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THE ASYMPTOTIC STRESS FIELD FOR FREE EDGE JOINTS UNDER SMALL - SCALE YIELDING CONDITIONS

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Abstract

In the complex engineering structures the use of bonded joints are often preferred to more traditional methods of fabrication such as bolts and welds since they are lighter and spread load more evenly. For determination the durability of such structures it is necessary to know well the stress field around stress concentrators. At the free edge of bonded joints between the adhesive and the adherend layers it is well-known that there exists an elastic singular stress field. However, little is known about the material behavior beyond the yield point. This paper presents the small-scale yielding plane-strain asymptotic field calculated for the interfacial free-edge joint singularity. Non-linear elasto-plastic materials with Ramberg-Osgood power-law hardening properties bonded to a rigid elastic substrate were considered. The leading order problem consists of five non-linear ordinary differential equations. The numerical solutions were obtained using a fourth order Runge - Kutta numerical method fitted to the governing equations. For exemplification the singularity orders for the interfacial free-edge joint with an angle of 60° are shown plotted against hardening exponent.

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1 Introduction

Interface-controlled fracture is one of the most important microscopic events leading to ultimate macroscopic rupture in many polycrystalline, composite, and ceramic materials. The elasto-plastic interfacial crack problem has received considerable attention in the last decade enabling a thorough understanding to be developed. Numerical solutions involving elasto-plastic behaviour at a traction-free crack tip for a Ramberg-Osgood hardening material have been developed by Shih and Asaro (1988, 1989) and Zywicz and Parks(1992) amongst others.

Failure of interfacial systems frequently initiates, however, at the free-edge joint of two materials, where a stress singularity also exists, leading to the development and propagation of an interface crack. The analysis of such interfacial free-edge stress fields is just as important, therefore, to our understanding of crack initiation and growth though in comparison to its counterpart the interface crack it has received far less attention. A description of the process leading to crack initiation assuming purely elastic behaviour is complicated by the difference in stress singularity orders and fields. Indeed, it has been shown by Klingbeil and Beuth (2000) that conflicting solutions are obtained if designing to prevent debond of the interfacial free-edge joint and/or to prevent propagation of an interfacial crack. Furthermore, the same limitations of the elastic solution apply to the interfacial free-edge solution as with the interfacial crack-tip, i.e. the stress and strains are unbounded. Relatively little effort has been paid to the elasto-plastic behaviour of the free-edge singularity except for the determination of plastic zone size and shape (Romeo and Ballarini, 1994, Yang, et al., 1997). The rigid-plastic slip-line field for the crack developed by Zywicz and Parks (1992) was used to show that the interfacial free-edge solution has some strong similarities with its counterpart the interfacial crack-tip for elastic-perfectly-plastic material. Zhang and Joseph (1998) presented a nonlinear plane strain finite element analysis in order to describe the singular stress fields in bi-material wedges, while Loghin and Joseph (2003) investigates the mixed mode fracture in power law hardening materials. Lazzarin et al., 2001 and Filippi et al., 2002 presented analytical approaches of asymptotic solutions for V shape notches in materials with Ramberg - Osgood non-linear material behaviour.

Marsavina and Nurse(2007) presented a comparison of the asymptotic structure of small-scale yielding for interfacial free-edge joint and crack-tip fields. They also proposed an expression for estimation the elasto-plastic singularity order for the interfacial free-edge joint s_{joint} on the form: $s_{joint} = 2s_{elastic}/(1+n)$, where $s_{elastic}$ is the elastic singularity order for the interfacial free-edge joint and n the hardening exponent.

In this paper, the asymptotic structure of the elasto-plastic stress field at the interfacial free-edge joint is considered for a Ramberg-Osgood hardening material and a rigid elastic material bonded perfectly to form a half plane. Within the framework of small-scale yielding (SSY) the singular fields for varying degrees of hardening are numerically calculated by developing asymptotic solutions to the fundamental equations of equilibrium and compatibility. The asymptotic structures of the stress and displacement field developed at the bonded free-edge joint are obtained using an approach similar to that of Sharma and Aravas (1993).

2 Formulation of the problem

A thorough analysis of the plane-strain interfacial free-edge joint (Fig. 1) is presented. The constitutive behaviour of the homogeneous isotropic elasto-

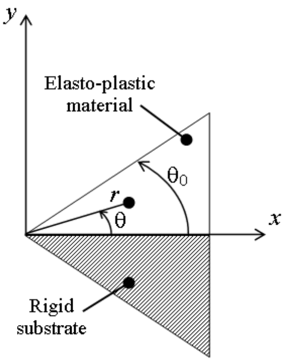

Figure 1: Geometry of interfacial free-edge joint and cylindrical coordinates

plastic material is characterised by the J_2 deformation theory for Ramberg-Osgood uniaxial stress-strain behaviour, i.e.:

$$\varepsilon_{ij} = \frac{1+\nu}{E} s_{ij} + \frac{1-2\nu}{3E}\sigma_{kk}\delta + \frac{3}{2}\alpha\varepsilon_0 \left(\frac{\sigma_e}{\sigma_0}\right)^{n-1} \frac{s_{ij}}{\sigma_0} \tag{1}$$

where ε_{ij} is the infinitesimal strain tensor, σ_0 is the yield stress, $\varepsilon_0 = \sigma_0/E$, and the deviatoric stress is given by:

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} \tag{2}$$

and the Mises equivalent stress is defined as:

$$\sigma_e = \left(\frac{3}{2} s_{ij} s_{ij}\right)^{\frac{1}{2}} \tag{3}$$

Also, n is the power-law hardening exponent ($1 \le n \le \infty$), E is the Young's

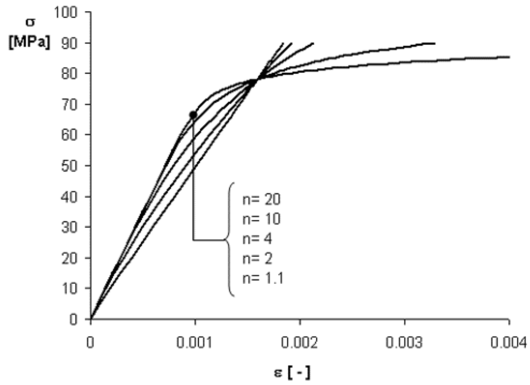

Figure 2: Non-linear Ramberg - Osgood material behavior

moduli, δ_{ij} is the Kronecker delta, and α is a material constant. If n=1 then the behaviour is purely elastic.

A cylindrical co-ordinate system is adopted and the equilibrium equations for infinitesimal linear strain theory can be expressed as:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{3}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \tag{4}$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{3}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{2\partial \sigma_{r\theta}}{r} = 0 \tag{5}$$

The strain-displacement equations are written as:

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} \tag{6}$$

$$\varepsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} \tag{7}$$

$$\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) \tag{8}$$

where $u = (u_r, u_\theta)$ is the displacement vector. Out-of-plane stresses and strains are assumed to vanish. Finally, the strain-compatibility equation is given by:

$$\left(\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} - \frac{1}{r}\frac{\partial}{\partial r}\right)\varepsilon_{rr} + \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)\varepsilon_{\theta\theta} - \left(\frac{2}{r^2}\frac{\partial}{\partial\theta} + \frac{2}{r}\frac{\partial^2}{\partial r\partial\theta}\right)\varepsilon_{r\theta} = 0 \quad (9)$$

For the problem of a homogeneous isotropic elasto-plastic material bonded perfectly to a rigid base, the following boundary conditions apply for the interfacial free-edge joint:

$$\sigma_{\theta\theta}(r,\theta_0) = 0, \sigma_{r\theta}(r,\theta_0) = 0 \quad (10)$$

$$u_r(r,0) = 0, u_\theta(r,0) = 0 \quad (11)$$

Existing solutions for the two geometries under elasto-plastic behaviour are described in the next section to provide a foundation for the new study. Selection of the upper material for this investigation is a comparatively arbitrary choice though that used throughout has elastic properties typical of modern adhesives and as shown later its singularity order under elastic conditions is approximately half that of the interfacial crack. Predictions for the asymptotic elasto-plastic behaviour at the interfacial free-edge joint satisfying the boundary conditions (10) and (11) are obtained using a similar approach to Sharma and Arawas (1993). The key parameter in the investigation is the hardening exponents and values were considered.

The fundamental question to be posed is what are the singularity orders for a joint angle θ and for different given hardening exponent n. To arrive at the answer the numerical results of an asymptotic analysis for the interfacial free-edge joint are presented next. Numerical calculations assume the upper domain is a Ramberg-Osgood hardening material that is perfectly bonded to a rigid elastic substrate. Field solutions to the plane-strain interfacial free-edge joint problem were obtained using the asymptotic approach of Sharma and Arawas (1993) for hardening cases n=1.1, 2, 4, 10, and 20. The value n=1.1 was chosen as it is just above unity (elastic behaviour). The asymptotic solutions to the problem are presented in two subsections. The first presents the radial variations of stress for the asymptotic fields. Finally, the second subsection presents the angular variations of stresses.

To obtain the asymptotic solution the problem is formulated in terms of the leading order stresses $\tilde{\sigma}^{(0)}$ in (12) and the corresponding displacement leading order expansion of the form:

$$\frac{\sigma(r,\theta)}{\sigma_0} = \left(\frac{\alpha\varepsilon_0\sigma_0 I_n r}{J}\right)^s \tilde{\sigma}^{(0)}(\theta) + Q\left(\frac{\sigma_0 r}{J}\right)^t \tilde{\sigma}^{(1)} + \dots \text{ as } r \longrightarrow 0 \quad (12)$$

$$\frac{\mathbf{u}(r,\theta)}{\alpha\varepsilon_0} = \left(\frac{\alpha\varepsilon_0\sigma_0 I_n}{J}\right)^{sn} r^s \tilde{\mathbf{u}}^{(0)}(\theta) + \ldots \text{ as } r \longrightarrow 0, \tag{13}$$

where $\tilde{\sigma}^{(0)}$ and $\tilde{\sigma}^{(1)}$ are normalised angular functions, $s < t < 0, J$ is the J-integral, and Q is the parameter controlling the magnitude of the second term. The quantity I_n is defined in Sharma and Arawas, 1991.

The expansions (12) and (13) are substituted into the governing equations of equilibrium and stress-strain relationship (Sharma and Arawas, 1991). Terms having like powers of r are collected and hierarchy of problems is obtained. The leading order problem that defines $\tilde{\sigma}^{(0)}$ and $\tilde{\mathbf{u}}^{(0)}$ consists of five non-linear ordinary differential equations:

$$\begin{gathered}
(s+1)\tilde{\sigma}_{rr}^{(0)} - \tilde{\sigma}_{\theta\theta}^{(0)} + \tfrac{\mathrm{d}\tilde{\sigma}_{r\theta}^{(0)}}{\mathrm{d}\theta} = 0 \\
\tfrac{\mathrm{d}\tilde{\sigma}_{\theta\theta}^{(0)}}{\mathrm{d}\theta} + (s+2)\tilde{\sigma}_{r\theta}^{(0)} = 0 \\
(sn+1)\tilde{\mathbf{u}}_r^{(0)} - \tfrac{3}{2}\tilde{\sigma}_e^{(0)n-1}\tilde{\mathbf{s}}_{rr}^{(0)} = 0 \\
\tilde{\mathbf{u}}_r^{(0)} + \tfrac{\mathrm{d}\tilde{\mathbf{u}}_\theta^{(0)}}{\mathrm{d}\theta} - \tfrac{3}{2}\tilde{\sigma}_e^{(0)n-1}\tilde{\mathbf{s}}_{\theta\theta}^{(0)} = 0 \\
\tfrac{1}{2}\left(\tfrac{\mathrm{d}\tilde{\mathbf{u}}_r^{(0)}}{\mathrm{d}\theta} + sn\tilde{\mathbf{u}}_\theta^{(0)}\right) - \tfrac{3}{2}\tilde{\sigma}_e^{(0)n-1}\tilde{\sigma}_{r\theta}^{(0)} = 0
\end{gathered} \tag{14}$$

A second order problem may be expressed as a linear eigenvalue problem to solve for the exponent t and the eigen-functions for the corresponding stresses $\sigma^{(1)}$ (and displacements $\mathbf{u}^{(1)}$). This has not been solved at this time as the focus of the paper is the leading order solution for the interfacial free-edge joint and its similarities with that of the crack tip. A fourth-order Runge-Kutta solution to the equations in (14) was obtained for different values of the hardening exponent n using the proprietary software *Mathcad* (v.2001). An iteration scheme was used to determine the solution s to the non-linear eigenvalue problem and the subsequent distributions for the stresses and displacements that satisfy (14) above, and the conditions (10) and (11). The shooting method was used prior to solve the system (14) in order to find the initial values for $\tilde{\sigma}_{\theta\theta}^{(0)}(r,0)$ and $\tilde{\sigma}_{r\theta}^{(0)}(r,0)$.

3 Numerical results

The results are shown for a joint angle of 60° ($\theta_0 = 60°$in Fig. 1), considering a non linear material behaviour described by eq. (1) and Fig. 2 considering the following hardening exponents: $n = 1.1$, 2, 4, 10 and 20.

3.1 Radial variations

Determination of the singularity order s was done using asymptotic analysis. The radial variation of the Von-Mises equivalent stress versus ratio between distance r and size of the plastic zone r_p in a log-log scale are plotted in Fig. 3 for $-4 < \log(r/rp) < 1$ and on a direction $\theta = 30°$.

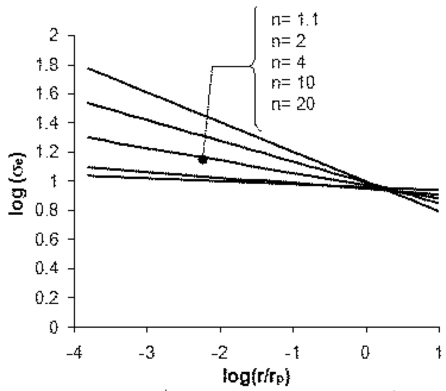

Figure 3: Radial variation of the asymptotic normalised plane-strain equivalent stress for the interfacial free-edge ($\theta_0 = 60°$) at $\theta = 30°$ for different hardening exponents

The asymptotic solution has been superimposed as a straight line of the appropriate gradient given by s to enable the region of singularity dominance to be determined. The singularity orders s, determined by the asymptotic solution, are plotted versus hardening exponent n, Fig. 4. It can be seen the tendency of decreasing the singularity with increasing the value of hardening coefficient.

3.2 Angular variations

The radial variation of the stresses $\tilde{\sigma}_{rr}^{(0)}\tilde{\sigma}_{\theta\theta}^{(0)}\tilde{\sigma}_{r\theta}^{(0)}$ for the five cases of hardening n = 1.1, 2, 4, 10 and 20 are plotted in Figs. 5 - 7 at a radius $\log(r/r_p) = -2$. The results are normalised to the maximum value of the equivalent stress in the angular variation. It could be observed that the free edge boundary condition for $\tilde{\sigma}_{\theta\theta}^{(0)}\tilde{\sigma}_{r\theta}^{(0)}$ at $\theta = \theta_0$ are fulfilled. The values of the stresses $\tilde{\sigma}_{rr}^{(0)}\tilde{\sigma}_{\theta\theta}^{(0)}\tilde{\sigma}_{r\theta}^{(0)}$ decrease with increasing the hardening exponent for all angles θ.

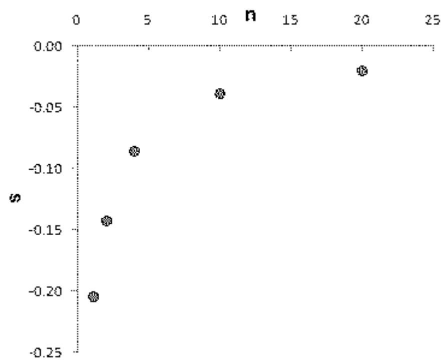

Figure 4: Singularity order versus hardening exponents for the interfacial free-edge ($\theta_0 = 60°$)

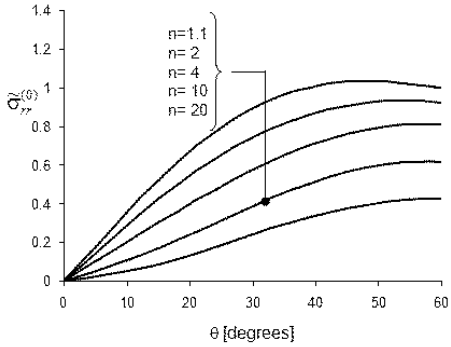

Figure 5: Angular variation of the asymptotic normalised plane-strain radial stress component $\tilde{\sigma}_{rr}^{(0)}$ for the interfacial free-edge joint ($\theta_0 = 60°$)

4 Conclusions

For an isotropic elasto - plastic non-linear material bonded to a rigid substrate the SSY asymptotic plane-strain behaviour at the interfacial free-edge joint has been identified for several values of the hardening exponent n. A non-linear system of five differential equations was solved numerically using a fourth order Runge - Kutta method in conjunction with the shooting method. Using an asymptotic analysis the cylindrical components of stresses have been determined for a geometry with $\theta_0 = 60°$. The singularity orders under elasto - plastic behaviour were identified and shown to be only dependent on the

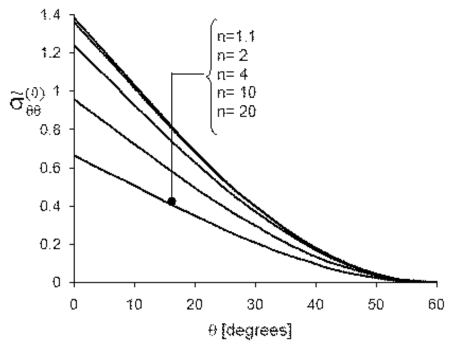

Figure 6: Angular variation of the asymptotic normalised plane-strain radial stress component $\tilde{\sigma}^{(0)}_{\theta\theta}$ for the interfacial free-edge joint ($\theta_0 = 60°$)

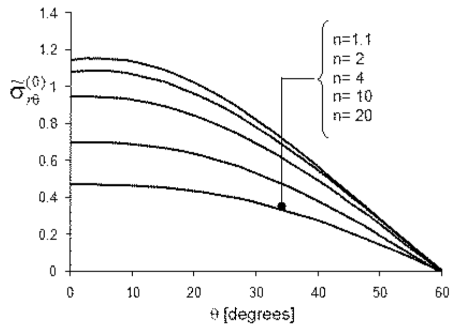

Figure 7: Angular variation of the asymptotic normalised plane-strain radial stress component $\tilde{\sigma}^{(0)}_{r\theta}$ for the interfacial free-edge joint ($\theta_0 = 60°$)

hardening exponent n and not on the elastic properties of the material.

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