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# GENERALIZED DERIVATIONS AND $C^*$-ALGEBRAS

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**Abstract**

Let $H$ be a separable infinite dimensional complex Hilbert space, and let $\mathcal{L}(H)$ denote the algebra of all bounded linear operators on $H$. Let $A, B \in \mathcal{L}(H)$. Define the generalized derivation $\delta_{A,B} : \mathcal{L}(H) \mapsto \mathcal{L}(H)$ by

$$\delta_{A,B}(X) = AX - XB.$$

The class of generalized $P$- symmetric operators is the class of all pairs of operators $A, B \in \mathcal{L}(H)$ such that $[TA = BT$ implies $A^*T = TB^*$, $T \in \mathcal{C}_1(H)]$ (*) (trace class operators), i.e, the pair $(A, B)$ satisfies the Fuglede-Putnam's theorem in $\mathcal{C}_1(H)$. In this paper we present new $C^*$-algebras generated by the pair $(A, B)$ satisfying (*). Other related results are also given.

## 1 Introduction

Let $H$ be a separable infinite dimensional complex Hilbert space, and let $\mathcal{L}(H)$ denotes the algebra of all bounded linear operators on $H$. Given $A, B \in \mathcal{L}(H)$, we define the generalized derivation

$\delta_{A,B} : \mathcal{L}(H) \mapsto \mathcal{L}(H)$ by $\delta_{A,B}(X) = AX - XB$. Note that $\delta_{A,A} = \delta_A$. In [2] J.Anderson , J.Bunce, J.A.Deddens and J.P.Williams show that, if $A$ is $D-$ symmetric, ( i.e., $\overline{\mathrm{ran}(\delta_A)} = \overline{\mathrm{ran}(\delta_{A^*})}$, where $\overline{ran(\delta_A)}$ is the closure of the range, $ran(\delta_A)$, of $\delta_A$ in the norm topology, then $AT = TA$, $T \in \mathcal{C}_1(H)$ (trace class operators) implies $A^*T = TA^*$. In order to generalize these results we

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initiated in [6, 8] the study of a more general class of $P$-symmetric operators, namely the class of pairs of operators $A, B \in \mathcal{L}(H)$ such that $BT = TA$, $T \in \mathcal{C}_1(H)$ implies $A^*T = TB^*$. We call such operators generalized $P$-symmetric operators. The set of all such pairs is denoted by $\mathcal{GF}_o(H)$, that is, the pair $(A, B)$ satisfies the Fuglede-Putnam's theorem in $\mathcal{C}_1(H)$. Since the study of generalized $P$-symmetric operators comes back to the study of operators $A, B \in \mathcal{L}(H)$ such that $\overline{\mathrm{ran}(\delta_{A,B})}^{w^*}$ is self-adjoint. It is natural to introduce the following sets

$$\mathcal{T}_0(A,B) = \{(C,D) \in \mathcal{L}(H) \times \mathcal{L}(H) : C\mathcal{L}(H) + \mathcal{L}(H)D \subset \overline{ran(\delta_{A,B})}^{w*}\}.$$

$$\mathcal{I}_0(A,B) = \{(C,D) \in \mathcal{L}(H) \times \mathcal{L}(H) : Cran(\delta_{A,B}) + ran(\delta_{A,B})D \subset \overline{ran(\delta_{A,B})}^{w*}\}.$$

$$\mathcal{B}_0(A,B) = \{(C,D) \in \mathcal{L}(H) \times \mathcal{L}(H) : ran(\delta_{C,D}) \subset \overline{ran(\delta_{A,B})}^{w*}\}.$$

It is known [12] that if $\mathcal{H}$ is of finite dimension, $C = D$ and $A = B$, then

$$\mathcal{T}_0(A) = \{0\},\ \mathcal{I}_0(A) = \{A\}^{'} \text{ and } \mathcal{B}_0(A) = \{A\}^{''},$$

where $\{A\}^{'}$ is the commutant of $A$ and $\{A\}^{''}$ is the bicommutant of $A$. In this paper we will prove that if the pair $(A, B)$ is generalized $P$-symmetric, then we have

(i) $\mathcal{T}_0(A,B)$, $\mathcal{I}_0(A,B)$ and $\mathcal{B}_0(A,B)$ are $C^*$-algebras $w^*$-closed in $\mathcal{L}(H) \times \mathcal{L}(H)$.

(ii) $\mathcal{T}_0(A,B)$ is a bilateral ideal of $\mathcal{I}_0(A,B)$.

(iii) $ran(\delta_{C,D}) \subset \overline{ran(\delta_{A,B})}^{w*}$ for all $C, D \in C^*(A,B)$, the $C^*$-algebra generated by the pair $(A, B)$ of operators such that $BT = TA$ implies $A^*T = TB^*$ for all $T \in \mathcal{C}_1(H)$. We also prove that if $(A, B)$ is generalized $P$-symmetric then $A^*ran(\delta_{A,B}) + ran(\delta_{A,B})B^* \subset \overline{ran(\delta_{A,B})}^{w*}$.

## 2 Preliminaries

**Definition 2.1.** *The Trace class operators, denoted by $\mathcal{C}_1(H)$, is the set of all compact operators $A \in \mathcal{L}(H)$, for which the eigenvalues of $(TT^*)^{\frac{1}{2}}$ counted according to multiplicity, are summable. The ideal $\mathcal{C}_1(H)$ of $\mathcal{L}(H)$ admits a trace function $tr(T)$, given by $tr(T) = \sum_n (Te_n, e_n)$ for any complete orthonormal system $(e_n)$ in $H$. As a Banach spaces $\mathcal{C}_1(H)$ can be identified with the dual of the ideal $K$ of compact operators by means of the linear isometry*

*$T \mapsto f_T$, where $f_T = tr(XT)$. Moreover $\mathcal{L}(H)$ is the dual of $\mathcal{C}_1(H)$, the ultra weakly continuous linear functionals on $\mathcal{L}(H)$ are those of the form $f_T$ for $T \in \mathcal{C}_1(H)$ and the weakly continuous linear functionals are those of the form $f_T$ with $T$ is of finite rank.*

**Definition 2.2.** *Given $A \in \mathcal{L}(H)$, the inner derivation*

$$\delta_A : \mathcal{L}(H) \to \mathcal{L}(H)$$

*is defined by*

$$\delta_{A,B}(X) = AX - XA, (X \in \mathcal{L}(H)).$$

**Definition 2.3.** *Let $A$ and $B$ be two operators in $\mathcal{L}(H)$. Then the generalized derivation*

$$\delta_{A,B} : \mathcal{L}(H) \to \mathcal{L}(H)$$

*is defined by*

$$\delta_{A,B}(X) = AX - XB, (X \in \mathcal{L}(H)).$$

**Definition 2.4.** *Let $A \in \mathcal{L}(H)$. Then $A$ is called $D$-symmetric if*

$$\overline{ran(\delta_A)} = \overline{\mathrm{ran}(\delta_{A^*})}.$$

**Definition 2.5.** *Let $A, B \in \mathcal{L}(H)$. The pair $(A, B)$ is called generalized $D$-symmetric pair of operators if*

$$\overline{\mathrm{ran}(\delta_{A,B})} = \overline{ran(\delta_{B^*,A^*})}.$$

*The set of all such pairs is denoted by $\mathcal{GS}(H)$. Here $\overline{ran(\delta_{A,B})}$ is the closure of the range, $\mathrm{ran}(\delta_{A,B})$, of $\delta_{A,B}$ in the norm topology.*

**Definition 2.6.** *Let $A \in \mathcal{L}(H)$. If*
*$AT = TA$ implies $A^*T = TA^*$, $\forall T \in \mathcal{C}_1(H)$, then $A$ is called $P$-symmetric.*

**Definition 2.7.** *Let $A, B \in \mathcal{L}(H)$, the pair $(A, B)$ of operators such that $BT = TA$ implies $A^*T = TB^*$ for all $T \in \mathcal{C}_1(H)$ is called a generalized $P$-symmetric pair of operators. The set of all such pairs is denoted by $\mathcal{GF}_o(H)$, that is, the pair $(A, B)$ satisfies the Fuglede-Putnam's theorem in $\mathcal{C}_1(H)$.*

Let $\mathcal{B}$ be a Banach space and let $\mathcal{S}$ be a subspace of $\mathcal{B}$. Denote by $\mathcal{B}'$ the set of all linear functionals, and set

$$\mathcal{B}^* = \left\{ f \in \mathcal{B}' : f \,\text{is bounded (norm-continuous)} \right\},$$

$$\mathrm{Ann}(S) = \{ f \in \mathcal{B}^* : f(s) = 0 \,\text{for all}\, s \in S \}.$$

In [6] the author proved the following theorem.

**Theorem 2.1.** *Let $A, B \in \mathcal{L}(H)$. Then $(A, B) \in \mathcal{GF}_0(H) \Leftrightarrow \overline{\mathrm{ran}(\delta_{A,B})}^{w*} = \overline{\mathrm{ran}(\delta_{B^*,A^*})}^{w*}$.*

## 3 Main Results

In the following theorem we will present some properties of $\mathcal{T}_0(A,B)$, $\mathcal{I}_0(A,B)$ and $\mathcal{B}_0(A,B)$.

**Theorem 3.1.** *Let $A, B \in \mathcal{L}(H)$. If the pair $(A,B)$ is generalized $P$-symmetric, then we have*

*(i) $\mathcal{T}_0(A,B)$, $\mathcal{I}_0(A,B)$ and $\mathcal{B}_0(A,B)$ are $C^*$-algebras $w^*$-closed in $\mathcal{L}(H) \times \mathcal{L}(H)$.*

*(ii) $\mathcal{T}_0(A,B)$ is a bilateral ideal of $\mathcal{I}_0(A,B)$.*

*(iii) $ran(\delta_{C,D}) \subset \overline{ran(\delta_{A,B})}^{w*}$ for all $C, D \in C^*(A,B)$, the $C^*$-algebra generated by the pair $(A,B) \in \mathcal{GF}_0(H)$.*

*Proof.* (i) Let $(C,D) \in \mathcal{T}_0(A,B)$. Since $C^*\mathcal{L}(H) = [\mathcal{L}(H)C]^* \subseteq \overline{ran(\delta_{A,B})}^{w*}$, it follows that $C^*\mathcal{L}(H) \subseteq \overline{ran(\delta_{A,B})}^{w*}$. By the same arguments as above we prove that $\mathcal{L}(H)D^* \subseteq \overline{ran(\delta_{A,B})}^{w*}$, that is, $(C^*, D^*) \in \mathcal{T}_0(A,B)$. If $C, D \in \mathcal{L}(H)$, then the linear maps $L_C X = CX$ and $R_D X = XD$ are $w^*$-continuous. Consequently $\mathcal{T}_0(A,B)$ is $w^*$-closed in $\mathcal{L}(H) \times \mathcal{L}(H)$. By the same arguments as above we prove that $\mathcal{I}_0(A,B)$ and $\mathcal{B}_0(A,B)$ are $C^*$-algebras $w^*$-closed in $\mathcal{L}(H) \times \mathcal{L}(H)$.

(ii) If $(C,D) \in \mathcal{I}_0(A,B)$ and $(E,F) \in \mathcal{T}_0(A,B)$, then for all $X \in \mathcal{L}(H)$ we have $X(CE) = (XC)E \in \overline{ran(\delta_{A,B})}^{w*} E \subset \overline{ran(\delta_{A,B})}^{w*}$. We have also $(DF)X = D(FX) \in \overline{ran(\delta_{A,B})}^{w*}$. This shows that $\mathcal{T}_0(A,B)$ is an ideal at right. Since $\mathcal{T}_0(A,B)$ is a $C^*$-algebra, it follows that $\mathcal{T}_0(A,B)$ is a bilateral ideal of $\mathcal{I}_0(A,B)$.

(iii) Assume that $(C,D) \in \mathcal{B}_0(A,B)$. Since $\mathcal{B}_0(A,B)$ is a $C^*$-algebra containing the pair $(A,B)$ and $(I,I)$, it contains $C^*(A,B)$. ☐

**Remark 3.1.** *In [8] the author proved that*

$$Ann(ran(\delta_{A,B})) = Ann(ran(E_{A,B})) \cap Ann(K(H)) \oplus \ker(\delta_{B,A}) \cap \mathcal{C}_1(H). \quad (2.1)$$

*Note that $\overline{ran(\delta_{A,B})}^{w*}$ is self-adjoint if and only if*

$$Ann(ran(\delta_{A,B})) \cap \mathcal{L}(H)'^{w*}$$

*is also self-adjoint. By using (2.1) we obtain in particular*

$$Ann(ran(\delta_{A,B})) \cap \mathcal{L}(H)'^{w*} \simeq \ker \delta_{B,A} \cap \mathcal{C}_1(H),$$

*where $\mathcal{L}(H)'^{w^*}$ is the set of the ultra-weakly continuous linear functionals on $\mathcal{L}(H)'$. Thus $(A,B) \in \mathcal{GF}_0(H)$ if and only if $\overline{ran(\delta_{A,B})}^{w*}$ is self-adjoint.*

**Theorem 3.2.** *Let $A, B \in \mathcal{L}(H)$. If $(A, B)$ is generalized $P$-symmetric, then*

$$B^* ran(\delta_{A,B}) + ran(\delta_{A,B})A^* \subset \overline{ran(\delta_{A,B})}^{w*}.$$

*Proof.* Assume that $(A, B)$ is generalized $P$-symmetric. Then it follows from Theorem 2.1 that:
$\overline{ran(\delta_{A,B})}^{w*} = \overline{ran(\delta_{B^*,A^*})}^{w*}$. But since
$B^*\delta_{B^*,A^*}(X) = \delta_{B^*,A^*}(B^*X)$ and $\delta_{B^*,A^*}(X)A^* = \delta_{B^*,A^*}(XA^*)$, we deduce that:
$B^* ran(\delta_{A,B}) \subset B^*\overline{ran(\delta_{A,B})}^{w*} = B^*\overline{ran(\delta_{B^*,A^*})}^{w*} \subseteq \overline{ran(\delta_{B^*,A^*})}^{w*} = \overline{ran(\delta_{A,B})}^{w*}$.
By the same arguments shown above we can prove that:
$ran(\delta_{A,B})A^* \subset \overline{ran(\delta_{A,B})}^{w*}$. This completes the proof. □

In [5] we proved that the direct sum of $D$-symmetric operators is also $D$-symmetric if $\sigma(A) \cap \sigma(B) = \phi$. By a slight modification in the proof of [5, Theorem 2.4] we can prove the following theorem.

**Theorem 3.3.** *Let $A$ and $B$ be two $P$-symmetric operators such that $\sigma(A) \cap \sigma(B) = \phi$. Then $A \oplus B$ is also $P$-symmetric.*

Note that the condition given in the previous theorem is necessary for $A \oplus B$ to be $P$-symmetric as we will show in the following example.

**Example 3.1.** *(i) Let $\Delta = \{z \in \mathbb{C} : |z| \leq 1\}$ and $H_1 = L^2(\Delta)$. Define $M \in \mathcal{L}(H)$ as follows :*

$$M : H_1 \to H_1,\ s.t., f \to Mf$$

*That is*

$$\forall z \in \Delta, Mf(z) = zf(z)$$

*(ii) Let $H_2$ be a separable complex Hilbert space, and let $(e_n)_n \in \mathbb{N}$ be an orthonormal basis of $H_2$. Consider the unilateral shift $S : H_2 \to H_2$, which is defined by*

$$Se_n = e_{n+1}, \forall n \in \mathbb{N}.$$

*Note that both $S$ and $M$ are normal and hence they are $P$-symmetric operators.*
*(iii)Define $\tau : H_2 \to H_1$ to be :*

$$\tau e_n(z) = z^n \chi_D,$$

*where $D = \{z \in \mathbb{C} : |z| \leq \alpha < 1\}$, and $\alpha$ is fixed. We claim that $\tau$ is a trace class operator, because*

$$\|\tau e_n(z)\|^2 = \|z^n\chi_D\|^2 = \int\int_D |z^n|^2 r dr d\theta = \int\int_D r^{2n+1}e^{2in\theta} dr d\theta = 2\pi(\frac{\alpha^{2n+2}}{2n+2}).$$

*Since* $|\alpha| < 1$,

$$\|\tau\| \leq \sum_{n=1}^{\infty} \|\tau e_n\| \leq \sqrt{2\pi} \frac{\alpha^{n+1}}{\sqrt{2n+2}} < \infty.$$

*Hence* $\tau$ *is a trace class operator.*

*(iv) Let*

$$A = \begin{pmatrix} M & 0 \\ 0 & S \end{pmatrix}$$

*and*

$$T = \begin{pmatrix} 0 & \tau \\ 0 & 0 \end{pmatrix}.$$

*Since* $\tau$ *is of trace class,* $T$ *is also of trace class. Note that* $AT = TA$*, but if* $T^*A = AT^*$*, then* $\tau^* M = S\tau^*$*. But the equation* $SX = XM$ *implies* $X = 0$ *[11]. This contradicts our hypothesis Thus* $A = M \oplus S$ *is not* $P$*-symmetric.*

The previous example is used in [2] to show that the direct sum of two $D$-symmetric operators is not in general $D$-symmetric.

In [5] we proved that the set $Ds = \{T + K : T$ is $D$-symmetric, $K$ compact $\}$ is norm-dense in $\mathcal{L}(\mathcal{H})$. By a slight modification in the proof of [5, Theorem 2.7] we can prove that the set $Ps = \{T + K : T$ is $P$-symmetric, $K$ compact $\}$ is also norm-dense $\mathcal{L}(\mathcal{H})$.

**Remark 3.2.** *It is known that the operator* $A, B \in B(H)$ *satisfy the Fuglede-Putnam's theorem if* $AX = XB$*,* $X \in B(H)$ *implies* $A^*X = XB^*$*. Thus our results are generalizations of Fuglede-Putnam's theorem in* $\mathcal{C}_1(H)$*. Recall [3] that if* $A$ *is normal or isometric, then* $A$ *is p-symmetric.*

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