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THE STRUCTURAL INFLUENCE OF THE FORCES OF THE STABILITY OF DYNAMICAL SYSTEMS

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In memoriam Acad. Lazar Dragos (1930 - 2009)

Abstract

In this paper, we consider the autonomous dynamical system linear or linearized with 2 degrees of freedom. In the system of equations of 4th degree, the structure generalized forces appear: $K(q)$ - the conservative forces, $N(q)$ - the non-conservative forces, $D(\dot{q})$ the dissipative forces, $G(\dot{q})$ the gyroscopically forces. In the linear system, these forces from the different structural combinations can produce the stability or the instability of the null solution. The theorems of Thomson - Tait - Cetaev (T-T-C) are known for the configurations (K, D, G). We introduce the non – conservative forces N, studying the stability with the Routh - Hurwitz criterion or constructing the Lyapunov function, obtaining some theorems with practical applications.

1 Introduction

In this section we study the structural influence of the term blocks on the stability of the null solution for the bi-dimensional system or equations with

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fourth degree, which is a linear or linearized system in first approximation for the nonlinear system ([2], [4]).

$$\begin{cases} \ddot{x}_1 + k_{11}x_1 + k_{12}x_2 + c_{11}\dot{x}_1 + c_{12}\dot{x}_2 + g_{11}\dot{x}_1 + g_{12}\dot{x}_2 + n_{11}x_1 + n_{12}x_2 = 0 \\ \ddot{x}_1 + k_{21}x_1 + k_{22}x_2 + c_{21}\dot{x}_1 + c_{22}\dot{x}_2 + g_{21}\dot{x}_1 + g_{22}\dot{x}_2 + n_{21}x_1 + n_{22}x_2 = 0 \end{cases} \tag{1}$$

In this system the matrix blocks $Kx, C\dot{x}, G\dot{x}, Nx$ are representing respectively the conservative (elastic) forces, the resistance (amortization) forces, the gyroscoplically forces and the non - conservative forces. The characteristic polynomial for the Routh - Hurwitz criterion will be ([1], [7]):

$$P(\lambda) = det \begin{vmatrix} \lambda^2 + (c_{11}+g_{11})\lambda + k_{11} + n_{11} & (c_{12}+g_{12})\lambda + k_{12} + n_{12} \\ (c_{21}+g_{21})\lambda + k_{21} + n_{21} & \lambda^2 + (c_{22}+g_{22})\lambda + k_{22} + n_{22} \end{vmatrix} = 0. \tag{2}$$

The system (1) can be put into the canonical form and making abstraction of the negative constant factor, the stated forces will be respectively side by the system (x_1, x_2): $\bar{F}(k_{11}x_1 + k_{12}x_2, k_{21}x_1 + k_{22}x_2)$, $\bar{C}(c_{11}\dot{x}_1 + c_{12}\dot{x}_2, c_{21}\dot{x}_1 + c_{22}\dot{x}_2)$, $\bar{G}(g_{11}\dot{x}_1 + g_{12}\dot{x}_2, g_{21}\dot{x}_1 + g_{22}\dot{x}_2)$, $\bar{N}(n_{11}x_1 + n_{12}x_2, n_{21}x_1 + n_{22}x_2)$.
Regarding this system there are classical contributions of the Lyapunov and the theorems of Thomson - Tait - Cetaev [4], Merkin [2] and Crandall [4]. Here we distinguish these results and we make other structural contribution by examples.
From the matricial calculus we know that any squared matrix can be decomposed in a sum of a symmetric matrix and an asymmetric one, $M = A + B$, where $A = \frac{1}{2}(M + M')$, $B = \frac{1}{2}(M - M')$.
From the decomposition theorem, and using the fact that the positional forces $\bar{K}$ are conservative with $rot\bar{K} = 0$, $\bar{K} = -gradU(q)$ and $rot\bar{N} \neq 0$ and $rot_{\dot{q}}\bar{C} = 0, \bar{C} = -grad_{\dot{q}}V(q), rot\bar{G} \neq 0$, we have the condition of symmetry and asymmetry $k_{ij} = k_{ji}, n_{ij} = -n_{ji}, g_{ij} = -g_{ji}, c_{ij} = c_{ji}, i, j = 1, 2$.
We have the relations:

$$\begin{cases} \ddot{x} + c_1\dot{x} + g\dot{y} + k_1x - py = X^s(0) \\ \ddot{y} + c_2\dot{y} - g\dot{x} + k_2y + px = Y^s(0). \end{cases} \tag{3}$$

The system (3) has the fourth degree, and the characteristic polynomial is:

$$P(\lambda) = det \begin{vmatrix} \lambda^2 + \lambda c_1 + k_1 & g\lambda - p \\ -g\lambda + p & \lambda^2 + c_2\lambda + k_2 \end{vmatrix} = 0, \tag{4}$$

$$P(\lambda) = \lambda^4 + \lambda^3(c_1+c_2) + \lambda^2(k_1+k_2+c_1c_2+g^2) + \lambda(c_1k_2+c_2k_1+2gp) + k_1k_2+n^2 = 0. \tag{5}$$

The system with constant coefficients (1) becomes:

$$\begin{cases} \ddot{x} + ax + ky + p\dot{x} + c\dot{y} + g\dot{y} - ny = 0 \\ \ddot{y} + kx + by + c\dot{x} + q\dot{y} - g\dot{x} + nx = 0 \end{cases}. \tag{6}$$

The mechanical justification of this configuration is obtained starting from the Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} = Q_1(q) + Q_2(\dot{q}) + Q_s(q,\dot{q}), \tag{7}$$

where T is the kinetic energy, Q_1, Q_2 are the generalized forces and Q_s is the generalized force with superior order terms. So, the mentioned decomposition becomes symbolic for the positional forces and the inertial forces:

$$\bar{Q}_1(q) = \bar{F}(q) + \bar{N}(q); \bar{Q}_2(\dot{q}) = \bar{C}(\dot{q}) + \bar{G}(\dot{q}), \tag{8}$$

where

$$\bar{F} = -gradU(q), \bar{C}(\dot{q}) = -grad_{\dot{q}}V(\dot{q}),$$

with:

- $U(q)$ - the potential energy,

- $U(\dot{q}) - \sum\sum c_{ij}\dot{q}_i\dot{q}_j$ - the dissipationfunction of with negative mechanical work,

- $\bar{N}(q)$ - the conservative forces,

- $\bar{G} = \sum\sum g_{ij}\dot{q}_i\dot{q}_j$ $(g_{ij} = -g_{ji})$, with null mechanical work,

The system is completely dissipative if $U(\dot{q})$ is a quadratique form positively defined and

$$\frac{d}{dt}(T+U) = -2H.$$

Theorem 1 *If $\bar{Q}_1 = -gradU(q) + N(q)$, then the expression of the Q_1 is:*

$$Q_1 = K(q) + N(q).$$

Theorem 2 *If $\bar{Q}(\dot{q}) = -gradH(\dot{q}) + \bar{D}(\dot{q})$, then the expression of Q_2 is:*

$$Q_2 = G(\dot{q}) + D(\dot{q}).$$

Based on the structure of forces $K(q), N(q), D(\dot{q}), G(\dot{q})$, we analyze the stability, making some combinations of these forces, and we obtain a series of theorems.

Thomson - Tait - Cetaev (T-T-C)has defined some theorems for the system K, D, G. The applications are for: the gyroscopes, bearings on the fluid support, the double pendulum, the electron in the magnetic field, the car equation and other examples from different domains making analogies for the system of the fourth degree.

In the next theorems we denote by:

- Σ - the system (ex. $\Sigma(K,D)$ - the system composed by K and D),

- S - the stable case,
- $A.S$ - the asymptotic case,
- I - the unstable case,

using directly the characteristic polynomial (5) (H-R) or the Lyapunov function for (6).

2 The study of the stability of the dynamical systems

To start with, we consider the stability of the equilibrium position in the point of minimum of the potential energy (the Theorem Lagrange-Dirichlet) with slight oscillations around this position. We denote with $K_0(q)$ the case of the cyclical coordinates in the plane phases, when a uniform movement related with these is obtained. Using the equations of Routh - Hurwitz - the case Lagrange - Poisson, for the solid with a fixed point (the gyroscop) we study the stability of the uniform movements.

Theorem 3 *If* $-\frac{\partial U}{\partial q} = K_0(q)$*, the dynamical system* $\sum(K_0(q))$ *is dynamically stable around* $q_0 = 0$*. (Ex.: the mathematical pendulum in the gravitation field.)*

$$S(K_0) \Rightarrow S(\sum K_0).$$

Observation 1 *If* K_0 *is stable, then* $S(\sum K_0, G)$ *is stable.*

Observation 2 *If* K_0 *is unstable, then* $S(\sum K_0, G)$ *is unstable.*

Theorem 4 *(T-T-C) In the conservative system, if the potential energy has an isolated minimum, then the system is simply stable around the minimal point: if* K_0 *is stable, then* $S(\sum K_0)$ *implies* $S(\sum(K_0 + (G, D)))$ *around the zero point.*

Theorem 5 *(T-T-C) If we have an isolated potential simply stable system then, by attaching to the system the dissipative forces, the the simple stability is kept. If the dissipation is total then the system becomes asymptotically stable: if* K_0 *is stable,* $S(\sum K_0) \Rightarrow \sum(K_0 + (G, D_c))A.S.$*,* D_c *is completely dissipative, and in this case the system is assymptotically stable.*

Theorem 6 *(T-T-C) If the dissipative forces is applied, then the instability is kept, in an instable potential regime in the neighborhood of a maximum point of the potential energy, when this energy is negative: if* K_0 *is unstable, then* $\sum(K_0 + G + D_c)$ *is unstable:* $I(K_0) \Rightarrow I\sum(K_0 + G + D_c)$*.*

The theorems 3–6 are verified directly applying the Routh - Hurwitz criterion for the system (13).

Theorem 7 *If G is stable, then G has a stable uniform movement (stability about $\dot{q}$). Ex.: the case Lagrange - Poisson, the Routh method, when the rotation angle is given and the precession angle cyclically implies the uniform movement.*

Theorem 8 *If $detG \neq 0$, the stability is conserved with respect to the coordinates and the speeds.*

Corollary 1 *If during the stable movement $detG = 0$, then the stability is lost with respect to the coordinates, but with the speeds.*

Corollary 2 *If the system is non linear under the acting on G and stable, with $detG \neq 0$, then the stability of the non linear system is not implicated.*

Theorem 9 *(T-T-C) The system $\sum(G + D_c)A.S.$ is conserved for the coordinates and speeds and for the non linear systems.*

Theorem 10 *If non - potential forces act, then the system is unstable: $\sum(N) \Rightarrow I(\sum(N))$. (see the Application 1)*

Theorem 11 *The system $\sum(N + D_c)$ is unstable.*

Theorem 12 *1. If the system $(K+N)$ is stable, then $\sum(K+N)$ is perturbed (can be stable and unstable).*

2. If the system $(K+G)$ is stable or unstable, then the system $(K+G+N)$ is perturbed (can be stable and unstable).

Theorem 13 *The dissipative forces can influenced on the stability $\sum(K+N)$:*
- if $\sum(K + N)$ is unstable, then $\sum(D + K + N)$ can be stabilized;
- if $\sum(K + N)$ is stable, then $\sum(K + D + N)$ can be destabilized.

Theorem 14 *If in the system K the two equations of second degree have equal frequencies and the system $\sum(K + N)$ is linear, then the stability is perturbed no matter of the nonlinear terms.*

Theorem 15 *If $(G + N)$ is unstable then $\sum(G + N + D)$ is stable.*

The theorem T-T-C does not apply always if the non potential forces (N) [2] do not appear.

Application 1 A system which has only the non potential forces is always unstable.

Such a system is represented by the equations:

$$\begin{aligned} \ddot{x} + py &= 0 \\ \ddot{y} - px &= 0. \end{aligned} \tag{9}$$

The characteristic equation is $\lambda^4+p^2=0$, with the solutions $\lambda=\pm\sqrt{2}/2(1\pm i)p$ having the roots with the real part positive, so we have the instability of the system.

Application 2 When the systems act by the conservative and non potential forces, the equations are:

$$\begin{aligned}\ddot{x}+k_1x+py&=0,\\ \ddot{y}+k_2y-px&=0.\end{aligned}\tag{10}$$

The characteristic equation $\lambda^4+(k_1+k_2)\lambda^2+k_1k_2+p^2=0$ must have the real and negative roots in λ^2. The stability domains are presented in the figures below:

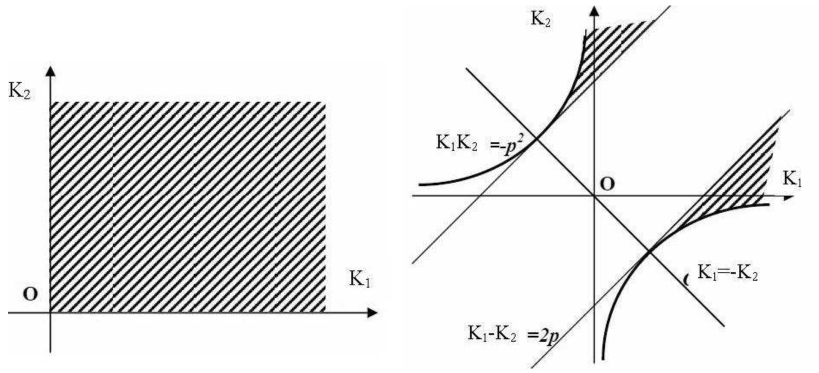

Figure 1: The domains of stability

Application 3 Introducing the dissipative forces in a system that acts by the conservative and non potential forces, we have (Fig. 1):

$$\begin{aligned}\ddot{x}+c_1\dot{x}+k_1x+py&=0,\\ \ddot{y}+c_2\dot{y}+k_2y-px&=0.\end{aligned}\tag{11}$$

The characteristic equation is:

$$\lambda^4+(c_1+c_2)\lambda^3+(k_1+k_2+c_1c_2)\lambda^2+(k_1c_2+k_2c_1)\lambda+k_1k_2+p^2=0.$$

Application 4 For the gyroscopic pendulum with 2 degrees of freedom x,y and 2 types of amortization [1] we have one linear stationary amortization $(c_s\dot{x},c_s\dot{y})$ and one rotational amortization, $(c_r(\dot{x}+\omega y),c_r(\dot{y}-\omega x)$, with the equations:

$$I_0\ddot{x}+(c_s+c_r)\dot{x}+J\omega\dot{y}+c_r\omega y-ax\delta=0,$$

$$I_0\ddot{y} + (b_s + b_r)\dot{y} - J\omega\dot{x} + c_r\omega x - ay\delta = 0,$$

with:
- I_0 the polar inertial momentum,
- J the axial inertial momentum,
- ω the rotation speed,
- $\delta = \pm 1$.

We note that the conservative forces are: $\bar{F}(-a\delta x, a\delta y)$, the amortization forces $(b_s + b_r)\dot{x}, (b_s + b_r)\dot{y}), \bar{G}(J\omega\dot{y}, -J\omega\dot{x})$, $\bar{N}(c_r\omega y, c_r\omega x)$.

From the Theorem of Thomson - Tait - Cetaev [3] it results that the movement will be unstable, being composed by a stable and an unstable movement.

The example of Crandall [3] shows that at the fast speed the movement is stable by using these amortizations, depends on the critical coefficient $r_o\frac{c_s}{c_r}$ and the critical speed of amortization is $\omega_c = \omega_p(1+r)$. On the other side, we notice the presence of the forces $\bar{N}$, which introduce the unstable zones and a stable zone from Merkin, which do not conserve the theorem (T-T-C).

Application 5 The cylindrical bearing with the rotor in the viscous fluid, with the center $C(x, y)$ [8].

The equations of stability of the rotor centre (K, G, N) are [7]:

$$\begin{cases} \ddot{x} + b\dot{x} + \omega^2 x - py = X \\ \ddot{y} + b\dot{y} + \omega^2 y + px = Y. \end{cases} \tag{12}$$

Here we have the forces: $K(\omega^2 x, \omega^2 y), G(b, b), N(-p, p)$, where N represents the aerodynamically forces produced by the rotor in the viscous fluid; the characteristic polynomial is:

$$P(\lambda) = \lambda^4 + 2b\lambda^3 + (2k^2 + b^2)\lambda^2 + 2bk^2\lambda + p^2 + k^4 = 0. \tag{13}$$

If $p = 0$ then the stability domain is in the first quadrant. If $p \neq 0$ then the stability disappears, so the non - potential forces ($p \neq 0$) can make the stability or can extend the stability (outside the first quadrant) (fig.1) [2].

Application 6 The gyroscope with two plans ([1], [2]).

The stability is kept by the horizontal plan with α angle and by the vertical plan with β angle, for the system.

$$\begin{cases} J\ddot{\alpha} + b\dot{\alpha} - H\dot{\beta} - p\beta = 0; (DGN) \\ J\ddot{\beta} + b\dot{\beta} + H\dot{\alpha} + p\alpha = 0; D(b, b); G(H, -H); N(p, -p). \end{cases} \tag{14}$$

Using the Hurwitz criterion of stability we have:

$$\Delta_3 = a_1 a_2 a_3 - a_0^2 a_3^2 - a_1^2 a_4 > 0; \Delta_2 > 0, \Delta_1 > 0, \alpha, \beta, p, H > 0,$$

$$\Delta_3 = 4pJ(H^2 + b^2)(bH - pJ) > 0.$$

For $b = 0$ the system in unstable, $\Delta_3 < 0$ and for $b \neq 0, b > \frac{pJ}{H}$, the system is asymptotically stable.

Application 7 The double pendulum with elastic articulations and a non conservative force (K, N) ([3]-[5]).

The governing equation are:

$$\begin{cases} a_{11}\ddot{\varphi}_1 + a_{12}\ddot{\varphi}_2 + l_1\varphi_1 - l_2\varphi_2 = 0 \\ a_{21}\ddot{\varphi}_1 + a_{22}\ddot{\varphi}_2 - c_1\varphi_1 - c_2\varphi_2 = 0 \end{cases} \tag{15}$$

For the asymptotical stability, the characteristic polynomial $a\lambda^4 + b\lambda^2 + c = 0$ satisfies the conditions: $b > 0, \delta = b^2 - 4ac > 0$.

Application 8 The automobile with automatic decompression [4].

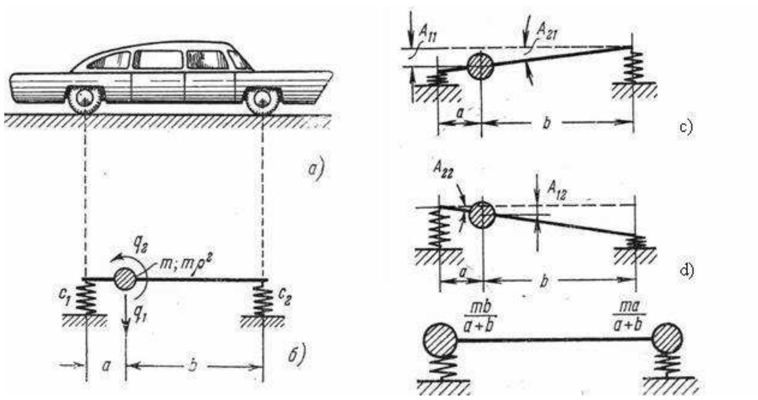

Figure 2: The stability of the automobile

The governing equations are:

$$\begin{aligned} m\ddot{x} + (k_1 + k_2)x + (k_1a - k_2b)y = 0, \\ m\rho^2\ddot{y} + (k_1a - k_2b)x + (k_1a^2 + k_2b^2)y = 0, \end{aligned} \tag{16}$$

where ρ is the inertial radius.

To make the decompression of the two equations and to have a noiseless automobile, we take: $k_1a - k_2b = 0$. This implies that $\rho^2 = \sqrt{ab}$ (Fig. 2).

In the Figures 2c and 2d the movements in caper and gallop are decomposed.

Finally we present the indeterminate coefficients method, building Lyapunov function for systems of forth degree. The characteristic polynomial is:

$$\lambda^4 + d\lambda^3 + c\lambda^2 + b\lambda + a = 0. \tag{17}$$

The H-R criterion for the A.S. of the null solutions gives: $d > 0, c > 0, a > 0, bcd - b^2 - d^2 > 0$.

Next we construct the Lyapunov function (V) for the system of 4th degree. We find some functions of four variables under the quadratic form:

$$V = \sum_{i,j=1}^{4} V_{ij} x_i y_j, W = \sum_{i,j=1}^{4} W_{ij} x_i y_j, \dot{V} = 2W, \tag{18}$$

where V_{ij} are unknown and W_{ij} are known.

Using the method of undetermined coefficients, V is obtained in the system having four equations with the unknowns V_{ij}. In the algebraic linear system we have the determinant D, we can choose $W = 2Dy^2$ and the identification $\dot{V} = 2Dy^2$.

It is noticed that $D = H_1 H_2 H_3 a$, where $H_1 = d > 0$, $h_2 = dc - b > 0$, $H_3 = bcd - a^2 d - b^2$.

The attached the system is:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= u \\ \dot{u} &= -ax - by - cz - du. \end{aligned}$$

Here we identify $\dot{V} = 2Dy^2$, so we have:

$$V = \frac{ac}{2}x^2 + adxy + \frac{c^2 - 2a}{2}y^2 + 2axz + cdyz + \frac{d^2 + c}{2}z^2 + u^2 + cyu + dzu + \frac{bd}{2}y^2 \tag{19}$$

$$\dot{V} = ady^2 - bcy^2 - 2byu - du^2. \tag{20}$$

For the nonlinear system we take the case when the term by is replaced by $\varphi(y), \varphi(0) = 0$:

$$V = E + db\frac{y^2}{2} \tag{21}$$

$$V = E + d\int_0^y \varphi(y)dy. \tag{22}$$

We consider the case of a system with two degrees of freedom, under the matricial form:

$$M\ddot{X} + C\dot{X} + KX = 0, \tag{23}$$

where M is the mass matrix, C is the absorption matrix, K is the potential elastic matrix.

$$
\begin{aligned}
m_1\dot{x}_1 c_{11}\dot{x}_1 + c_12\dot{x}_2 + k_{11}x_1 + k_{12}x_2 = 0 \\
m_2\dot{x}_2 + c_{21}\dot{x}_1 + c_{22}x_2 + k_{21}x_1 + k_{22}x_2 = 0.
\end{aligned} \tag{24}
$$

Passing at the system of fourth degree, we get:

$$
\dot{x} = u, \dot{y} = v. \tag{25}
$$

The characteristic polynomial $p = P(r)$ is:

$$
P(r) = det \begin{vmatrix} m_1r^2 + c_{11}r + k_{11} & c_{12}r + k_{12} \\ c_{21}r + k_{21} & m_2r^2 + c_{22}r + k_{22} \end{vmatrix}. \tag{26}
$$

By developing in series, we obtain the polynomial of 4^{th} degree for which we apply the above theory to find the function V.

Others applications of this kind, regarding the stability study of the dynamical systems and their automatic regulation appeared in [5]-[9].

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