therefore $2x_0v+vy+yv=0$, where $x=x_0+y$, with $x_0\in K, y=x_1f_1+x_2f_2+x_3f_3+x_4f_4+x_5f_5+x_6f_6+x_7f_7, t\,(y)=0$.

As $vy+yv\in K$, the previous equality is equivalent to $x_0=0$ and $vy+yv=0$, that is $x_0=0$ and $a_1x_1+a_2x_2+a_3x_3+a_4x_4+a_5x_5+a_6x_6+a_7x_7=0$.□

Proposition 3.11. *Let* $a,b\in\mathbb{O}\,(\alpha,\beta,\gamma)$ *with*
$a=a_0+a_1f_1+a_2f_2+a_3f_3+a_4f_4+a_5f_5+a_6f_6+a_7f_7$,
$b=b_0+b_1f_1+b_2f_2+b_3f_3+b_4f_4+b_5f_5+b_6f_6+b_7f_7$.
i)([Ti; 99], Theorem 3.3.) *The equation*

$$ax=\bar{x}b \tag{3.5.}$$

has non-zero solutions if and only if $n\,(a)=n\,(b)$. *In this case, if* $a+\bar{b}\neq 0$, *then* (3.5.) *has a solution of the form* $x=\lambda\,(\bar{a}+b)\,,\lambda\in K$.

ii) If $a+\bar{b}=0$, *then the general solution of the equation* (3.5.) *can be written in the form* $x=x_0+x_1e_1+x_2e_2+x_3e_3+x_4e_4+x_5e_5+x_6e_6+x_7e_7$, *where* $a_0x_0-\alpha a_1x_1-\beta a_2x_2-\alpha\beta a_3x_3-\gamma a_4x_4-\alpha\gamma a_5x_5-\beta\gamma a_6x_6-\alpha\beta\gamma a_7x_7$=0.

Proof. We suppose that (3.5.) has a non-zero solution, $x\in\mathbb{O}\,(\alpha,\beta,\gamma)$. Then we have $ax=\bar{x}b$ and $n\,(ax)=n\,(\bar{x}b)$, $n\,(a)\,n\,(x)=n\,(x)\,n\,(b)$, therefore $n\,(a)=n\,(b)$.

Conversely, we suppose that $n\,(a)=n\,(b)$. Let use take $y=\bar{a}+b$; we obtain $ay-\bar{y}a=a\,(\bar{a}+b)-\left(a+\bar{b}\right)b=a\bar{a}+ab-ab-\bar{b}b=n\,(a)-n\,(b)=0$.

If $a+\bar{b}=0$, then $b=-\bar{a}$ and the equation (3.5.) becomes $ax+\overline{ax}=0$, that is $t\,(ax)=0$. But $t\,(ax)=a_0x_0-\alpha a_1x_1-\beta a_2x_2-\alpha\beta a_3x_3-\gamma a_4x_4-\alpha\gamma a_5x_5-\beta\gamma a_6x_6-\alpha\beta\gamma a_7x_7$.□

Proposition 3.12. *Let* $a\in\mathbb{O}\,(\alpha,\beta,\gamma)\,,a\notin K$. *If there exists* $r\in K$ *such that* $n\,(a)=r^2$, *then* $a=\bar{q}rq^{-1}$, *where* $q=r+\bar{a}$.

Proof. By hypothesis, we have $a(r+\bar{a}) = ar + a\bar{a} = ar + n(a) = ar + r^2 = (a+r)r$. As $\bar{q} = r + a$ it results that $\bar{q}r = aq$.□

Proposition 3.13. *Let $a \in \mathbb{O}(\alpha,\beta,\gamma)$ with $a \notin K$, such that there exist $r, s \in K$ with properties $n(a) = r^4$ and $n(r^2+\bar{a}) = s^2$. Then the quadratic equation*

$$x^2 = a \tag{3.6.}$$

has two solutions of the form $x = \pm \frac{r(r^2+a)}{n(r^2+\bar{a})}$.

Proof. From *Proposition 3.12.,* it results that a has the form $a = \bar{q}r^2q^{-1}$, where $q = r^2 + \bar{a}$. As $q^{-1} = \frac{\bar{q}}{n(q)}$, we obtain $a = r^2\bar{q}q^{-1} = r^2\bar{q}\frac{\bar{q}}{n(q)} = r^2\frac{\bar{q}^2}{s^2} = \left(\frac{r}{s}\bar{q}\right)^2$, therefore $x_1 = \frac{r}{s}\bar{q}, x_2 = -\frac{r}{s}\bar{q}$ are the solutions.□

Corollary 3.14. *Let a, b, c be in $\mathbb{O}(\alpha,\beta,\gamma)$ such that ab and $b^2 - c \notin K$. If ab and $b^2 - c$ satisfy the conditions in Proposition 3.13., then the equations $xax = b$ and $x^2 + bx + xb + c = 0$ have solutions.*

Proof. $xax = b \Longleftrightarrow (ax)^2 = ab$ and $x^2 + bx + xb + c = 0 \Longleftrightarrow (x+b)^2 = b^2 - c$.□

Corollary 3.15. *If $b, c \in \mathbb{O}(\alpha,\beta,\gamma), b, c \notin K, c \in \mathcal{A}(b)$ with $\frac{b^2}{4} - c \neq 0$ and there exists $r \in K$ such that $n\left(\frac{b^2}{4} - c\right) = r^2$, and $n\left(r^2 + \frac{\bar{b}^2}{4} - \bar{c}\right) = s^2, s \neq 0$ then the equation*

$$x^2 + bx + c = 0 \tag{3.7.}$$

has a solution in $\mathbb{O}(\alpha,\beta,\gamma)$.

Proof. Let $x_0 \in \mathbb{O}(\alpha,\beta,\gamma)$ be a solution of the equation (3.7.). As $x_0^2 = t(x_0)x_0 - n(x_0)$ and $x_0^2 + bx_0 + c = 0$, it results that $t(x_0)x_0 - n(x_0) + bx_0 + c = 0$, therefore $(t(x_0) + b)x_0 = c + n(x_0)$. As $t(x_0) + b \neq 0, t(x_0), n(x_0) \in K, 1 \in \mathcal{A}(b,c)$, it results that $t(x_0) + b$ and $c + n(x_0) \in \mathcal{A}(b,c)$. Therefore $x_0 \in \mathcal{A}(b,c)$. Since $c \in \mathcal{A}(b)$, it results that $\mathcal{A}(b,c) = \mathcal{A}(b)$ is commutative, therefore x_0 commutes with every element of $\mathcal{A}(b,c)$. Then the equation (3.7.) can also be written under the form: $\left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4} + c = 0$.□

§ 4. EQUATIONS IN ALGEBRAS OBTAINED BY THE CAYLEY-DICKSON PROCESS OF DIMENSION ≥ 8

In this section, A denotes an algebra obtained by the Cayley-Dickson process and having $\dim A = n, n \gneqq 8$. Let $\{e_1, e_2, ..., e_n\}$ be a basis of A.

Proposition 4.1. *Let $a, b \in A$ with $t(a) = t(b)$ and $n(a - a_0) = n(b - b_0)$.*
i) If $b \neq \bar{a}$, then the equation

$$ax = xb \tag{4.1.}$$

has a solution of the form $x = \theta(n(a - a_0) + n(b - b_0))$, where $\theta \in K$ is arbitrary.

ii) If $b = \bar{a}$, then the equation (4.1.) *has the general solution of the form $x = x_1e_1 + x_2e_2 + ... + x_ne_n$, with $f(a, x) = 0$ where $f : A \times A \to K$ is the associated bilinear form.*

Proof. *i)* Let $x_1 = a - a_0 + b - b_0$. We denote $a - a_0 = v, b - b_0 = w$, with $t(v) = t(w) = 0$ and $n(v) = n(w)$; then we have $ax_1 - x_1b = (a_0 + v)(a - a_0 + b - b_0) - (a - a_0 + b - b_0)(b_0 + w) = a_0(a - a_0) +$
$+a_0(b - b_0) + v(a - a_0) + v(b - b_0) - (a - a_0)b_0 - (b - b_0)b_0 - (a - a_0)w -$
$-(b - b_0)w = v^2 + vw - vw - w^2 = 0$.

ii) If $b = \bar{a}$, then the equation (4.1.) becomes $ax = x\bar{a}$,therefore
$(a_0 + v)(x_0 + y) - (x_0 + y)(a_0 - v) = 0, vx_0 + vy + x_0v + yv = 0$ and $2vx_0 + vy + yv = 0$. As $vy + yv \in K$ (in *Proposition 1.11.*), it results that $x_0 = 0$,therefore $vy + yv = 0$, where $x = x_0 + y$, with $t(y) = 0$ and we obtain (by *Proposition 1.11.*) $f(a, x) = 0$.□

Remark 4.2. Since A is not an alternative algebra, we obtain that the element $x_2 = n(a - a_0) - (a - a_0)(b - b_0)$ is not a solution for the equation (4.1.)

Proposition 4.3. *Let $a, b \in A$.*
i) ([Ti; 99] Theorem 4.3.) *The equation*

$$ax = \bar{x}b \tag{4.2.}$$

has non-zero solutions if $n(a) = n(b)$. In this case, if $a + \bar{b} \neq 0$, then (4.2.) *has a solution of the form $x = \lambda(\bar{a} + b), \lambda \in K$.*

ii) If $a + \bar{b} = 0$, then the general solution for the equation (4.2.) *can be written under the form $x = x_0 + x_1e_1 + ... + x_ne_n$, where $t(ax) = 0$ and t is the trace*

Proof. *i)*We suppose that $n(a) = n(b)$. Let $y = \bar{a} + b$ and we obtain $ay - \bar{y}a = a(\bar{a} + b) - (a + \bar{b})b = a\bar{a} + ab - ab - \bar{b}b = n(a) - n(b) = 0$.

ii) If $a + \bar{b} = 0$,then $b = -\bar{a}$ and the equation (4.2.) becomes $ax + \bar{a}x = 0$, that is $t(ax) = 0$.□

Proposition 4.4 *Let* $a \in A, a \notin K$. *If there exists* $r \in K$ *such that* $n(a) = r^2$, *then* $a = \bar{q}rq^{-1}$, *where* $q = r + \bar{a}$.

Proof. By hypothesis, we have $a(r+\bar{a}) = ar + a\bar{a} = ar + n(a) = ar + r^2 = (a+r)r$. As $\bar{q} = r + a$, it results $\bar{q}r = aq$.□

Proposition 4.5. *Let* $a \in A$ *with* $a \notin K$,*such that there exist* $r, s \in K$ *with the property* $n(a) = r^4$ *and* $n(r^2 + \bar{a}) = s^2$. *Then the quadratic equation*

$$x^2 = a \tag{4.3.}$$

has two solutions of the form $x = \pm \frac{r(r^2+a)}{n(r^2+\bar{a})}$.

Proof. From *Proposition 4.4.,* it results that a has the form $a = \bar{q}r^2q^{-1}$, where $q = r^2 + \bar{a}$. As $q^{-1} = \frac{\bar{q}}{n(q)}$, we obtain that $a = r^2\bar{q}q^{-1} = r^2\bar{q}\frac{\bar{q}}{n(q)} = r^2\frac{\bar{q}^2}{s^2} = \left(\frac{r}{s}\bar{q}\right)^2$, therefore $x_1 = \frac{r}{s}\bar{q}$ and $x_2 = -\frac{r}{s}\bar{q}$ are solutions.□

Corollary 4.6. *Let* $a, b, c \in A$ *such that* ab *and* $b^2 - c \notin K$. *If* ab *and* $b^2 - c$ *satisfy the hypothesis of Proposition 4.5., then the equation* $x^2 + bx + xb + c = 0$ *has solutions.*

Proof. $x^2 + bx + xb + c = 0 \Longleftrightarrow (x+b)^2 = b^2 - c$.□

Remark 4.7. Since, generally, the equation $xax = b$ cannot be written in the form $(ax)(ax) = ab$ in A, we cannot solve this equation by using *Proposition 4.5.*

Corollary 4.8. *If* $b, c \in A, b, c \notin K, by = yb, \forall y \in A$ *with* $\frac{b^2}{4} - c \neq 0$ *and there exists* $r \in K$ *such that* $n\left(\frac{b^2}{4} - c\right) = r^4$, $n\left(r^2 + \frac{\bar{b}^2}{4} - \bar{c}\right) = s^2$, $s \neq 0$, *then the equation*

$$x^2 + bx + c = 0, \tag{4.4.}$$

has solutions in A .

Proof. Let $x_0 \in A$ be a solution of the equation (4.4.). Since $by = yb, \forall y \in A$, then the equation (4.4.) can be also written as $\left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4} + c = 0$ and then we get the result from *Proposition 4.5.*□

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