An. Şt. Univ. Ovidius Constanţa Vol. 17(2), 2009, 5–18

OPERATION-SEPARATION AXIOMS IN BITOPOLOGICAL SPACES

S.M. Al-Areefi

Abstract

In this paper, the concept of pairwise $\gamma - T_0$, weak pairwise $\gamma - T_1, \gamma_i - T_1$, pairwise $\gamma - T_1$, weak pairwise $\gamma - T_2, \gamma_i - T_2$, pairwise $\gamma - T_2$, strong pairwise $\gamma - T_2$, pairwise $\gamma - R_0$ and pairwise $\gamma - R_1$ are introduced and studied as a unification of several characterizations and properties of T_0, T_1, T_2, R_0 and R_1 in bitopological spaces

1. Introduction

As a continuation of the study of operations in bitopological spaces introduced in [8], the aim of this paper is to introduce and study some separation axioms in bitopological spaces using the concept of operations on such spaces. In Section 2, we give a brief account of the definitions and results obtained in [8] which we need in this article. Section 3 is devoted to introduce the concepts of pairwise $\gamma - T_0$, weak pairwise $\gamma - T_1, \gamma_i - T_1$, pairwise $\gamma - T_1$, weak pairwise $\gamma - T_2, \gamma_i - T_2$, pairwise $\gamma - T_2$ and strong pairwise $\gamma - T_2$ and study some of their properties. Finally, the concepts of pairwise $\gamma - R_0$ and pairwise $\gamma - R_1$ are introduced and studied in Section 4.

Throughout the paper, (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or briefly, X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of X, by $i - int(A)$ and $i - cl(A)$, we denote respectively the interior and the closure of A with respect to τ_i (or σ_i) for $i = 1, 2$. Also $i, j = 1, 2$ and $i \neq j$. By id we mean the identity and nbd is the abbreviation of neighborhood. Also $X \setminus A = A^c$ is the complement of A in X.

Key Words: Bitopological spaces, operation and separation axioms
Mathematics Subject Classification: 54C55, 54D10, 54D15
Received: March, 2009
Accepted: September, 2009

2. Preliminary

Here, we give a brief account of the definitions and results obtained in [8] which we need in this paper.

Definition 2.1. Let (X, τ_1, τ_2) be a bitopological space. An operation γ on $\tau_1 \cup \tau_2$ is a mapping $\gamma : \tau_1 \cup \tau_2 \to P(X)$ such that $V \subset V^\gamma$ for each $V \in \tau_1 \cup \tau_2$, where V^γ denotes the value of γ at V. The operators $\gamma(V) = V, \gamma(V) = j - cl(V)$ and $\gamma(V) = i - int(j - cl(V))$ for $V \in \tau_i$ are operations in $\tau_1 \cup \tau_2$.

Definition 2.2. A subset A of a bitopological space (X, τ_1, τ_2) will be called a γ_i-*open set* if for each $x \in A$, there exists a τ_j- open set U such that $x \in U$ and $U^\gamma \subset A$. $\tau_{i\gamma}$ will denote the set of all γ_i-open sets. Clearly we have $\tau_{i\gamma} \subset \tau_i$. A subset B of X is said to be γ_i-*closed* if $X \setminus B$ is γ_i-*open* in X.

If $\gamma(u) = u$ (resp. $j - cl(u)$) and $j - cl(i - int(u))$) for each $u \in T_i$ then, the concept of γ_i-open sets coincides with the concept of τ_i-open (resp. $ij - o - open$ [3] and $cj - \delta-$open [3] sets.

Definition 2.3. An operation γ on $\tau_1 \cup \tau_2$ in (X, τ_1, τ_2) is said to be *i-regular* if for every pair U, V of τ_i-open nbds of each point $x \in X$, there exists a τ_i-open nbd W of x such that $W^\gamma \subset U^\gamma \cap V^\gamma$. If γ is 1-regular and 2-regular then it is called pairwise regular or, simply, regular.

Definition 2.4. An operation γ on (X, τ_1, τ_2) is said to be i-open if for every τ_i-open neighborhood U of each $x \in X$, there exists a γ_i-*open* set S such that $x \in S$ and $S \subset U^\gamma$. If γ is 1-open and 2-open then it is called pairwise open or, simply, open.

Example 2.5. *Let $X = \{a, b, c\}$ and let $\tau_1 = \tau_2 = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. Let $\gamma : \tau_1 \cup \tau_2 \to P(X)$ be an operation defined by $\gamma(B) = i - cl(B)$ and $\lambda : \tau_1 \cup \tau_2 \to P(X)$ be an operation defined by $\lambda(B) = i - int(j - cl(B))$. Then we have $\tau_{i\gamma} = \{\phi, X\}$ and $\tau_{1\lambda} = \tau_i$. It is easy to see that γ is i-regular but it is not i-open on (X, τ_1, τ_2).*

Example 2.6. *Let $X = \{a, b, c\}$ and let $\tau_1 = \tau_2 = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. For $b \in X$ we define an operation $\gamma : \tau_1 \cup \tau_2 \to P(X)$ by*

$$\gamma(A) = A^\gamma = \begin{cases} A & \text{if } b \in A \\ i - cl(A) & \text{if } b \notin A \end{cases}$$

Then the operation $\gamma : \tau_1 \cup \tau_2 \to P(X)$ is not regular on $\tau_1 \cup \tau_2$. In fact, let $U = \{a\}$ and $V = \{a, b\}$ be i-open neighborhoods of a, then $U^\gamma \cap V^\gamma = \{a, c\} \cap \{a, c\} = \{a\}$ and $W^\gamma \not\subset U^\gamma \cap V^\gamma$ for any i-open neighborhood W of a.

Definition 2.7. Let γ, μ be two operations on $\tau_1 \cup \tau_2$ in (X, τ_1, τ_2). Then (X, τ_1, τ_2) is called $(\gamma, \mu)_i$-*regular* if for each $x \in X$ and each τ_i-open nbd V of x, there exists a τ_i-open set U containing x such that $U^\gamma \subset V^\mu$. If $\mu = id$, then the $(\gamma, \mu)_i$-regular space is called γ_i-regular. If (X, τ_1, τ_2) is $(\gamma, \mu)_1$-regular and $(\gamma, \mu)_2$-regular, then it is called pairwise (γ, μ)-*regular*. Moreover, we can show that this operation is i-open on $\tau_1 \cup \tau_2$.

In the rest of this paper, instead of "γ operation on $\tau_1 \cup \tau_2$ in (X, τ_1, τ_2)", we shall say "γ operation on (X, τ_1, τ_2)".

Proposition 2.8. *Let γ be an operation on a bitopological space (X, τ_1, τ_2). Then (X, τ_1, τ_2) is a γ_i-*regular *space if and only if $\tau_i = \tau_{i\gamma}$ holds.*

Proposition 2.9. *Let γ be an i-regular operation on a bitopological space (X, τ_1, τ_2), then*

(i) If A and B are γ_i-open sets of X, then $A \cap B$ is γ_i-open.

(ii) $\tau_{i\gamma}$ is a topology on X.

Remark 2.10. If γ is not i-regular, then the above proposition is not true in general. In Exmaple 2.6, γ is not i-regular. In fact $\tau_{i\gamma} = \{\phi, X, \{b\}, \{a, b\}\{a, c\}\}$ which is not a topology on X.

Definition 2.11. Let γ be an operation on a bitopological space (X, τ_1, τ_2). A point $x \in X$ is in the γ_i-closure of a set $A \subset X$ if $U^\gamma \cap A \neq \phi$ for each τ_i-open nbd U of x. The γ_i-closure of the set A is denoted by $cl_{i\gamma}(A)$. A subset A of X is said to be γ_i-*closed* (in the sense of $cl_{i\gamma}(A)$) if $cl_{i\gamma}(A) \subset A$.

Definition 2.12. Let $A \subset (X, \tau_1, \tau_2)$. For the family $\tau_{i\gamma}$, we define a set $\tau_{i\gamma} - cl(A)$ as follows:

$$\tau_{i\gamma} - cl(A) = \cap\{F : A \subset F, X \setminus F \in \tau_{i\gamma}\}.$$

Proposition 2.13. *For a point $x \in X$, $x \in \tau_{i\gamma} - cl(A)$ if and only if $V \cap A \neq \phi$ for any $V \in \tau_{i\gamma}$ such that $x \in V$.*

Remark 2.14. It is easily shown that for any subset A of (X, τ_1, τ_2), $A \subset \tau_i - cl(A) \subset cl_{i\gamma}(A) \subset \tau_{i\gamma} - cl(A)$.

Theorem 2.15. *Let γ be an operation on a bitopological space (X, τ_1, τ_2), then*

(i) $cl_{i\gamma}(A)$ is τ_i-closed in (X, τ_1, τ_2).

(ii) If (X, τ_1, τ_2) is γ_i-regular, then $cl_{i\gamma}(A) = \tau_i - cl(A)$ holds.

(iii) If γ is open, then $cl_{i\gamma}(A) = \tau_{i\gamma} - cl(A)$ and $cl_{i\gamma}(cl_{i\gamma}(A)) = cl_{i\gamma}(A)$ hold and $cl_{i\gamma}(A)$ is γ_i-closed (in the sense of Definition 2.11).

Theorem 2.16. *Let γ be an operation on a bitopological space (X, τ_1, τ_2) and $A \subset X$, then the following are equivalent:*

(a) A is γ_i-open in (X, τ_1, τ_2).

(b) $cl_{i\gamma}(X \setminus A) = X \setminus A$ (i.e., $X \setminus A$ is γ_i-closed in the sense of Definition 2.11).

(c) $\tau_{i\gamma} - cl(X \setminus A) = X \setminus A$.

(d) $X \setminus A$ is γ_i-closed (in the sense of Definition 2.12) in (X, τ_1, τ_2).

Lemma 2.17. *If γ is a regular operation on (X, τ_1, τ_2), then $cl_{i\gamma}(A \cup B) = cl_{i\gamma}(A) \cup cl_{i\gamma}(B)$, for any subsets A and B of X.*

Corollary 2.18. *If γ is a regular and open operation on (X, τ_1, τ_2), then $cl_{i\gamma}$ satisfies the Kuratowski closure axioms.*

Definition 2.19. A mapping $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be $(\gamma, \beta)_i$-*continuous* if for each point x of X and each σ_i-open set V containing $f(x)$, there exists a τ_i-open set U such that $x \in U$ and $f(U^\gamma) \subset V^\beta$. If f is $(\gamma, \beta)_i$-continuous for $i = 1, 2$, then it is called pairwise (γ, β)-*continuous*.

Example 2.20. *If $(\gamma, \beta) = (id, id)$ (resp. $id, j - cl), (j - cl, id), (j - cl - j - cl), (id, i - int \circ j - cl), (i - int \circ j - cl, id), (i - int \circ j - cl, j - cl), (j - cl, i - int \circ j - cl)$ and $(i - int \circ j - cl, i - int \circ j - cl)$) then pairwise (γ, β)-continuity coincides with pairwise continuity [7] (resp. pairwise weak continuity [5], pairwise strong θ-continuity [3], pairwise θ-continuity [4], pairwise almost continuity [5], pairwise super continuity [9], pairwise weak θ-continuity [11], pairwise almost strong θ-continuity [3] and pairwise δ-continuity [3]).*

Proposition 2.21. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a $(\gamma, \beta)_i$-continuous mapping, then*

(i) $f(cl_{i\gamma}(A)) \subset cl_{i\beta}(f(A))$ for every $A \subset X$.

(ii) for any β_i-closed set B of Y, $f^{-1}(B)$ is γ_i-closed in X, i.e., for any $U \in \sigma_{i\beta}$, $f^{-1}(U) \in \tau_{i\gamma}$.

Definition 2.22. Let γ be an operation on a bitopological space (X, τ_1, τ_2). A subset $K \subset X$ is said to be γ_i-*compact* if for every τ_i-open cover $\mathbf{C}$ of K, there exists a finite subfamily $\{G_1, \ldots, G_n\}$ of $\mathbf{C}$ such that $K \subset \bigcup_{r=1}^{n} G_r^\gamma$.

Example 2.23. *If $\gamma = id$ (resp. $j - cl$ and $i - int \circ j - cl$) then γ_i compactness coincides with τ_i-compactness (resp. ij-almost-compactness [11] and ij-near-compactness [3]).*

Theorem 2.24. *Let γ be an operation on a bitopological space (X, τ_1, τ_2). If X is γ_i-compact, then every cover of X by γ_i-open sets has a finite subcover. If γ is open, then the converse is true.*

3. Pairwise $\gamma - T_k$ Spaces

Definition 3.25. Let γ be an operation on a bitopological space (X, τ_1, τ_2). Then X is called *pairwise* $\gamma - T_0$ if for any two distinct points of X, there exists a subset which is either γ_i-open or γ_j-open containing one of the points but not the other.

Example 3.26. *In Definition 3.25, if $\gamma = id$, then we obtain the definition of pairwise T_0 spaces [6]. If $\gamma(U) = j - cl(U)$ (resp. $i - int(j - cl(U))$ for $U \in \tau_i$, then pairwise $\gamma - T_0$ spaces are called pairwise $\theta - T_0$ (resp. pairwise $\delta - T_0$). If $\gamma(U) = i - int(i - cl(U))$ for $U \in \tau_i$, then we obtain pairwise rT_0 spaces [2].*

Theorem 3.27. *Let γ be an operation on a bitopological space $(X.\tau_1, \tau_2)$. If X is pairwise $\gamma - T_0$, then for any two distinct points x and y of X, there exists a subset U which is either τ_i-open or τ_j-open containing one of them (say x) such that $y \notin U^\gamma$. If γ is open, then the converse is true.*

Proof. Let $x, y \in X$ such that $x \neq y$, then there exists a set U which is (say) $\tau_{i\gamma}$-open such that $x \in U$ and $y \notin U$. Then there exists a τ_i-open set G containing x such that $G^\gamma \subset U$. Obviously, $y \notin G^\gamma$. Now, if γ is open, let $x, y \in X$ such that $x \neq y$. Then, by assumption, there exists a τ_i-open set U such that $x \in U$ and $y \notin U^\gamma$. Since γ is open, there exists a $\tau_{i\gamma}$-open set S such that $x \in S$ and $S \subset U^\gamma$. Obviously, $y \notin S$. This shows that X is pairwise $\gamma - T_0$. □

Theorem 3.28. *Let γ be an operation on a bitopological space (X, τ_1, τ_2). Then X is pairwise $\gamma - T_0$ if and only if for each two distinct points x and y of X, either $\tau_{i\gamma} - cl\{x\} \neq \tau_{i\gamma} - cl\{y\}$ or $\tau_{j\gamma} - cl\{x\} \neq \tau_{j\gamma} - cl\{y\}$.*

Proof. Let X be a pairwise $\gamma - T_0$ space and $x, y \in X$ such that $x \neq y$. Suppose U is a γ_i-open set containing x but not y. Then $y \in \tau_{i\gamma} - cl\{y\} \subset X \setminus U$ and so $x \notin \tau_{i\gamma} - cl\{y\}$. Hence $\tau_{i\gamma} - cl\{x\} \neq \tau_{i\gamma} - cl\{y\}$. Conversely, let $x, y \in X$ such that $x \neq y$. Then either $\tau_{i\gamma} - cl\{x\} \neq \tau_{i\gamma} - cl\{y\}$ or $\tau_{j\gamma} - cl\{x\} \neq \tau_{j\gamma} - cl\{y\}$. In the former case, let $z \in X$ such that $z \in \tau_{i\gamma} - cl\{y\}$ and $z \notin \tau_{i\gamma} - cl\{x\}$. We assert that $y \notin \tau_{iy} - cl\{x\}$. If $y \in \tau_{i\gamma} - cl\{x\}$, then $\tau_{i\gamma} - cl\{y\} \subset \tau_{i\gamma} - cl\{x\}$, so $z \in \tau_{i\gamma} - cl\{y\} \subset \tau_{i\gamma} - cl\{x\}$, a contradiction. Hence $y \notin \tau_{i\gamma} - cl\{x\}$ and therefore $U = X \setminus \tau_{i\gamma} - cl\{x\}$ is a γ_i-open set containing y but not x. The case $\tau_{j\gamma} - cl\{x\} \neq \tau_{j\gamma} - cl\{y\}$ can be dealt with similarly. □

Corollary 3.29. *A bitopological space (X, τ_1, τ_2) is pairwise $\theta - T_0$ if and only if for each two distinct points x and y of X, either $ij - cl_\theta\{x\} \neq ij - cl_\theta\{y\}$ or $ji - cl_\theta\{x\} \neq ji - cl_\theta\{y\}$.*

Corollary 3.30. *A bitopological space (X, τ_1, τ_2) is pairwise $\delta - T_0$ if and only if each two distinct points x and y of X, either $ij - cl_\delta\{x\} \neq ij - cl_\delta\{y\}$ or $ji - cl_\delta\{x\} \neq ji - cl_\delta\{y\}$.*

Definition 3.31. Let γ be an operation on a bitopological space (X, τ_1, τ_2). Then X is called weakly pairwise $\gamma - T_1$ if for any two distinct points x and y of X, there exists a γ_i-open set U and a γ_j-open set V such that either $x \in U \setminus V$ and $y \in V \setminus U$ or $y \in U \setminus V$ and $x \in V \setminus U$.

Example 3.32. *In Definition 3.31, if $\gamma = id$, then we obtain the definition of weakly pairwise T_1 [10]. If $\gamma(U) = j - cl(U)$ (resp. $i - int(j - cl(U))$) for $U \in \tau_i$, then weakly pairwise $\gamma - T_1$ spaces are called weakly pairwise $\theta - T_1$ (resp. weakly pairwise $\delta - T_1$). If $\gamma(U) = i - int(i - cl(U))$ for $U \in \tau_i$, then we obtain weakly pairwise rT_1 spaces [2].*

Theorem 3.33. *Let γ be an operation on a bitopological space (X, τ_1, τ_2). Then the following are equivalent:*

(i) (X, τ_1, τ_2) *is weakly pairwise* $\gamma - T_1$.

(ii) $\tau_{1\gamma} - cl\{x\} \cap \tau_{2\gamma} - cl\{x\} = \{x\}$ *for every* $x \in X$.

(iii) For every $x \in X$, the intersection of all γ_1-open nbds and all γ_2-open nbds of x is $\{x\}$.

Proof. (i) ⇒ (ii): Let $x \in X$ and $y \in \tau_{1\gamma} - cl\{x\} \cap \tau_{2\gamma} - cl\{x\}$, where $y \neq x$. Since X is weakly pairwise $\gamma - T_1$, there exists a γ_1-open set U such that $y \in U$, $x \notin U$ or there exists a γ_2-open set V such that $y \in V$, $x \notin V$. In either case, $y \notin \tau_{1\gamma} - cl\{x\} \cap \tau_{2\gamma} - cl\{x\}$. Hence $\{x\} = \tau_{1\gamma} - cl\{x\} \cap \tau_{2\gamma} - cl\{x\}$.

(ii) ⇒ (iii): If $x, y \in X$ such that $x \neq y$, then $x \notin \tau_{1\gamma} - cl\{y\} \cap \tau_{2\gamma} - cl\{y\}$, so there is a γ_1-open set or a γ_2-open set containing x but not y. Therefore y does not belong to the intersection of all γ_1-open nbds and all γ_2-open nbds of x.

(iii) ⇒ (i): Let x and y be two distinct points of X. By (iii), y does not belong to a γ_1-nbd or a γ_2-nbd of x. Therefore there exists a γ_1-open or a γ_2-open set containing x but not y. Hence x is a weakly pairwise $\gamma - T_1$ space. □

Corollary 3.34. *For a bitopological space (X, τ_1, τ_2), the following are equivalent:*

(i) (X, τ_1, τ_2) *is weakly pairwise* $\theta - T_1$.

(ii) $12 - cl_\theta\{x\} \cap 21 - cl_\theta\{x\} = \{x\}$ *for every* $x \in X$.

(iii) For every $x \in X$*, the intersection of all* $12 - \theta$*-open nbds and all* $21 - \theta$*-open nbds of* x *is* $\{x\}$.

Corollary 3.35. *For* $x \in X$*, the intersection of all* $12 - \delta$*-open neighborhoods and all* $21 - \theta$*-open neighborhoods of* x *is* $\{x\}$.

Theorem 3.36. *Let* γ *be an operation on a bitopological space* (X, τ_1, τ_2)*. If* X *is weakly pairwise* $\gamma - T_1$*, then for any two distinct points* x *and* y *of* X*, there exists a* τ_i*-open set* U *and a* τ_j*-open set* V *such that either* $x \in U,\ y \notin U^\gamma$ *and* $y \in V,\ = x \notin V^\gamma$ *or* $y \in U,\ x \notin U^\gamma$ *and* $x \in V, y \notin V^\gamma$*. If* γ *is open, then the converse is true.*

Proof. Similar to that of Theorem 3.27. □

Definition 3.37. Let γ be an operation on a bitopological space (X, τ_1, τ_2). Then X is called $\gamma_i - T_1$ if for any two distinct points x and y of X, there exist two γ_i-open sets U and V such that $x \in U \setminus V$ and $y \in V \setminus U$. If X is $\gamma_1 - T_1$ and $\gamma_2 - T_1$, then it is called pairwise $\gamma - T_1$.

Example 3.38. *In Definition 3.37, if* $\gamma = id$*, then we obtain the definition of pairwise* T_1 *[14]. If* $\gamma(U) = j - cl(U)$ *(resp.* $i - int(j - cl(U))$*) for* $U \in \tau_i$*, then pairwise* $\gamma - T_1$ *spaces are called pairwise* $\theta - T_1$ *(resp. pairwise* $\delta - T_1$*) and* $\gamma_i - T_1$ *spaces are called* $ij - \theta - T_1$ *(resp.* $ij - \delta - T_1$*) spaces.*

Theorem 3.39. *Let* γ *be an operation on a bitopological space* (X, τ_1, τ_2)*. Then* X *is* $\gamma_i - T_1$ *if and only if* $\tau_{i\gamma} - cl\{x\} = \{x\}$*, for every* $x \in X$.

Proof. Let $y \notin \{x\}$, then $y \neq x$ and there exists a $\tau_{i\gamma}$-open set U such that $y \in U$ and $x \notin U$. Therefore $y \notin \tau_{i\gamma} - cl\{x\}$ and so $\tau_{i\gamma} - cl\{x\} = \{x\}$. Conversely, let $x, y \in X$ such that $x \neq y$. Since $\tau_{i\gamma} - cl\{x\} = \{x\}$ and $\tau_{i\gamma} - cl\{y\} = \{y\}$, then there exists a γ_i-open set U and γ_i-open set V such that $x \in U \setminus V$ and $y \in V \setminus U$. Thus X is $\gamma_i - T_1$. □

Corollary 3.40. *Let* γ *be an operation on a bitopological space* (X, τ_1, τ_2)*. Then* X *is pairwise* $\gamma - T_1$ *if and only if* $\tau_{1\gamma} - cl\{x\} = \{x\} = \tau_{2\gamma} - cl\{x\}$*, for every* $x \in X$.

Corollary 3.41. *A bitopological space* (X, τ_1, τ_2) *is pairwise* T_1 *if and only if* $\tau_1 - cl\{x\} = \{x\} = \tau_2 - cl\{x\}$*, for every* $x \in X$.

Corollary 3.42. *A bitopological space (X, τ_1, τ_2) is pairwise $\theta - T_1$ if and only if $12 - cl_\theta\{x\} = \{x\} = 21 - cl_\theta\{x\}$, for every $x \in X$.*

Corollary 3.43. *A bitopological space (X, τ_1, τ_2) is pairwise $\delta - T_1$ if and only if $12 - cl_\delta\{x\} = \{x\} = 21 - cl_\delta\{x\}$, for every $x \in X$.*

Definition 3.44. Let γ be an operation on a bitopological space (X, τ_1, τ_2). Then X is called *weakly pairwise* $\gamma - T_2$ if for any two distinct points x and y of X, there exist a γ_i-open set U and a disjoint γ_j-open set V such that $x \in U$ and $y \in V$ or $x \in V$ and $y \in U$.

Example 3.45. *If $\gamma = id$ in Definition 3.44, we obtain the definition weakly pairwise T_2 [15]. In case $\gamma(U) = j - cl(U)$ (resp. $i - int(j - cl(U))$) for $U \in \tau_i$, then weakly pairwise $\gamma - T_2$ is called weakly pairwise $\theta - T_2$ (resp. weakly pairwise $\delta - T_2$). If $\gamma(U) = i - int(i - cl(U))$ for $U \in \tau_i$, then we obtain a pairwise semi rT_2 [2].*

Definition 3.46. Let γ be an operation on a bitopological space (X, τ_1, τ_2). Then X is called $\gamma_i - T_2$ if for any two distinct points x and y of X, there exist two disjoint γ_i-open sets U and V such that $x \in U$ and $y \in V$. If X is $\gamma_1 - T_2$ and $\gamma_2 - T_2$, then it is called pairwise $\gamma - T_2$.

Definition 3.47. Let γ be an operation on a bitopological space (X, τ_1, τ_2). Then X is called *strongly pairwise* $\gamma - T_2$ if for any two distinct points x and y of X, there exist a γ_i-open set U and a disjoint γ_j-open set V such that $x \in U$ and $y \in V$.

Example 3.48. *If $\gamma = id$ in Definition 3.47, we obtain the definition pairwise T_2 [7]. In case $\gamma(U) = j - cl(U)$ (resp. $i - int(j - cl(U))$) for $U \in \tau_i$, then strongly pairwise $\gamma - T_2$ is called* strongly pairwise $\theta - T_2$ *(resp. strongly pairwise $\delta - T_2$). If $\gamma(U) = i - int(i - cl(U))$ for $U \in \tau_i$, then we obtain pairwise rT_2 [2].*

Theorem 3.49. *Let γ be an operation on a bitopological space (X, τ_1, τ_2). Then X is strongly pairwise $\gamma - T_2$ if and only if the intersection of all γ_i-closed γ_j-nbd of each point of X is reduced to that point.*

Proof. Let X be strongly pairwise $\gamma - T_2$ and $x \in X$. To each $y \in X$, $x \neq y$, there exist a γ_i-open set G and a γ_j-open set H such that $x \in H$, $y \in G$ and $G \cap H = \phi$. Since $x \in H \subset X \setminus G$, therefore $X \setminus G$ is γ_i-closed γ_j-nbd of x to which y does not belong. Consequently, the intersection of all γ_i-closed γ_j-nbds of X is reduced to $\{x\}$. Conversely, let $x, y \in X$ such that $x \neq y$, then by hypothesis, there exists a γ_i-closed γ_j-nbd U of x such that $y \notin U$. Now, there exists a γ_i-open set G such that $x \in G \subset U$. Thus G is a γ_i-open set, $X \setminus U$ is a γ_j-open set, $x \in G$, $y \in X \setminus U$ and $G \cap X \setminus U = \phi$. Hence X is strongly pairwise $\gamma - T_2$. □

Theorem 3.50. *Let γ be an operation on a bitopological space (X, τ_1, τ_2). Then X is $\gamma_i - T_2$ if and only if the intersection of all γ_i-closed γ_i-nbds of each point of X is reduced to that point.*

Proof. Similar to that of Theorem 3.49. □

Theorem 3.51. *Let γ be a regular operation on a strongly pairwise $\gamma - T_2$ space (X, τ_1, τ_2) and $A \subset X$ a γ_i-compact. Then A is γ_j-closed.*

Proof. If $A = X$, then A is obviously γ_j-closed. If $A \neq X$, then there is a point $x \in X \setminus A$. Since X is strongly pairwise $\gamma - T_2$, for every $y \in A$, there exist a γ_j-open set U_y and a γ_i-open set V_y such that $x \in U_y,\ y \in V_y$ and $U_y \cap V_y = \phi$. Then $\{V_y : y \in A\}$ is a γ_i-open cover of A which is γ_i-compact, then there exists a finite subfamily $V_{y_1}, \ldots, V_{y_n}$ such that $A \subset \bigcup_{r=1}^{n} V_{y_r}$. Let $U = \bigcap_{r=1}^{n} U_{y_r}$ and $V = \bigcup_{r=1}^{n} V_{y_r}$. Since γ is regular, therefore U is a γ_j-open set, V is a γ_i-open set, $x \in U$, $A \subset V$ and $U \cap U = \phi$. Thus $x \in U \subset X \setminus A$ and so $X \setminus A$ is γ_j-open and so A is γ_j-closed. □

Corollary 3.52. *If A is an ij-almost-compact subset of a strongly pairwise $\theta - T_2$ space (X, τ_1, τ_2), then A is $ji - \theta$-closed.*

Corollary 3.53. *If A is an ij-nearly-compact subset of a strongly pairwise $\delta - T_2$ space (X, τ_1, τ_2), then A is $ji - \delta$-closed.*

Theorem 3.54. *Let γ (resp. β) be an operation on a bitopological space (X, τ_1, τ_2) (resp. (Y, σ_1, σ_2)) and $f : X \to Y$ be a pairwise (γ, β)-continuous injection. If Y is pairwise $\beta - T_0$ (resp. weak pairwise $\beta - T_1, \beta_i - T_1$, pairwise $\beta - T_1$, weak pairwise $\beta - T_2, \beta_i - T_2$, pairwise $\beta - T_2$ and strong pairwise $\beta - T_2$), then X is pairwise $\gamma - T_0$ (resp. weak pairwise $\gamma - T_1, \gamma_i - T_1$, pairwise $\gamma - T_1$, weak pairwise $\gamma - T_2, \gamma_i - T_2$, pairwise $\gamma - T_2$ and strong pairwise $\gamma - T_2$).*

Proof. Suppose that Y is strong pairwise $\beta - T_2$ and let $x, y \in X$ such that $x \neq y$. Then there exist a $\sigma_{i\beta}$-open set U and a $\sigma_{j\beta}$-open set V such that $f(x) \in U$, $f(y) \in V$ and $U \cap V = \phi$. Since f is pairwise (γ, β)-continuous, by Proposition 2.21, $f^{-1}(U)$ is $\tau_{i\gamma}$-open and $f^{-1}(V)$ is $\tau_{j\gamma}$-open. Also, $x \in f^{-1}(U), y \in f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \phi$. This show that X is strong pairwise $\gamma - T_2$. The proofs of the other cases are similar. □

For different choices for γ and β in Theorem 3.54, we can write down a lot of results, for example, we have the following

Corollary 3.55. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is pairwise weakly continuous injection and Y is a strong pairwise $\theta - T_2$ space, then X is pairwise Hausdorff.*

Corollary 3.56. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_2, \sigma_2)$ is pairwise almost continuous injection and Y is a strong pairwise $\delta - T_2$ space, then X is pairwise Hausdorff.*

Corollary 3.57. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_2, \sigma_2)$ is pairwise strongly θ-continuous injection and Y is a pairwise Hausdorff space, then X is strong pairwise $\theta - T_2$.*

Corollary 3.58. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_2, \sigma_2)$ is pairwise super continuous injection and Y is a pairwise Hausdorff space, then X is strong pairwise $\delta - T_2$.*

Corollary 3.59. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_2, \sigma_2)$ is pairwise θ-continuous injection and Y is a strong pairwise $\theta - T_2$ space, then X is strong pairwise $\theta - T_2$.*

Corollary 3.60. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_2, \sigma_2)$ is pairwise weakly θ-continuous injection and Y is a strong pairwise $\theta - T_2$ space, then X is strong pairwise $\delta - T_2$.*

Definition 3.61. Let γ be an operation on a bitopological space (X, τ_1, τ_2) and $H \subset X$, then the relative operation $\gamma^* : \tau_1/H \cup \tau_2/H \to P(H)$ is an operation on $(H, \tau_1/H, \tau_2/H)$ defined by $(G \cap H)^{\gamma^*} = G^\gamma \cap H$ for each $G \in \tau_1 \cup \tau_2$, where $\tau_i/H = \{U \cap H : U \in \tau_i\}$.

Lemma 3.62. *Let γ be an operation on a bitopological space (X, τ_1, τ_2), $H \subset X$ and γ^* be the relative operation in the subspace $(H, \tau_1/H, \tau_2/H)$. If U is γ_i-open in X, then $U \cap H$ is γ_i^*-open in H.*

Theorem 3.63. *Every subspace of a pairwise $\gamma - T_0$ (resp. weak pairwise $\gamma - T_1, \gamma_i - T_1$, pairwise $\gamma - T_1$, weak pairwise $\gamma - T_2, \gamma_i - T_2$, pairwise $\gamma - T_2$ and strong pairwise $\gamma - T_2$) is a pairwise $\gamma^* - T_0$ (resp. weak pairwise $\gamma^* - T_1, \gamma_i^* - T_1$, pairwise $\gamma^* - T_1$, weak pairwise $\gamma^* - T_2, \gamma_i^* - T_2$, pairwise $\gamma^* - T_2$ and strong pairwise $\gamma^* - T_2$).*

Proof. Follows directly from Lemma 3.62. □

Remark 3.64. The argument of Theorem 3.36 is true for the cases of weak pairwise $\gamma - T_1, \gamma_i - T_1$, pairwise $\gamma - T_1$, weak pairwise $\gamma - T_2, \gamma_i - T_2$, pairwise $\gamma - T_2$ and strong pairwise $\gamma - T_2$ spaces.

For different choices for γ in Remark 3.64, we can write down a lot of results, for example, we have the following

Corollary 3.65. *A bitopological space* (X, τ_1, τ_2) *is weak pairwise* $\delta - T_1$ *if and only if for any two distinct points* x *and* y *of* X*, there exist a* τ_i*-open set* U *and a* τ_j*-open set* V *such that either* $x \in U, y \notin i - int(j - cl(U))$ *and* $y \in V, x \notin j - int(i - cl(V))$ *or* $y \in U$, $x \notin i - int(j - cl(U))$ *and* $x \in V, y \notin j - int(i - cl(V))$.

Corollary 3.66. *A bitopological space* (X, τ_1, τ_2) *is weak pairwise* $\delta - T_2$ *if and only if for any two distinct points* x *and* y *of* X*, there exist a* τ_i*-open set* U *and a* τ_j*-open set* V *such that either* $x \in U$ *and* $y \in V$ *or* $y \in U$*, and* $x \in V$ *and* $i - int(j - cl(U)) \cap j - int(i - cl(V)) = \phi$.

Corollary 3.67. *A bitopological space* (X, τ_1, τ_2) *is strong pairwise* $\delta - T_2$ *if and only if for any two distinct points* x *and* y *of* X*, there exist a* τ_i*-open set* U *and a* τ_j*-open set* V *such that* $x \in U, y \in V$*, and* $i - int(j - cl(U)) \cap j - int(i - cl(V)) = \phi$.

Theorem 3.68. *Let* γ *be an open and regular operation on a bitopological space* (X, τ_1, τ_2)*. Then* (X, τ_1, τ_2) *is pairwise* $\gamma - T_0$ *(resp. weak pairwise* $\gamma - T_1$*, pairwise* $\gamma - T_1$*, weak pairwise* $\gamma - T_2$ *and strong pairwise* $\gamma - T_2$*) if and only if* $(X, \tau_{1\gamma}, \tau_{2\gamma})$ *is pairwise* T_0 *(resp. weak pairwise* T_1*, pairwise* T_1*, weak pairwise* T_2*, and strong pairwise* T_2*).*

Proof. It is straight forward by Proposition 2.9 and Remark 3.64 □

For different choices for γ in Theorem 3.68, we can write down a lot of results, for example, we have the following

Corollary 3.69. *A bitopological space* (X, τ_1, τ_2) *is pairwise* $\delta - T_0$ *if and only if* $(X, \tau_{1\delta}, \tau_{2\delta})$ *is pairwise* T_0.

Corollary 3.70. *A bitopological space* (X, τ_1, τ_2) *is pairwise* $\delta - T_1$ *if and only if* $(X, \tau_{1\delta}, \tau_{2\delta})$ *is pairwise* T_1.

Corollary 3.71. *A bitopological space* (X, τ_1, τ_2) *is strong pairwise* $\delta - T_2$ *if and only if* $(X, \tau_{1\delta}, \tau_{2\delta})$ *is strong pairwise* T_2.

We finish this section by investigating general operator approaches of closed graphs of mappings.

Definition 3.72. [13] Let (X, τ_1, τ_2) and $(Y, \sigma_, \sigma_2)$ be two bitopological spaces. The cross product of the two spaces X and Y is defined to be the space $(X \times Y, \mu_1, \mu_2)$, where $\mu_i = \tau_i \times \sigma_j$.

Definition 3.73. Let γ (resp. β) be an operation on the bitopological space (X, τ_1, τ_2) (resp. (Y, σ_1, σ_2)). An operation $\rho : \tau_1 \times \sigma_2 \cup \tau_2 \times \sigma_1 \rightarrow P(X \times Y)$ is said to be associated with γ and β if $(U \times V)^\rho = U^\gamma \times V^\beta$ for each $U \in \tau_i$ and $V \in \sigma_j$. ρ is called regular with respect to γ and β if for each $(x,y) \in X \times Y$ and each μ_j-open nbd W of (x,y), there exist a τ_i-open nbd U of x and a σ_j-open nbd V of y such that $W^\rho \subset U^\gamma \times V^\beta$.

Theorem 3.74. *Let ρ be an operation on $X \times X$ associated with γ and γ. If $f : X \to Y$ is a pairwise (γ, β)-continuous mapping and (Y, σ_1, σ_2) is a strong pairwise $\beta - T_2$ space, then the set $A = \{(x,y) \in X \times X : f(x) = f(y)\}$ is a ρ_i-closed subset of $X \times X$.*

Proof. We show that $cl_{i\rho}(A) \subset A$. Let $(x,y) \in X \times X \setminus A$. Then there exist a σ_i-open set U and a σ_j-open set V such that $f(x) \in U, f(y) \in V$ and $U^\beta \cap V^\beta = \phi$. Moreover, there exist a τ_i-open set W and a τ_j-open set S such that $x \in W, y \in S$ and $f(W^\gamma) \subset U^\beta$ and $f(S^\gamma) \subset V^\beta$. Therefore, $(W \times S)^\rho \cap A = \phi$. This shows that $(x,y) \notin cl_{i\rho}(A)$ and so $cl_{i\rho}(A) \subset A$. □

Corollary 3.75. *Let ρ be an operation on $X \times X$ associated with γ and γ which is regular with respect to γ and γ. A space X is strong pairwise $\gamma - T_2$ if and only if the diagonal set $\Delta = \{(x,x) : x \in X\}$ is ρ_i-closed in $X \times Y$.*

Theorem 3.76. *Let ρ be an operation on $X \times Y$ associated with γ and β. If $f : X \to Y$ is a pairwise (γ, β)-continuous mapping and (Y, σ_1, σ_2) is a strong pairwise $\beta - T_2$ space, then the graph of f, $G(f) = \{(x, f(x)) \in X \times Y\}$ is a ρ_i-closed subset of $X \times Y$.*

Proof. The proof is similar to that of Theorem 3.74 □

References

[1] Abd El-Monsef, M.E. Kozae, A.M. and Taher, B.M., Compactification in bitopological spaces, *Arab J. for Sci. Engin.*, 22(1A)(1997), 99–105.

[2] Arya, S.P. and Nour, T.M., Separation axioms for bitopological spaces, *Indian J. Pure Appl. Math.*, 19(1)(1988), 42–50.

[3] Banerjee, G.K., On pairwise almost strongly θ-continuous mappings, *Bull. Calcutta Math. Soc.*, 79(1987), 314–320.

[4] Bose, S. and Sinha, D., Almost open, almost closed, θ-continuous and almost quasi compact mappings in bitopological spaces, *Bull. Calcutta Math. Soc.*, 73(1981), 345–356.

[5] Bose, S. and Sinha, D., Pairwise almost continuous map and weakly continuous map in bitopological spaces, *Bull. Calcutta Math. Soc.*, 74(1982), 195–205.

[6] Fletcher, P., Hoyle, H.B. III and Patty, C.W., The comparison of topologies, *Duke Math. J.*, 36(2)(1969), 325–331.

[7] Kelly, J.C., Bitopological spaces, *Proc. London Math. Soc.*, (3)13(1963), 7–89.

[8] Khedr, F.H., Operations on bitopological spaces (submitted).

[9] Khedr, F.H. and Al-Shibani, A.M., On pairwise super continuous mappings in bitoplogical spaces, *Internat. J. Math. Math. Sci.*, (4)14(1991), 715–722.

[10] Misra, D., N., and Dube, K.K., Pairwise R_0-spaces, *Ann de la Soc. Sci*, Bruxells, T. 87(1)(1973), 3–15.

[11] Mukherjee, M.N., On pairwise almost compactness and pairwise H-closedness in a bitopologicla space, *Ann. Soc. Sci. Bruxelles*, T. 96, 2(1982), 98–106.

[12] Murdeshware, M.G. and Namipally, S.A., Quasi-uniform topological spaces, *Monograph, P. Noordoff Ltd.* (1966).

[13] Nandi, J.N. and Mukherjee, M.N., Bitopological δ-cluster sts and some applications, *Indiana J. Pure Appl. Math.*, 29(8)(1998), 837–848.

[14] Reilly, I.L., On bitopological separation properties, *Nanta Math.*, 5(2)(1972), 14–25.

[15] Smithson, R.E., Mutlifunctions and bitopological spaces, *J. Nat. Sci. Math.*, 11(2)(1971), 191–198.

[16] Swart, J., Total disconnectedness in bitopological spaces and product bitopological spaces, *Proc. Kon. Ned. Akad Wetench A 74 = Indag. Math.*, 33(2)(1971), 135–145.

Faculty of Science
Department of Mathematics
P.O. Box 838, Dammam 31113, Kingdom of Saudi Arabia
Email: s.alareefi@yahoo.com