An. Şt. Univ. Ovidius Constanţa Vol. 17(3), 2009, 1–13

SOME NEW FOUR - POINT QUADRATURE FORMULAS

Ana Maria Acu, Arif Rafiq and Florin Sofonea

Abstract

In this paper we present a new family of four-point quadrature formulas of close type. These quadrature formulas can be considered as generalizations of Gauss, Newton, Simpson and Lobatto quadrature formulas for different classes of functions. The optimal quadrature formulas in the sense of minimal errors are obtained. An analysis of error inequalities for different classes of functions is also given.

1 Introduction

In our paper, we intend to introduce a family of four-point quadrature formulas. For this purpose, we consider the following two classes of functions

$$H^{n,p}[a,b]=\left\{ f\in C^{n-1}[a,b], f^{(n-1)}\text{abs. cont.}, \int_a^b \left|f^{(n)}(x)\right|^p dx<\infty\right\},$$
$$1\le p<\infty,$$
$$H^{n,\infty}[a,b]=\left\{ f\in C^{n-1}[a,b], f^{(n-1)}\text{abs. cont.}, \sup_{x\in[a,b]} \left|f^{(n)}(x)\right|<\infty\right\},$$

The norms on these spaces of functions are the usual ones:

$$\left\|f^{(n)}\right\|_p:=\left\{\int_a^b \left|f^{(n)}(x)\right|^p dx\right\}^{1/p}, \ 1\le p<\infty,\ f\in H^{n,p}[a,b],$$
$$\left\|f^{(n)}\right\|_\infty:=\sup_{x\in[a,b]}\left|f^{(n)}(x)\right|,\ f\in H^{n,\infty}[a,b].$$

Key Words: Quadrature rule, Error inequalities, Numerical integration.
Mathematics Subject Classification: 65D30 , 65D32.
Received: April 2009
Accepted: October 2009

In order to compare our results to others obtained by some authors, we recall them. In [3], N. Ujevic obtained the following optimal quadrature formula in the sense of minimal error:

Theorem 1.[3] *Let $I\subset\mathbb{R}$ be an open interval such that $[-1,1]\subset I$ and let $f:I\to\mathbb{R}$ be a twice differentiable function such that $f''\in L_2(1,1)$. Then we have*

$$\int_{-1}^{1} f(t)dt = f(\sqrt{6}-3)+f(3-\sqrt{6})+\mathcal{R}[f], \tag{1}$$

and

$$|\mathcal{R}[f]|\le\sqrt{\frac{98}{5}-8\sqrt{6}}\left\|f^{(2)}\right\|_2\simeq 0.0639\left\|f^{(2)}\right\|_2. \tag{2}$$

In [4], F. Zafar, N.A. Mir presented a family of four-point quadrature formulas, a generalization of Gauss two-point, Newton Simpson and Lobatto four-point quadrature formula for twice-differentiable mapping.

Theorem 2.[4] *Let $I\subset\mathbb{R}$ be an open interval such that $[-1,1]\subset I$ and let $f:I\to\mathbb{R}$ be a twice differentiable function such that $f''(t)$ is bounded and integrable. Then,*

$$\begin{aligned}\int_{-1}^{1} f(t)dt &= \Big[hf(-1)+(1-h)f(-4+4h+2\sqrt{3-6h+4h^2})\\ &+ (1-h)f(4-4h-2\sqrt{3-6h+4h^2})+hf(1)\Big]+\mathcal{R}[f],\end{aligned} \tag{3}$$

where $|\mathcal{R}[f]|\le 2\Delta(h)\left\|f^{(2)}\right\|_\infty$, $h\in\left[0,\frac{1}{2}\right]$, and $\Delta(h)$ is defined as

$$\begin{aligned}\Delta(h) &= \frac{52}{3}h^3-44h^2+\frac{83}{2}h-\frac{83}{6}+8(1-h)^2\sqrt{4h^2-6h+3}\\ &+\frac{2}{3}\left[8h^2-14h+7-4(1-h)\sqrt{4h^2-6h+3}\right]^{\frac{3}{2}}.\end{aligned}$$

2 Main results

Our goal is to give a new family of four-point quadrature rule. The estimation of the remainder term can be obtained and it is better than in other formulas.

Theorem 3. *If $f\in H^{3,2}[-1,1]$ and $h\in\left(-\infty,\frac{1}{3}\right]$, then we have*

$$\int_{-1}^{1} f(t)dt=hf(-1)+(1-h)f\left(-\sqrt{\frac{3h-1}{3(h-1)}}\right)+(1-h)f\left(\sqrt{\frac{3h-1}{3(h-1)}}\right)+hf(1)+\mathcal{R}[f], \tag{4}$$

$$|\mathcal{R}[f]| \leq \sqrt{\Delta_3(h)} \left\| f^{(3)} \right\|_2 \tag{5}$$

where

$$\Delta_3(h) = -\frac{2}{8505(h-1)^2} \Big[-1134h^4 + 2646h^3 - 2097h^2 + 666h - 74 + \sqrt{3}\sqrt{\frac{3h-1}{h-1}} \left(378h^4 - 1008h^3 + 924h^2 - 336h + 42\right) \Big].$$

Proof. We seek a quadrature formula of the type

$$-\int_{-1}^{1} f(t)dt + hf(-1) + (1-h)f(x) + (1-h)f(y) + hf(1) = \int_{-1}^{1} p_3(t) f^{(3)}(t)dt, \tag{6}$$

where $x, y \in [-1, 1]$, $x < y$, $h \in (-\infty, \frac{1}{3}]$. We define

$$p_3(t) = \begin{cases} \frac{1}{6}(t-a_1)^3 + \frac{1}{2}(t-a_2)^2 + a_3, & t \in [-1, x], \\ \frac{1}{6}(t-b_1)^3 + \frac{1}{2}(t-b_2)^2 + b_3, & t \in [x, y], \\ \frac{1}{6}(t-c_1)^3 + \frac{1}{2}(t-c_2)^2 + c_3, & t \in [y, 1], \end{cases}$$

where $a_i, b_i, c_i \in \mathbb{R}$, $i \in \{1, 2, 3\}$ are parameters which have to be determined. Integrating by parts, we obtain

$$\begin{aligned}
&\int_{-1}^{1} p_3(t) f^{(3)}(t)dt = \int_{-1}^{x} \left[\frac{1}{6}(t-a_1)^3 + \frac{1}{2}(t-a_2)^2 + a_3 \right] f^{(3)}(t)dt \\
&+ \int_{x}^{y} \left[\frac{(t-b_1)^3}{6} + \frac{(t-b_2)^2}{2} + b_3 \right] f^{(3)}(t)dt + \int_{y}^{1} \left[\frac{(t-c_1)^3}{6} + \frac{(t-c_2)^2}{2} + c_3 \right] f^{(3)}(t)dt \\
&= \left[\frac{1}{6}(x-a_1)^3 + \frac{1}{2}(x-a_2)^2 + a_3 - \frac{1}{6}(x-b_1)^3 - \frac{1}{2}(x-b_2)^2 - b_3 \right] f''(x) \\
&+ \left[\frac{1}{6}(y-b_1)^3 + \frac{1}{2}(y-b_2)^2 + b_3 - \frac{1}{6}(y-c_1)^3 - \frac{1}{2}(y-c_2)^2 - c_3 \right] f''(y) \\
&- \left[\frac{(-1-a_1)^3}{6} + \frac{(-1-a_2)^2}{2} + a_3 \right] f''(-1) + \left[\frac{(1-c_1)^3}{6} + \frac{(1-c_2)^2}{2} + c_3 \right] f''(1) \\
&+ \left[-\frac{(x-a_1)^2}{2} + a_2 + \frac{(x-b_1)^2}{2} - b_2 \right] f'(x) + \left[-\frac{(y-b_1)^2}{2} + b_2 + \frac{(y-c_1)^2}{2} - c_2 \right] f'(y) \\
&+ \left[\frac{1}{2}(-1-a_1)^2 - 1 - a_2 \right] f'(-1) - \left[\frac{1}{2}(1-c_1)^2 + 1 - c_2 \right] f'(1)
\end{aligned}$$

$$+(-a_1+b_1)f(x)+(-b_1+c_1)f(y)+a_1f(-1)+(2-c_1)f(1)-\int_{-1}^{1}f(t)dt.$$

We require that

$$\frac{1}{6}(x-a_1)^3+\frac{1}{2}(x-a_2)^2+a_3-\frac{1}{6}(x-b_1)^3-\frac{1}{2}(x-b_2)^2-b_3=0,$$
$$\frac{1}{6}(y-b_1)^3+\frac{1}{2}(y-b_2)^2+b_3-\frac{1}{6}(y-c_1)^3-\frac{1}{2}(y-c_2)^2-c_3=0,$$
$$\frac{1}{6}(-1-a_1)^3+\frac{1}{2}(-1-a_2)^2+a_3=0,\ \frac{1}{6}(1-c_1)^3+\frac{1}{2}(1-c_2)^2+c_3=0,$$
$$-\frac{1}{2}(x-a_1)^2+a_2+\frac{1}{2}(x-b_1)^2-b_2=0,\ -\frac{1}{2}(y-b_1)^2+b_2+\frac{1}{2}(y-c_1)^2-c_2=0,$$
$$\frac{1}{2}(-1-a_1)^2-1-a_2=0,\ \frac{1}{2}(1-c_1)^2+1-c_2=0,$$

$-a_1+b_1=1-h,\ -b_1+c_1=1-h,\ a_1=h,\ 2-c_1=h.$

From the above relations we find

$$a_1=h,\ a_2=-\frac{1}{2}+h+\frac{1}{2}h^2,\ a_3=\frac{1}{24}-\frac{1}{3}h^3-\frac{1}{4}h^2-\frac{1}{8}h^4,$$

$$b_1=1,\ b_2=h+(1-h)\sqrt{\frac{3h-1}{3(h-1)}},\ b_3=\frac{1}{3}\left[-3h+2+3(h-1)\sqrt{\frac{3h-1}{3(h-1)}}\right]h,$$

$$c_1=2-h,\ c_2=\frac{3}{2}-h+\frac{1}{2}h^2,\ c_3=\frac{1}{24}-\frac{1}{4}h^2+\frac{1}{3}h^3-\frac{1}{8}h^4,$$

$$x=-\sqrt{\frac{3h-1}{3(h-1)}},\ y=\sqrt{\frac{3h-1}{3(h-1)}}.$$

Therefore we have

$$\int_{-1}^{1}f(t)dt=hf(-1)+(1-h)f\left(-\sqrt{\frac{3h-1}{3(h-1)}}\right)+(1-h)f\left(\sqrt{\frac{3h-1}{3(h-1)}}\right)+hf(1)+\mathcal{R}[f],$$

where

$$\mathcal{R}[f]=-\int_{-1}^{1}p_3(t)f^{(3)}(t)dt, \tag{7}$$

$$p_3(t)=\begin{cases} \frac{1}{6}t^3-\frac{1}{2}t^2h+\frac{1}{2}t^2+\frac{1}{2}t-th+\frac{1}{6}-\frac{1}{2}h,\ t\in\left[-1,-\sqrt{\frac{3h-1}{3(h-1)}}\right], \\ \frac{1}{6}\left(-6h+2\sqrt{3}(h-1)\sqrt{\frac{3h-1}{h-1}}+t^2+3\right)t,\ t\in\left(-\sqrt{\frac{3h-1}{3(h-1)}},\sqrt{\frac{3h-1}{3(h-1)}}\right), \\ \frac{1}{6}t^3-\frac{1}{2}t^2+\frac{1}{2}t^2h+\frac{1}{2}t-th-\frac{1}{6}+\frac{1}{2}h,\ t\in\left[\sqrt{\frac{3h-1}{3(h-1)}},1\right]. \end{cases}$$

The remainder term (7) has the following evaluation

$$|\mathcal{R}[f]|\le\|p_3\|_2\left\|f^{(3)}\right\|_2,$$

but

$$\begin{aligned}\|p_3\|_2^2 &= \int_{-1}^{-\sqrt{\frac{3h-1}{3(h-1)}}}\frac{1}{36}\left[t^3-3t^2h+3t^2+3t-6th+1-3h\right]^2dt \\ &+\int_{-\sqrt{\frac{3h-1}{3(h-1)}}}^{\sqrt{\frac{3h-1}{3(h-1)}}}\frac{1}{36}\left[-6h+2\sqrt{3}\sqrt{\frac{3h-1}{h-1}}h+t^2-2\sqrt{3}\sqrt{\frac{3h-1}{h-1}}+3\right]^2t^2dt \\ &+\int_{\sqrt{\frac{3h-1}{3(h-1)}}}^{1}\frac{1}{36}\left[t^3+3t^2h-3t^2+3t-6th-1+3h\right]^2dt \\ &= -\frac{2}{8505(h-1)^2}\left[-1134h^4+2646h^3-2097h^2+666h-74\right. \\ &+\left.\sqrt{3}\sqrt{\frac{3h-1}{h-1}}\left(378h^4-1008h^3+924h^2-336h+42\right)\right].\end{aligned}$$

From the above relations we obtain the estimation (5).

Remark 1. *For $h=0$ we obtain the Gauss two-point quadrature formula*

$$\int_{-1}^{1}f(t)dt=f\left(-\frac{\sqrt{3}}{3}\right)+f\left(\frac{\sqrt{3}}{3}\right)+\mathcal{R}[f],$$

where

$$|R[f]|\le\sqrt{\frac{148}{8505}-\frac{4}{405}\sqrt{3}}\left\|f^{(3)}\right\|_2\simeq 0.0172\left\|f^{(3)}\right\|_2.$$

Remark 2. *For $h=\frac{1}{6}$ we get Lobatto four-point quadrature formula as follows*

$$\int_{-1}^{1}f(t)dt=\frac{1}{6}\left[f(-1)+5f\left(-\frac{\sqrt{5}}{5}\right)+5f\left(\frac{\sqrt{5}}{5}\right)+f(1)\right]+\mathcal{R}[f],$$

where

$$|\mathcal{R}[f]| \leq \sqrt{-\frac{1}{675}\sqrt{5}+\frac{79}{23625}}\left\|f^{(3)}\right\|_2 \simeq 0.0056\left\|f^{(3)}\right\|_2.$$

Remark 3. *For $h=\dfrac{1}{4}$ we get Newton Simpson′ s rule as follows:*

$$\int_{-1}^{1} f(t)dt = \frac{1}{4}\left[f(-1)+3f\left(-\frac{1}{3}\right)+3f\left(\frac{1}{3}\right)+f(1)\right]+\mathcal{R}[f],$$

where

$$|\mathcal{R}[f]| \leq \sqrt{\frac{11}{153090}}\left\|f^{(3)}\right\|_2 \simeq 0.0085\left\|f^{(3)}\right\|_2.$$

Remark 4. *From the above special cases, (4) may be considered as a generalization of Gauss two-point, Newton Simpson and Lobatto four-point quadrature formula for the class of functions $H^{3,2}[-1,1]$.*

From the class of quadrature formulas (4) we will determine the optimal quadrature formula in the sense of minimal error.

Corollary 1. *If $f \in H^{3,2}[-1,1]$, then one has the following optimal quadrature rule*

$$\int_{-1}^{1} f(t)dt = \frac{2-\sqrt{2}}{3}f(-1)+\frac{\sqrt{2}+1}{3}f\left(1-\sqrt{2}\right)+\frac{\sqrt{2}+1}{3}f\left(\sqrt{2}-1\right)+\frac{2-\sqrt{2}}{3}f(1)+\mathcal{R}[f], \tag{8}$$

$$|\mathcal{R}[f]| \leq \sqrt{\frac{4}{945}\frac{5\sqrt{2}-7}{(1+\sqrt{2})^3}}\left\|f^{(3)}\right\|_2 \simeq 0.0046\left\|f^{(3)}\right\|_2. \tag{9}$$

Proof. We seek h such that $\Delta_3(h) \to min$. For that purpose, we calculate

$$\Delta_3'(h) = -\frac{4}{1215}\cdot\frac{1}{(h-1)^3\sqrt{\frac{3h-1}{h-1}}}\left[\sqrt{3}\left(162h^4-459h^3+441h^2-165h+21\right)\right.$$
$$\left. + \sqrt{\frac{3h-1}{h-1}}\left(-162h^4+513h^3-567h^2+252h-37\right)\right].$$

From the equation $\Delta_3'(h)=0$ we find the solution $h=\dfrac{2-\sqrt{2}}{3}$. We have

$$\Delta_3\left(\frac{2-\sqrt{2}}{3}\right) = \frac{4}{945}\frac{5\sqrt{2}-7}{(1+\sqrt{2})^3}.$$

We conclude that $h = \dfrac{2-\sqrt{2}}{3}$ is the point of global minimum and we obtain the quadrature formula (8) and the estimation of the remainder term (9).

In the following Theorem we obtain the estimation of the remainder term of our quadrature formula for the functions from $H^{2,2}[-1,1]$. Moreover, we will compare this estimation with the result obtained by N. Ujevic in Theorem 1.

Theorem 4. *If $f \in H^{2,2}[-1,1]$ and $h \in (-\infty, \frac{1}{3}]$, then we have*

$$\int_{-1}^{1} f(t)dt = hf(-1) + (1-h)f\left(-\sqrt{\frac{3h-1}{3(h-1)}}\right) + (1-h)f\left(\sqrt{\frac{3h-1}{3(h-1)}}\right) + hf(1) + \mathcal{R}[f], \tag{10}$$

$$|\mathcal{R}[f]| \leq \sqrt{\Delta_2(h)}\left\|f^{(2)}\right\|_2 \tag{11}$$

where

$$\Delta_2(h) = \frac{2}{135(h-1)}\left[-90h^3 + 180h^2 - 120h + 17 + \sqrt{\frac{9h-3}{h-1}}\left(30h^3 - 70h^2 + 50h - 10\right)\right].$$

Proof. We define

$$p_2(t) = \begin{cases} \frac{1}{2}t^2 - th + t + \frac{1}{2} - h, t \in \left[-1, -\sqrt{\frac{3h-1}{3(h-1)}}\right] \\ \frac{1}{6}\left(-6h + 2\sqrt{3}(h-1)\sqrt{\frac{3h-1}{h-1}} + 3t^2 + 3\right), t \in \left(-\sqrt{\frac{3h-1}{3(h-1)}}, \sqrt{\frac{3h-1}{3(h-1)}}\right) \\ \frac{1}{2}t^2 + th - t + \frac{1}{2} - h, t \in \left[\sqrt{\frac{3h-1}{3(h-1)}}, 1\right] \end{cases}$$

Since

$$\int_{-1}^{1} p_2(t)f''(t)dt = \int_{-1}^{1} f(t)dt - \left[hf(-1) + (1-h)f\left(-\sqrt{\frac{3h-1}{3(h-1)}}\right) + (1-h)f\left(\sqrt{\frac{3h-1}{3(h-1)}}\right) + hf(1)\right],$$

we can write

$$\int_{-1}^{1} f(t)dt = hf(-1) + (1-h)f\left(-\sqrt{\frac{3h-1}{3(h-1)}}\right) + (1-h)f\left(\sqrt{\frac{3h-1}{3(h-1)}}\right) + hf(1) + \mathcal{R}[f],$$

where

$$\mathcal{R}[f] = \int_{-1}^{1} p_2(t) f''(t)dt. \tag{12}$$

The remainder term (12) has the following evaluation

$$|\mathcal{R}[f]| \le \|p_2\|_2 \, \|f''\|_2 \, ,$$

but

$$\begin{aligned}
\|p_2\|_2^2 &= \int_{-1}^{-\sqrt{\frac{3h-1}{3(h-1)}}} \left(\frac{1}{2}t^2 - th + t + \frac{1}{2} - h\right)^2 dt \\
&+ \int_{-\sqrt{\frac{3h-1}{3(h-1)}}}^{\sqrt{\frac{3h-1}{3(h-1)}}} \frac{1}{36}\left(-6h + 2\sqrt{3}(h-1)\sqrt{\frac{3h-1}{h-1}} + 3t^2 + 3\right)^2 dt \\
&+ \int_{\sqrt{\frac{3h-1}{3(h-1)}}}^{1} \left(\frac{1}{2}t^2 + th - t + \frac{1}{2} - h\right)^2 dt \\
&= \frac{2}{135(h-1)}\left[-90h^3 + 180h^2 - 120h + 17 + \sqrt{\frac{9h-3}{h-1}}\left(30h^3 - 70h^2 + 50h - 10\right)\right].
\end{aligned}$$

From the above relations we obtain the estimation (11).

Remark 5. *For $h = 0$ we obtain the Gauss two-point quadrature formula*

$$\int_{-1}^{1} f(t)dt = f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right) + \mathcal{R}[f],$$

where

$$|R[f]| \le \sqrt{-\frac{34}{135} + \frac{4}{27}\sqrt{3}} \left\|f^{(2)}\right\|_2 \simeq 0.0689 \left\|f^{(2)}\right\|_2 .$$

Remark 6. *For $h = \frac{1}{6}$ we get Lobatto four-point quadrature formula as follows*

$$\int_{-1}^{1} f(t)dt = \frac{1}{6}\left[f(-1) + 5f\left(-\frac{\sqrt{5}}{5}\right) + 5f\left(\frac{\sqrt{5}}{5}\right) + f(1)\right] + \mathcal{R}[f],$$

where

$$|\mathcal{R}[f]| \le \sqrt{-\frac{11}{135} + \frac{1}{27}\sqrt{5}} \left\|f^{(2)}\right\|_2 \simeq 0.0365 \left\|f^{(2)}\right\|_2 . \tag{13}$$

Remark 7. *For $h = \dfrac{1}{4}$ we get Newton Simpson$'$ s rule as follows:*

$$\int_{-1}^{1} f(t)dt = \frac{1}{4}\left[f(-1) + 3f\left(-\frac{1}{3}\right) + 3f\left(\frac{1}{3}\right) + f(1)\right] + \mathcal{R}[f],$$

where

$$|\mathcal{R}[f]| \le \sqrt{\frac{1}{810}}\left\|f^{(2)}\right\|_2 \simeq 0.0351\left\|f^{(2)}\right\|_2. \tag{14}$$

Remark 8. *From the above special cases, (10) may be considered as a generalization of Gauss two-point, Newton Simpson and Lobatto four-point quadrature formula for the class of functions $H^{2,2}[-1,1]$.*

Remark 9. *We see that the estimation (13) and (14) are better then the estimation obtained by N. Ujevic in Theorem 1.*

Corollary 2. *If $f \in H^{2,2}[-1,1]$, then one has the following optimal quadrature rule*

$$\begin{aligned}\int_{-1}^{1} f(t)dt &= \left(\frac{3}{4}-\frac{\sqrt[3]{9}}{12}-\frac{\sqrt[3]{3}}{4}\right)f(-1)+\left(\frac{1}{4}+\frac{\sqrt[3]{9}}{12}+\frac{\sqrt[3]{3}}{4}\right)f\left(-\sqrt{\frac{-5+\sqrt[3]{9}+3\sqrt[3]{3}}{3+\sqrt[3]{9}+3\sqrt[3]{3}}}\right)\\ &+\left(\frac{1}{4}+\frac{\sqrt[3]{9}}{12}+\frac{\sqrt[3]{3}}{4}\right)f\left(\sqrt{\frac{-5+\sqrt[3]{9}+3\sqrt[3]{3}}{3+\sqrt[3]{9}+3\sqrt[3]{3}}}\right)\left(\frac{3}{4}-\frac{\sqrt[3]{9}}{12}-\frac{\sqrt[3]{3}}{4}\right)f(1)+\mathcal{R}[f], \end{aligned}\tag{15}$$

$$|\mathcal{R}[f]| \le \sqrt{\Delta_2\left(\frac{3}{4}-\frac{\sqrt[3]{9}}{12}-\frac{\sqrt[3]{3}}{4}\right)}\left\|f^{(2)}\right\|_2 \simeq 0.0313\left\|f^{(2)}\right\|_2, \tag{16}$$

where

$$\begin{aligned}\Delta_2\left(\frac{3}{4}-\frac{\sqrt[3]{9}}{12}-\frac{\sqrt[3]{3}}{4}\right) &= \frac{1}{180}\cdot\frac{1}{(3+\sqrt[3]{9}+3\sqrt[3]{3})^2}\Big[-\sqrt[3]{3873}-129-\sqrt[3]{253^2}\\ &+\sqrt{-5+\sqrt[3]{9}+3\sqrt[3]{3}}\sqrt{3+\sqrt[3]{9}+3\sqrt[3]{3}}\cdot\left(35\sqrt[3]{3}+105+25\sqrt[3]{9}\right)\Big].\end{aligned}$$

Proof. We seek h such that $\Delta_2(h) \to min$. For that purpose, we calculate

$$\begin{aligned}\Delta_2'(h) &= \frac{2}{27}\frac{1}{\sqrt{\frac{3h-1}{h-1}}(h-1)^2}\Big[36\sqrt{3}h^3 - 78\sqrt{3}h^2 + 52\sqrt{3}h - 10\sqrt{3}\\ &+ \sqrt{\frac{3h-1}{h-1}}(-36h^3 + 90h^2 - 72h + 17)\Big].\end{aligned}$$

From the equation $\Delta_2'(h)=0$ we find the point of global minimum $h=\frac{3}{4}-\frac{\sqrt[3]{9}}{12}-\frac{\sqrt[3]{3}}{4}$ and we obtain the quadrature formula (15) and the estimation of the remainder term (16).

Remark 10. *In Figure 1. we can see the graphics of functions* $f(h)=\sqrt{\Delta_3(h)}$, $g(h)=\sqrt{\Delta_2(h)}$ *and the points* $(h,\sqrt{\Delta_3(h)})$, $h\in\left\{0,\frac{1}{6},\frac{1}{4},\frac{2-\sqrt{2}}{3}\right\}$ *and* $(h,\sqrt{\Delta_2(h)})$, $h\in\left\{0,\frac{1}{6},\frac{1}{4},\frac{3}{4}-\frac{\sqrt[3]{9}}{12}-\frac{\sqrt[3]{3}}{4}\right\}$.

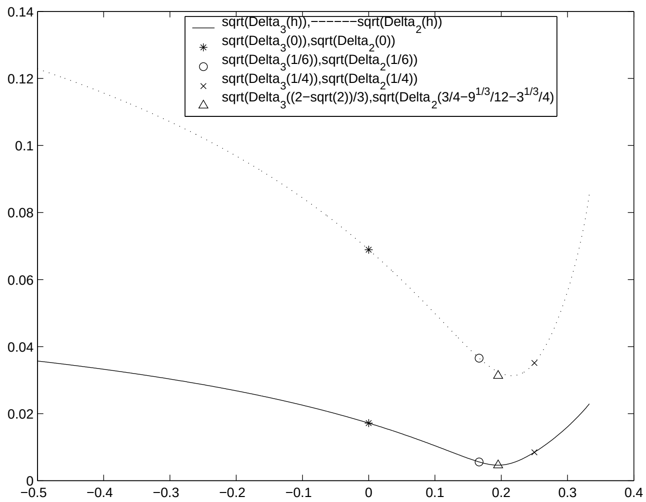

Figure 1.

In the following Theorem we obtain the estimation of the remainder term of our quadrature formula for the functions from $H^{2,\infty}[-1,1]$.

Theorem 5. *If* $f\in H^{2,\infty}[-1,1]$ *and* $h\in\left(-\infty,\frac{1}{3}\right]$*, then we have*

$$\int_{-1}^{1}f(t)dt=hf(-1)+(1-h)f\left(-\sqrt{\frac{3h-1}{3(h-1)}}\right)+(1-h)f\left(\sqrt{\frac{3h-1}{3(h-1)}}\right)+hf(1)+\mathcal{R}[f], \tag{17}$$

$$|\mathcal{R}[f]|\le\Delta(h)\left\|f^{(2)}\right\|_\infty \tag{18}$$

where

$$\Delta(h) = $$

$$= \begin{cases} -\frac{4}{27}\left[3-6h+2\sqrt{\frac{9h-3}{h-1}}(h-1)\right]\sqrt{-9+18h+6\sqrt{\frac{9h-3}{h-1}}(1-h)}, h \leq 0, \\ \frac{8}{3}h^3-\frac{4}{9}\sqrt{-9+18h+6\sqrt{\frac{9h-3}{h-1}}(1-h)}\cdot\left[\frac{2}{3}(h-1)\sqrt{\frac{9h-3}{h-1}}-2h+1\right], h \in \left(0, \frac{1}{4}\right), \\ \frac{8}{3}h^3,\ h \in \left[\frac{1}{4}, \frac{1}{3}\right]. \end{cases}$$

Proof. From the proof of Theorem 4 we have the quadrature formula (17) and the remainder term

$$\mathcal{R}[f] = \int_{-1}^{1} p_2(t) f^{(2)}(t)dt$$

has the following evaluation

$$|\mathcal{R}[f]| \leq \|p_2\|_1 \cdot \left\|f^{(2)}\right\|_\infty,$$

with

$$\|p_2\|_1 = -\frac{4}{27}\left[3-6h+2\sqrt{\frac{9h-3}{h-1}}(h-1)\right]\sqrt{-9+18h+6\sqrt{\frac{9h-3}{h-1}}(1-h)}, h \leq 0,$$

$$\|p_2\|_1 = \frac{8}{3}h^3-\frac{4}{9}\sqrt{-9+18h+6\sqrt{\frac{9h-3}{h-1}}(1-h)}\cdot\left[\frac{2}{3}(h-1)\sqrt{\frac{9h-3}{h-1}}-2h+1\right], h \in \left(0, \frac{1}{4}\right),$$

$$\|p_2\|_1 = \frac{8}{3}h^3,\ h \in \left[\frac{1}{4}, \frac{1}{3}\right].$$

Remark 11. *For $h = 0$ we obtain the Gauss two-point quadrature formula*

$$\int_{-1}^{1} f(t)dt = f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right) + \mathcal{R}[f],$$

where

$$|R[f]| \leq -\frac{4}{9}\sqrt{-9+6\sqrt{3}}\left(-\frac{2}{3}\sqrt{3}+1\right)\left\|f^{(2)}\right\|_\infty \simeq 0.0811\left\|f^{(2)}\right\|_\infty.$$

Remark 12. *For $h = \frac{1}{6}$ we get Lobatto four-point quadrature formula as follows*

$$\int_{-1}^{1} f(t)dt = \frac{1}{6}\left[f(-1) + 5f\left(-\frac{\sqrt{5}}{5}\right) + 5f\left(\frac{\sqrt{5}}{5}\right) + f(1)\right] + \mathcal{R}[f],$$

where

$$|\mathcal{R}[f]| \leq \left(\frac{1}{81} + \frac{4}{27}(\sqrt{5}-2)\sqrt{-6+3\sqrt{5}}\right)\left\|f^{(2)}\right\|_\infty \simeq 0.0418\left\|f^{(2)}\right\|_\infty .$$

Remark 13. *For $h = \dfrac{1}{4}$ we get Newton Simpson′ s rule as follows:*

$$\int_{-1}^{1} f(t)dt = \frac{1}{4}\left[f(-1) + 3f\left(-\frac{1}{3}\right) + 3f\left(\frac{1}{3}\right) + f(1)\right] + \mathcal{R}[f],$$

where

$$|\mathcal{R}[f]| \leq \frac{1}{24}\left\|f^{(2)}\right\|_\infty \simeq 0.0417\left\|f^{(2)}\right\|_\infty .$$

Remark 14. *From the graphics of functions $f(h) = \pm\sqrt{\dfrac{3h-1}{3(h-1)}}$, $h \leq \dfrac{1}{3}$, $g(h) = \pm(4-4h-2\sqrt{3-6h+4h^2})$, $h \in \left[0, \dfrac{1}{2}\right]$, we see that the set of values of nodes from our quadrature formula include the set of values of nodes from quadrature formula (3) obtained by F. Zafar and N.A. Mir.*

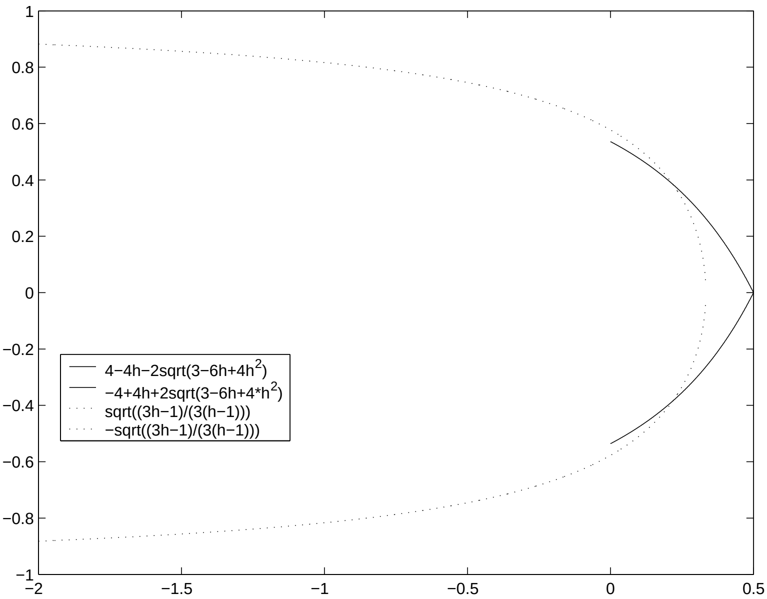

Figure 2.

References

[1] A. M. Acu, A. Baboş, *An error analysis for a quadrature formula*, The 14-the International Conference The Knowledge-based organization Technical Sciences Computer Science, Modelling & Simulation and E-learning Technologies. Physics, Mathematics and Chemistry. Conference Proceedings 8, Sibiu, 27-29 November, 2008, ISSN: 1843-6722, 290-298

[2] A. Branga, *Natural splines of Birkhoff type and optimal approximation*, J. Comput. Analysis Appl., 7(2005), no. 1, 81-88.

[3] N. Ujevic, *Error inequalities for a quadrature formula of open type*, Rev. Colombiana de Mat., 37(2003), 93-105.

[4] F. Zafar, N.A.Mir, *Some generalized error inequalities and applications*, Journal of Inequalities and Applications, Volume 2008, Article ID 845934, 15 pages.

Ana Maria Acu and Florin Sofonea
Lucian Blaga" University, Department of Mathematics
E-mail: acuana77@yahoo.com, sofoneaflorin@yahoo.com

Arif Rafiq
COMSATS Institute of Information Technology,
Department of Mathematics,
Defense Road, Off Raiwind Road, Lohore-Pakistan e-mail:
arafiq@comsats.edu.pk