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EQUILIBRIA OF FREE ABSTRACT FUZZY ECONOMIES

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Abstract

In this paper, we introduce the concept of free abstract fuzzy economy and, using Wu's existence theorem of maximal elements for lower semicontinuous correspondences [26] and Kim and Lee's existence theorems of best proximity pairs [14], we prove the existence of fuzzy equilibrium pairs for free abstract fuzzy economies first with upper semicontinuous and then with lower semicontinuous constraint correspondences and $Q_\theta-$ majorized preference correspondences.

1 Introduction

In the last years, the classical model of abstract economy was generalized by many authors. This model was proposed in his pioneering works by Debreu [5] or later by Shafer and Sonnenschein [22], Yannelis and Prahbakar [27]. For example, Vind [25] defined the social system with coordination, Yuan [28] proposed the model of the general abstract economy. Kim and Tan [15] defined the generalized abstract economies. Also Kim [9] obtained a generalization of the quasi fixed-point theorem due to Lefebvre [17], and as an application, he proved an existence theorem of equilibrium for a generalized quasi-game with infinite number of agents.

In [14] Kim and Lee defined the free abstract economy and proved existence theorems of best proximity pairs and equilibrium pairs. Their theorems for best proximity pairs generalizes the previous results due to Srinivasan and

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Veeramani [23], [24], Sehgal and Singh [21], Reich [20]. Their existence theorems of equilibrium pairs refer to free abstract economies with upper semicontinuous constraint correspondences and preference correspondences with open lower sections. Using Park's fixed point theorem for acyclic factorizable multifunctions, Kim [10] generalized Kim and Lee's results.

L. Zadeh initiated the theory of fuzzy sets [29] as a framework for phenomena which can not be characterized precisely. In [11] the authors introduced the concept of a fuzzy game and proved the existence of equlibrium for 1-person fuzzy game. Also the existence of equilibrium points of fuzzy games was studied in [3], [4], [11],[12], [13]. Fixed point theorems for fuzzy mappings were proved in [2], [6].

In this paper we introduce the concept of free abstract fuzzy economy and use Kim and Lee's existence theorems of best proximity pairs [14] and Wu's existence theorem of maximal elements for lower semicontinuous correspondences [26] to prove the existence of fuzzy equilibrium pairs for free abstract fuzzy economies first with upper semicontinuous and then with lower semicontinuous constraint correspondences and $Q_\theta-$majorized preference correspondences. The $Q_\theta-$majorized correspondences were introduced by Liu and Cai in [18].

The paper is organized in the following way: Section 2 contains preliminaries and notation. The equilibrium pair theorems are stated in Section 3.

2 Preliminaries and notation

Throughout this paper, we shall use the following notation and definitions:

Let A be a subset of a topological space X.

1. $\mathcal{F}(A)$ denotes the family of all non-empty finite subsets of A.
2. 2^A denotes the family of all subsets of A.
3. cl A denotes the closure of A in X.
4. If A is a subset of a vector space, co$A$ denotes the convex hull of A.
5. If F, $T : A \to 2^X$ are correspondences, then co$T$, cl T, $T \cap F : A \to 2^X$ are correspondences defined by $(\text{co}T)(x) = \text{co}T(x)$, $(\text{cl}T)(x) = \text{cl}T(x)$ and $(T \cap F)(x) = T(x) \cap F(x)$ for each $x \in A$, respectively.
6. The graph of $T : X \to 2^Y$ is the set $\text{Gr}(T) = \{(x, y) \in X \times Y \mid y \in T(x)\}$

7. The correspondence $\overline{T}$ is defined by $\overline{T}(x) = \{y \in Y : (x, y) \in \mathrm{cl}_{X\times Y}\mathrm{Gr}(T)\}$ (the set $\mathrm{cl}_{X\times Y}\mathrm{Gr}(T)$ is called the adherence of the graph of T).

It is easy to see that $\mathrm{cl}T(x) \subset \overline{T}(x)$ for each $x \in X$.

Definition 1. Let X, Y be topological spaces and $T : X \to 2^Y$ be a correspondence

1. T is said to be *upper semicontinuous* if for each $x \in X$ and each open set V in Y with $T(x) \subset V$, there exists an open neighborhood U of x in X such that $T(y) \subset V$ for each $y \in U$.
2. T is said to be *lower semicontinuous* (shortly l.s.c) if for each x$\in X$ and each open set V in Y with $T(x) \cap V \neq \emptyset$, there exists an open neighborhood U of x in X such that $T(y) \cap V \neq \emptyset$ for each $y \in U$.
3. T is said to have *open lower sections* if $T^{-1}(y) := \{x \in X : y \in T(x)\}$ is open in X for each $y \in Y$.

Lemma 1. [28]. *Let X and Y be two topological spaces and let A be a closed subset of X. Suppose $F_1 : X \to 2^Y$, $F_2 : X \to 2^Y$ are lower semicontinuous such that $F_2(x) \subset F_1(x)$ for all $x \in A$. Then the correspondence $F : X \to 2^Y$ defined by*

$$F(x) = \begin{cases} F_1(x), & \text{if } x \notin A, \\ F_2(x), & \text{if } x \in A \end{cases}$$

is also lower semicontinuous.

Definition 2. [18] Let X be a topological space and Y be a non-empty subset of a vector space E, $\theta : X \to E$ be a mapping and $T : X \to 2^Y$ be a correspondence.

1. T is said to be *of class Q_θ* (or Q) if
   (a) for each $x \in X$, $\theta(x) \notin \mathrm{cl}T(x)$ and
   (b) T is lower semicontinuous with open and convex values in Y;
2. A correspondence $T_x : X \to 2^Y$ is said to be a *Q_θ-majorant of T at x* if there exists an open neighborhood $N(x)$ of x such that
   (a) For each $z \in N(x)$, $T(z) \subset T_x(z)$ and $\theta(z) \notin \mathrm{cl}T_x(z)$
   (b) T_x is l.s.c. with open and convex values;

3. T is said to be Q_θ-*majorized* if for each $x \in X$ with $T(x) \neq \emptyset$ there exists a Q_θ-majorant T_x of T at x.

Theorem 1. [18] Let X be regular paracompact topological vector space and Y be a nonempty subset of a vector space E. Let $\theta : X \to E$ and $T : X \to 2^Y \setminus \{\emptyset\}$ be Q_θ-majorized. Then there exists a correspondence $\varphi : X \to 2^Y$ of class Q_θ such that $T(x) \subset \varphi(x)$ for each $x \in X$.

Definition 3. Let X, Y be topological spaces and $T : X \to 2^Y$ be a correspondence. An element $x \in X$ is named *maximal element* for T if $T(x) = \Phi$.

For each $i \in I$, let X_i be a nonempty subset of a topological space E_i and $T_i : X := \prod\limits_{i \in I} X_i \to 2^{Y_i}$ a correspondence. Then a point $x \in X$ is called a maximal element for the family of correspondences $\{T_i\}_{i \in I}$ if $T_i(x) = \emptyset$ for all $i \in I$.

Notation. Let X and Y be any two subsets of a normed space E with a norm $\| \cdot \|$, and the metric $d(x, y)$ is induced by the norm.We use the following notation:

Prox$(X, Y) := \{(x, y) \in X \times Y : d(x, y) = d(X, Y) = \inf\{d(x, y) : x \in X, y \in Y\}\}$;

$X_0 := \{x \in X : d(x, y) = d(X, Y)$ for some $y \in Y\}$;

$Y_0 := \{y \in Y : d(x, y) = d(X, Y)$ for some $x \in X\}$.

If X and Y are non-empty compact and convex subsets of a normed linear space, then it is easy to see that X_0 and Y_0 are both non-empty compact and convex.

Let I be a finite (or an infinite) index set. For each $i \in I$, let X and Y_i be nonempty subsets of a normed space E with a norm $\| \cdot \|$, and the metric $d(x, y)$ is induced by the norm. Then, we can use the following notation: for each $i \in I$,

$X^0 := \{x \in X :$ for each $i \in I$, $\exists\, y_i \in Y_i$ such that

$$d(x, y_i) = d(X, Y_i) = \inf\{d(x, y) : x \in X, y \in Y_i\}\};$$

$Y_i^0 := \{y \in Y_i :$ there exists $x \in X$ such that $d(x, y) = d(X, Y_i)\}$.

When $\mid I \mid = 1$, it is easy to see that $X_0 = X^0$ and $Y_0 = Y_i^0$.

Notation. Let E and F be two Hausdorff topological vector spaces and $X \subset E$, $Y \subset F$ be two nonempty convex subsets. We denote by $\mathcal{F}(Y)$ the collection of fuzzy sets on Y. A mapping from X into $\mathcal{F}(Y)$ is called a fuzzy mapping. If $F : X \to \mathcal{F}(Y)$ is a fuzzy mapping, then for each $x \in X$, $F(x)$ (denoted by F_x in this sequel) is a fuzzy set in $\mathcal{F}(Y)$ and $F_x(y)$ is the degree of membership of point y in F_x.

A fuzzy mapping $F : X \to \mathcal{F}(Y)$ is called convex, if for each $x \in X$, the fuzzy set F_x on Y is a fuzzy convex set, i.e., for any $y_1,y_2 \in Y$, $t \in [0,1]$, $F_x(ty_1 + (1-t)y_2) \geq \min\{F_x(y_1), F_x(y_2)\}$.

In the sequel, we denote by

$(A)_q = \{y \in Y : A(y) \geq q\}$, $q \in [0,1]$ the q-cut set of $A \in \mathcal{F}(Y)$.

3 The existence of equilibrium pairs for free abstract economies

Let I be a nonempty set (the set of agents). For each $i \in I$, let X_i be a non-empty set of manufacturing commodities, and Y_i be a non-empty set of selling commodities. Define $X := \prod_{i\in I} X_i$; let $A_i : X \to \mathcal{F}(Y_i)$ be the fuzzy constraint correspondence, $P_i : Y := \prod_{i\in I} Y_i \to \mathcal{F}(Y_i)$ the fuzzy preference correspondence, $a_i : X \to (0,1]$ fuzzy constraint function and $p_i : Y \to (0,1]$ fuzzy preference function. We consider that X_i and Y_i are non-empty subsets of a normed linear space E.

Definition 4. A *free abstract fuzzy economy* is defined as an ordered family $\Gamma = (X_i, Y_i, A_i, P_i, a_i, p_i)_{i\in I}$.

Definition 5. A *fuzzy equilibrium* pair for Γ is defined as a pair of points $(\overline{x}, \overline{y}) \in X \times Y$ such that for each $i \in I$, $\overline{y_i} \in (A_{i\overline{x}})_{a_i(\overline{x})}$ with $d(\overline{x_i}, \overline{y_i}) = d(X_i, Y_i)$ and $(A_{i\overline{x}})_{a_i(\overline{x})} \cap (P_{i\overline{x}})_{p_i(\overline{x})} = \emptyset$, where $(A_{i\overline{x}})_{a_i(\overline{x})} = \{z \in Y_i : A_{i\overline{x}}(z) \geq a_i(\overline{x})\}$ and $(P_{i\overline{x}})_{p_i(\overline{x})} = \{z \in Y_i : P_{i\overline{x}}(z) \geq p_i(\overline{x})\}$.

If $A_i, P_i : X \to 2^{Y_i}$ are classical correspondences then we get the definition of free abstract economy and equilibrium pair defined by W.K. Kim and K. H. Lee in [14].

Whenever $X_i = X$ for each $i \in I$, for the simplicity, we may assume $A_i : X \to \mathcal{F}(Y_i)$ instead of $A_i : \prod_{i\in I} X_i \to \mathcal{F}(Y_i)$ for the free abstract fuzzy economy $\Gamma = (X, Y_i, A_i, P_i, a_i, p_i)_{i\in I}$ and equilibrium pair. In particular, when $I = \{1, 2...n\}$, we may call Γ a free n-person fuzzy game.

The economic interpretation of an *equilibrium* pair for Γ is based on the requirement that for each $i \in I$, minimize the travelling cost $d(x_i, y_i)$, and also, maximize the preference P_{i_y} on the constraint set A_{i_y}. Therefore, it is contemplated to find a pair of points $(\overline{x}, \overline{y}) \in X \times Y$ such that for each $i \in I$, $\overline{y_i} \in (A_{i\overline{x}})_{a_i(\overline{x})}$ and $(A_{i\overline{x}})_{a_i(\overline{x})} \cap (P_{i\overline{x}})_{p_i(\overline{x})} = \emptyset$ and $\| \overline{x_i} - \overline{y_i} \| = d(X_i, Y_i)$, where $d(X_i, Y_i) = \inf\{\| \overline{x_i} - \overline{y_i} \| | \, x_i \in X_i, \; y_i \in Y_i\}$.

When in adition $X_i = Y_i$ and $A_i, P_i : X \to 2^{Y_i}$ are classical correspondences for each $i \in I$, then the previous definitions can be reduced to the standard definitions of equilibrium theory in economics due to Debreu [5], Shafer and Sonnenshein [22] or Yannelis and Prabhakar [27].

To prove our equilibrium theorems we need the following results.

Definition 6 [14] Let X and Y be two non-empty subsets of a normed linear space E, and let $T : X \to 2^Y$ be a correspondence. Then the pair $(\overline{x}, T(\overline{x})$ is called the *best proximity pair* [14] for T if $d(\overline{x}, T(\overline{x})) = d(\overline{x}, \overline{y}) = d(X, Y)$ for some $\overline{y} \in T(\overline{x})$. Then the best proximity pair theorem seeks an appropriate solution which is optimal. In fact, the best proximity pair theorem analyzes the conditions which the problem of minimizing the real-valued function $x \to d(x, T(x))$ has a solution.

W. K. Kim and K. H. Lee gave [14] the following theorem of existence of best proximity pairs.

Theorem 2. For each $i \in I = \{1, ...n\}$, let X and Y be non-empty compact and convex subsets of a normed linear space E, and let $T_i : X \to 2^{Y_i}$ be an upper semicontinuous correspondence in X^0 such that $T_i(x)$ is nonempty closed and convex subset of Y_i for each $x \in X$. Assume that $T_i(x) \cap Y_i^0 \neq \emptyset$ for each $x \in X^0$.

Then there exists a system of best proximity pairs $\{\overline{x_i}\} \times T_i(\overline{x_i}) \subseteq X \times Y_i$, *i.e.*, for each $i \in I$, $d(\overline{x_i}, T(\overline{x_i})) = d(X, Y_i)$.

Definition 7 [14]. The set $\mathcal{A}_x = \{y \in Y : y \in A(x)$ and $d(x, y) = d(X, Y)\}$ is named *the best proximity set* of the correspondence $A : X \to 2^Y$ at x.

In general, $\mathcal{A}_x$ might be an empty set. If $(x, A(x))$ is a proximity pair for A and $A(x)$ is compact, then $\mathcal{A}_x$ must be non-empty.

Theorem 3 is an existence theorem for maximal elements that is Theorem 7 in [17].

Theorem 3. [26] *Let $\Gamma = (X_i, P_i)_{i \in I}$ be a qualitative game where I is an index set such that for each $i \in I$, the following conditions hold:*

1) X_i is a nonempty convex compact metrizable subset of a Hausdorff locally convex topological vector space E and $X := \prod_{i \in I} X_i$,

2) $P_i : X \to 2^{X_i}$ is lower semi-continuous;

4) *for each $x \in X$, $x_i \notin \text{clco} P_i(x)$*

Then there exists a point $\overline{x} \in X$ such that $P_i(\overline{x}) = \emptyset$ for all $i \in I$, i.e. $\overline{x}$ is a maximal element of Γ.

We state some new equilibrium existence theorems for free abstract fuzzy economies with a finite set of players.

Theorem 4 is an existence theorem of pair equilibrium for a free n person fuzzy game with upper semi-continuous constraint correspondences and Q_θ-majorized preference correspondences.

Theorem 4. *Let $\Gamma = (X, Y_i, A_i, P_i, a_i, p_i)_{i\in I}$ be a free n-person fuzzy game such that for each $i \in I = \{1, 2...n\}$:*

(1) *X and Y_i are non-empty compact and convex subsets of normed linear space E;*

(2) *$A_i : X \to \mathcal{F}(Y_i)$ is such that $x \to (A_{i_x})_{a_i(x)} : X \to 2^{Y_i}$ is upper semicontinuous in X^0, $(A_{i_x})_{a_i(x)}$ is a nonempty, closed convex subset of Y_i, $(A_{i_x})_{a_i(x)} \cap Y_i^0 \neq \emptyset$* for each $x \in X^0$ *for each $x \in X$.*

(3) *$P_i : Y := \prod_{i\in I} Y_i \to \mathcal{F}(Y_i)$ is such that $y \to (P_{i_y})_{p_i(y)} : Y \to 2^{Y_i}$* is *$Q_{\pi_i}-$majorized*;

(4) $(P_{i_y})_{p_i(y)}$ *is nonempty for each $y \in Y$*;

Then there exists a fuzzy equilibrium pair of points $(\overline{x}, \overline{y}) \in X \times Y$ such that for each $i \in I$, $\overline{y_i} \in (A_{i_{\overline{x}}})_{a_i(\overline{x})}$ with $d(\overline{x_i}, \overline{y_i}) = d(X_i, Y_i)$ and $(A_{i_{\overline{x}}})_{a_i(\overline{x})} \cap (P_{i_{\overline{y}}})_{p_i(\overline{y})} = \emptyset$.

Proof. Since $x \to (A_{i_x})_{a_i(x)}$ satisfies the whole assumption of Theorem 2 for each $i \in I$, there exists a point $\overline{x} \in X$ satisfying the system of best proximity pairs, i.e. $\{\overline{x}\} \times (A_{i_{\overline{x}}})_{a_i(\overline{x})} \subseteq X \times Y_i$ such that $d(\overline{x}, (A_{i_{\overline{x}}})_{a_i(\overline{x})}) = d(X, Y_i)$ for each $i \in I$. Let $\mathcal{A}_i := \left\{y_i \in (A_{i_{\overline{x}}})_{a_i(\overline{x})} \ / \ d(\overline{x}, y_i) = d(X, Y_i)\right\}$ the non-empty best proximity set of the correspondence $x \to (A_{i_x})_{a_i(x)}$. The set $\mathcal{A}_i$ is nonempty, closed and convex.

Since $y \to (P_{i_y})_{p_i(y)}$ is Q_{π_i}-majorized for each $i \in I$, by Theorem 1, we have that there exists a correspondence $\varphi_i : Y \to 2^{Y_i}$ of class Q_{π_i} such that $(P_{i_y})_{p_i(y)} \subset \varphi_i(y)$ for each $y \in Y$. Then, φ_i is lower semicontinuous with open, convex values and $\pi_i(y) \notin \mathrm{cl}\varphi_i(y)$ for each $y \in Y$.

For each $i \in I$ define a correspondence

$\Phi_i : Y \to 2^{Y_i}$ by

$$\Phi_i(y) := \begin{cases} \varphi_i(y), & \text{if } y_i \notin \mathcal{A}_i, \\ (A_{i_{\overline{x}}})_{a_i(\overline{x})} \cap \varphi_i(y), & \text{if } y_i \in \mathcal{A}_i. \end{cases}$$

By Lemma 1, Φ_i is lower semicontinuous, has convex values, and $\pi_i(y) \notin cl(\Phi_i(y))$. By applying Theorem 3 to $(Y_i, \Phi_i)_{i\in I}$, there exists a maximal element $\overline{y} \in Y$ such that $\Phi_i(\overline{y}) = \emptyset$ for each $i \in I$. For each $y \in Y$ with $y_i \notin \mathcal{A}_i$, $\Phi_i(y)$ is a non-empty subset of Y_i because $(P_{i_y})_{p_i(y)} \neq \emptyset$. We have that $\overline{y}_i \in \mathcal{A}_i$ and $(A_{i_{\overline{x}}})_{a_i(\overline{x})} \cap \varphi_i(\overline{y}) = \emptyset$. Since $(P_{i_{\overline{y}}})_{p_i(\overline{y})} \subset \varphi_i(\overline{y})$, it follows that $(A_{i_{\overline{x}}})_{a_i(\overline{x})} \cap (P_{i_{\overline{y}}})_{p_i(\overline{y})} = \emptyset$. Hence, $\overline{y}_i \in \mathcal{A}_i$, i.e. $\overline{y}_i \in (A_{i_{\overline{x}}})_{a_i(\overline{x})}$ and $d(\overline{x}, \overline{y}_i) = d(X, Y_i)$ for each $i \in I$. Then $(\overline{x}, \overline{y})$ is a fuzzy equilibrium pair for Γ. □

Corrolary 1 is an existence result of pair equilibrium for a free n person fuzzy game with corespondences $x \to (P_{i_x})_{p_i(x)}$ being lower semicontinuous.

Corolarry 1. *Let $\Gamma = (X, Y_i, A_i, P_i, a_i, p_i)_{i\in I}$ be a free n-person fuzzy game such that for each $i \in I = \{1, 2...n\}$:*

(1) *X and Y_i are non-empty compact and convex subsets of normed linear space E;*

(2) *$A_i : X \to \mathcal{F}(Y_i)$ is such that $x \to (A_{i_x})_{a_i(x)} : X \to 2^{Y_i}$ is upper semicontinuous in X^0, $(A_{i_x})_{a_i(x)}$ is a nonempty, closed convex subset of Y_i, $(A_{i_x})_{a_i(x)} \cap Y_i^0 \neq \emptyset$* for each *$x \in X^0$ for each $x \in X$.*

(3) *$P_i : Y := \prod_{i \in I} Y_i \to \mathcal{F}(Y_i)$ is such that $y \to (P_{i_y})_{p_i(y)} : Y \to 2^{Y_i}$ is lower semicontinuous with nonempty open convex values and $y_i \notin (P_{i_y})_{p_i(y)}$ for each $y \in Y$;*

Then there exists a fuzzy equilibrium pair of points $(\overline{x}, \overline{y}) \in X \times Y$ such that for each $i \in I$, $\overline{y_i} \in (A_{i_{\overline{x}}})_{a_i(\overline{x})}$ with $d(\overline{x_i}, \overline{y_i}) = d(X_i, Y_i)$ and $(A_{i_{\overline{x}}})_{a_i(\overline{x})} \cap (P_{i_{\overline{y}}})_{p_i(\overline{y})} = \emptyset$.

The second corollary is an equilibrium existence result for a n person fuzzy game.

Corolarry 2. *Let $\Gamma = (X_i, A_i, P_i)_{i \in I}$ be a n-person fuzzy game such that for each $i \in I = \{1, 2...n\}$:*

(1) *X_i is non-empty compact and convex subsets of normed linear space E;*

(2) *$A_i : X := \prod_{i \in I} X_i \to \mathcal{F}(X_i)$ is such that $x \to (A_{i_x})_{a_i(x)} : X \to 2^{X_i}$* is *upper semicontinuous and each $(A_{i_x})_{a_i(x)}$ is a nonempty closed convex subset of X_i;*

(3) *$P_i : X \to \mathcal{F}(X_i)$ is such that $x \to (P_{i_x})_{p_i(x)} : X \to 2^{X_i}$* is *$Q_{\pi_i}-$majorized;*

(4) *$(P_{i_x})_{p_i(x)}$ is nonempty for each $x \in X$;*

Then there exists a fuzzy equilibrium pair $(\overline{x}, \overline{y}) \in X$ such that for each $i \in I$, $\overline{y_i} \in (A_{i_{\overline{x}}})_{a_i(\overline{x})}$ and $(A_{i_{\overline{x}}})_{a_i(\overline{x})} \cap (P_{i_{\overline{y}}})_{p_i(\overline{y})} = \emptyset$.

Theorem 5 is an existence theorem of pair equilibrium for a free n person fuzzy game with lower semi-continuous constraint correspondences and Q_θ-majorized preference correspondences.

Theorem 5. *Let $\Gamma = (X, Y_i, A_i, P_i, a_i, p_i)_{i \in I}$ be a free n-person fuzzy game such that for each $i \in I = \{1, 2...n\}$:*

(1) *X and Y_i are non-empty compact and convex subsets of normed linear space E;*

(2) *$A_i : X \to \mathcal{F}(Y_i)$ is such that $x \to (A_{i_x})_{a_i(x)} : X \to 2^{Y_i}$ is lower semicontinuous in X^0 such that each $(A_{i_x})_{a_i(x)}$ is a nonempty, closed convex subset of Y_i and $(A_{i_x})_{a_i(x)} \subset Y_i^0 \neq \emptyset$* for each *$x \in X^0$ for each $x \in X$.*

(3) *$P_i : Y := \prod_{i \in I} Y_i \to \mathcal{F}(Y_i)$ is such that $y \to (P_{i_y})_{p_i(y)} : Y \to 2^{Y_i}$* is *$Q_{\pi_i}-$majorized;*

(4) *$(P_{i_y})_{p_i(y)}$ is nonempty for each $y \in Y$;*

Then there exists a fuzzy equilibrium pair of points $(\overline{x}, \overline{y}) \in X \times Y$ *such that for each* $i \in I$, $\overline{y_i} \in (A_{i_{\overline{x}}})_{a_i(\overline{x})}$ *with* $d(\overline{x_i}, \overline{y_i}) = d(X_i, Y_i)$ and $(A_{i_{\overline{x}}})_{a_i(\overline{x})} \cap (P_{i_{\overline{y}}})_{p_i(\overline{y})} = \emptyset$.

Proof. By Theorem 1.1 in Michael [19], for each $i \in I$, there exists an upper semicontinuous correspondence $H_i : X \to 2^{Y_i}$ with nonempty values such that $H_i(x) \subset (A_{i_x})_{a_i(x)}$ for all $x \in X$. Let be $S_i(x) = \overline{co}H_i(x) \subset (A_{i_x})_{a_i(x)}$. The correspondence S_i satisfies the hypothesis of Theorem 2, then we get a best proximity pair $\{\overline{x}\} \times S_i(\overline{x}) \subseteq X \times Y_i$ for S_i, i.e. $d(\overline{x}, S_i(\overline{x}) = d(X, Y_i)$. Let $\mathcal{S}_i := \{y_i \in S_i(\overline{x}) \ / \ d(\overline{x}, y_i) = d(X, Y_i)\}$ the non-empty best proximity set of the correspondence S_i. The set $\mathcal{S}_i$ is nonempty, closed and convex.

Since $y \to (P_{i_y})_{p_i(y)}$ is Q_{π_i}-majorized, by Theorem 1, we have that there exists a correspondence $\varphi_i : Y \to 2^{Y_i}$ of class Q_{π_i} such that $(P_{i_y})_{p_i(y)} \subset \varphi_i(y)$ for each $y \in Y$. Then, φ_i is lower semicontinuous with open, convex values and $\pi_i(y) \notin \text{cl}\varphi_i(y)$ for each $y \in Y$.

For each $i \in I$ define the correspondence $\Phi_i : Y \to 2^{Y_i}$ by

$$\Phi_i(y) := \begin{cases} \varphi_i(y), & \text{if } y_i \notin \mathcal{S}_i, \\ (A_{i_{\overline{x}}})_{a_i(\overline{x})} \cap \varphi_i(y), & \text{if } y_i \in \mathcal{S}_i. \end{cases}$$

By Lemma 1, Φ_i is lower semicontinuous, and also have convex values, and $\pi_i(y) \notin \text{cl}\Phi_i(y)$ for each $y \in Y$. By applying Theorem 3 to $(Y_i, \Phi_i)_{i \in I}$, there exists a maximal element $\overline{y} \in Y$ such that $\Phi_i(\overline{y}) = \emptyset$ for all $i \in I$. Since $(P_{i_y})_{p_i(y)} \neq \emptyset$, it follows that for each $y \in Y$ with $y_i \notin \mathcal{S}_i$, $\Phi_i(y)$ is a non-empty subset of Y_i for each $i \in I$. We have $\overline{y}_i \in \mathcal{S}_i$ and $(A_{i_{\overline{x}}})_{b_i(\overline{x})} \cap \varphi_i(\overline{y}) = \emptyset$. Since $(P_{i_{\overline{y}}})_{p_i(\overline{y})} \subset \varphi_i(\overline{y})$ we have that $(A_{i_{\overline{x}}})_{a_i(\overline{x})} \cap (P_{i_{\overline{y}}})_{p_i(\overline{y})} = \emptyset$. Hence, $\overline{y}_i \in S_i(\overline{x}) \subset (A_{i_{\overline{x}}})_{a_i(\overline{x})}$ such that $d(\overline{x}, \overline{y}_i) = d(X, Y_i)$ for each $i \in I$. Then $(\overline{x}, \overline{y})$ is a fuzzy equilibrium pair for Γ. $\square$

If the corespondences $x \to (P_{i_x})_{p_i(x)}$ *are lower semicontinuous, we get the following corrolary.*

Corrolary 3. *Let* $\Gamma = (X, Y_i, A_i, P_i, a_i, p_i)_{i \in I}$ *be a free n-person fuzzy game such that for each* $i \in I = \{1, 2...n\}$:

(1) X *and* Y_i *are non-empty compact and convex subsets of normed linear space E;*

(2) $A_i : X \to \mathcal{F}(Y_i)$ *is such that* $x \to (A_{i_x})_{a_i(x)} : X \to 2^{Y_i}$ *is lower semicontinuous in* X^0 *such that each* $(A_{i_x})_{a_i(x)}$ *is a nonempty, closed convex subset of* Y_i *and* $(A_{i_x})_{a_i(x)} \subset Y_i^0 \neq \emptyset$ for each $x \in X^0$ *for each* $x \in X$.

(3) $P_i : Y := \prod_{i \in I} Y_i \to \mathcal{F}(Y_i)$ *is such that* $y \to (P_{i_y})_{p_i(y)} : Y \to 2^{Y_i}$ *is lower semicontinuous with nonempty open convex values and* $y_i \notin (P_{i_y})_{p_i(y)}$ *for each* $y \in Y$;

Then there exists a fuzzy equilibrium pair of points $(\overline{x}, \overline{y}) \in X \times Y$ *such that for each* $i \in I$, $\overline{y_i} \in (A_{i_{\overline{x}}})_{a_i(\overline{x})}$ *with* $d(\overline{x_i}, \overline{y_i}) = d(X_i, Y_i)$ and $(A_{i_{\overline{x}}})_{a_i(\overline{x})} \cap (P_{i_{\overline{y}}})_{p_i(\overline{y})} = \emptyset$.

Corollary 4 is an equilibrium existence result for a n person fuzzy game.

Corrolary 4 *Let* $\Gamma = (X_i, A_i, P_i, a_i, p_i)_{i\in I}$ *be an n-person fuzzy game such that for each* $i \in I = \{1, 2...n\}$:

(1) X_i *are non-empty compact and convex subsets of normed linear space E;*

(2) $A_i : X \to \mathcal{F}(X_i)$ *are such that* $x \to (A_{i_x})_{a_i(x)} : X \to 2^{Y_i}$ *is lower semicontinuous in* X^0, *each* $(A_{i_x})_{a_i(x)}$ *is a nonempty, closed convex subset of* Y_i, $(A_{i_x})_{a_i(x)} \subset Y_i^0 \neq \emptyset$ for each $x \in X^0$ *for each* $x \in X$.

(3) $P_i : X := \prod\limits_{i\in I} X_i \to \mathcal{F}(X_i)$ *is such that* $x \to (P_{i_x})_{p_i(x)} : X \to 2^{X_i}$ is $Q_{\pi_i}-$*majorized*;

(4) $(P_{i_x})_{p_i(x)}$ *is nonempty for each* $x \in X$;

Then there exists a fuzzy equilibrium pair of points $(\overline{x}, \overline{y}) \in X \times X$ *such that for each* $i \in I$, $\overline{y_i} \in (A_{i_{\overline{x}}})_{a_i(\overline{x})}$ *and* $(A_{i_{\overline{x}}})_{a_i(\overline{x})} \cap (P_{i_{\overline{y}}})_{p_i(\overline{y})} = \emptyset$.

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