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Univalence preserving integral operators defined by generalized Al-Oboudi differential operators

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Abstract

In this paper, we investigate sufficient conditions for the univalence of an integral operator defined by generalized Al-Oboudi differential operator.

1 Introduction

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, and $\mathcal{S} = \{f \in \mathcal{A} : f \text{ is univalent in } \mathbb{U}\}$.

The following definition of fractional derivative by Owa [8] (also by Srivastava and Owa [14]) will be required in our investigation.

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The fractional derivative of order γ is defined, for a function f, by

$$D_z^\gamma f(z) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^\gamma} d\xi \quad (0 \leq \gamma < 1), \tag{1.2}$$

where the function f is analytic in a simply connected region of the complex z-plane containing the origin, and the multiplicity of $(z-\xi)^{-\gamma}$ is removed by requiring $\log(z-\xi)$ to be real when $z-\xi > 0$.

It readily follows from (1.2) that

$$D_z^\gamma z^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\gamma)} z^{k-\gamma} \quad (0 \leq \gamma < 1,\ k \in \mathbb{N} = \{1, 2, \ldots\}).$$

Using $D_z^\gamma f$, Owa and Srivastava [9] introduced the operator $\Omega^\gamma : \mathcal{A} \to \mathcal{A}$, which is known as an extension of fractional derivative and fractional integral, as follows:

$$\begin{aligned} \Omega^\gamma f(z) &= \Gamma(2-\gamma) z^\gamma D_z^\gamma f(z), \quad \gamma \neq 2, 3, 4, \ldots \\ &= z + \sum_{k=2}^\infty \frac{\Gamma(k+1)\Gamma(2-\gamma)}{\Gamma(k+1-\gamma)} a_k z^k. \end{aligned} \tag{1.3}$$

Note that

$$\Omega^0 f(z) = f(z).$$

In [3], Al-Oboudi and Al-Amoudi defined the linear multiplier fractional differential operator $D_\lambda^{n,\gamma}$ as follows:

$$\begin{aligned} D^0 f(z) &= f(z), \\ D_\lambda^{1,\gamma} f(z) &= (1-\lambda)\,\Omega^\gamma f(z) + \lambda z\,(\Omega^\gamma f(z))' \\ &= D_\lambda^\gamma (f(z)), \quad \lambda \geq 0,\ 0 \leq \gamma < 1, \qquad (1.4) \\ D_\lambda^{2,\gamma} f(z) &= D_\lambda^\gamma \left(D_\lambda^{1,\gamma} f(z) \right), \\ &\vdots \\ D_\lambda^{n,\gamma} f(z) &= D_\lambda^\gamma \left(D_\lambda^{n-1,\gamma} f(z) \right), \quad n \in \mathbb{N}. \qquad (1.5) \end{aligned}$$

If f is given by (1.1), then by (1.3), (1.4) and (1.5), we see that

$$D_\lambda^{n,\gamma} f(z) = z + \sum_{k=2}^\infty \Psi_{k,n}(\gamma, \lambda)\, a_k z^k, \quad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \tag{1.6}$$

where

$$\Psi_{k,n}(\gamma,\lambda)=\left[\frac{\Gamma(k+1)\Gamma(2-\gamma)}{\Gamma(k+1-\gamma)}\left(1+(k-1)\lambda\right)\right]^{n}. \tag{1.7}$$

Remark 1.1. (**i**) When $\gamma=0$, we get Al-Oboudi differential operator [2].

(**ii**) When $\gamma=0$ and $\lambda=1$, we get Sălăgean differential operator [13].

(**iii**) When $n=1$ and $\lambda=0$, we get Owa-Srivastava fractional differential operator [9].

By using the generalized Al-Oboudi differential operator $D_\lambda^{n,\gamma}$, we introduce the following integral operator:

Definition 1.1 *Let $n\in\mathbb{N}_0, m\in\mathbb{N}$, $\beta\in\mathbb{C}$ with $\Re(\beta)>0$ and $\alpha_i\in\mathbb{C}$ $(i\in\{1,\dots,m\})$. We define the integral operator*

$$I_\beta^{n,\gamma}(f_1,\dots,f_m):\mathcal{A}^m\to\mathcal{A},$$

$$I_\beta^{n,\gamma}(f_1,\dots,f_m)(z)=\left\{\beta\int_0^z t^{\beta-1}\prod_{i=1}^{m}\left(\frac{D_\lambda^{n,\gamma}f_i(t)}{t}\right)^{\alpha_i}dt\right\}^{\frac{1}{\beta}}\quad(z\in\mathbb{U}), \tag{1.8}$$

where $D_\lambda^{n,\gamma}$ is the generalized Al-Oboudi differential operator.

Remark 1.2. (**i**) For $m\in\mathbb{N}$, $\beta\in\mathbb{C}$, $\Re(\beta)>0$, $\alpha_i\in\mathbb{C}$ and $D_\lambda^{0,\gamma}f_i(z)=D_0^{1,0}f_i(z)=f_i(z)\in\mathcal{S}$ $(i\in\{1,\dots,m\})$, we have the integral operator

$$I_\beta(f_1,\dots,f_m)(z)=\left\{\beta\int_0^z t^{\beta-1}\prod_{i=1}^{m}\left(\frac{f_i(t)}{t}\right)^{\alpha_i}dt\right\}^{\frac{1}{\beta}}$$

which was introduced in [4].

(**ii**) For $m\in\mathbb{N}$, $\beta=1$, $\alpha_i\in\mathbb{C}$ and $D_\lambda^{0,\gamma}f_i(z)=D_0^{1,0}f_i(z)=f_i(z)\in\mathcal{S}$ $(i\in\{1,\dots,m\})$, we have the integral operator

$$I(f_1,\dots,f_m)(z)=\int_0^z\left(\frac{f_1(t)}{t}\right)^{\alpha_1}\cdots\left(\frac{f_m(t)}{t}\right)^{\alpha_m}dt$$

which was studied in [4].

(**iii**) For $n\in\mathbb{N}_0$, $m\in\mathbb{N}$, $\beta=1$, $\alpha_i\in\mathbb{C}$ and $D_\lambda^{n,0}f_i(z)=D^nf_i(z)$ $(i\in\{1,\dots,m\})$, we have the integral operator

$$I^n(f_1,\dots,f_m)(z)=\int_0^z\left(\frac{D^nf_1(t)}{t}\right)^{\alpha_1}\cdots\left(\frac{D^nf_m(t)}{t}\right)^{\alpha_m}dt$$

which was studied in [5].

(**iv**) For $n = 0$, $m = 1$, $\beta = 1$, $\alpha_1 = 1$, $\alpha_2 = \alpha_3 = \cdots = \alpha_m = 0$ and $D_\lambda^{0,\gamma} f_1(z) = D_0^{1,0} f_1(z) = f(z) \in \mathcal{A}$, we have Alexander integral operator

$$I(f)(z) = \int_0^z \frac{f(t)}{t} dt$$

which was introduced in [1].

(**v**) For $n = 0$, $m = 1$, $\beta = 1$, $\alpha_1 = \alpha \in [0,1]$, $\alpha_2 = \alpha_3 = \cdots = \alpha_m = 0$ and $D_\lambda^{0,\gamma} f_1(z) = D_0^{1,0} f_1(z) = f(z) \in \mathcal{S}$, we have the integral operator

$$I(f)(z) = \int_0^z \left(\frac{f(t)}{t} \right)^\alpha dt$$

which was studied in [6].

To discuss our problems, we have to recall here the following results.

General Schwarz Lemma [7]**.** *Let the function f be regular in the disk $\mathbb{U}_R = \{z \in \mathbb{C} : |z| < R\}$, with $|f(z)| < M$ for fixed M. If $f(z)$ has one zero with multiplicity order bigger than m for $z = 0$, then*

$$|f(z)| \leq \frac{M}{R^m} |z|^m \quad (z \in \mathbb{U}_R).$$

The equality can hold only if

$$f(z) = e^{i\theta} (M/R^m) z^m,$$

where θ is constant.

Theorem A [10]**.** *Let α be a complex number with $\Re(\alpha) > 0$ and $f \in \mathcal{A}$. If $f(z)$ satisfies*

$$\frac{1 - |z|^{2\Re(\alpha)}}{\Re(\alpha)} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1,$$

for all $z \in \mathbb{U}$, then the integral operator

$$F_\alpha(z) = \left\{ \alpha \int_0^z t^{\alpha-1} f'(t) dt \right\}^{\frac{1}{\alpha}}$$

is in the class $\mathcal{S}$.

Theorem B [11]**.** *Let α be a complex number with $\Re(\alpha) > 0$ and $f \in \mathcal{A}$. If $f(z)$ satisfies*

$$\frac{1 - |z|^{2\Re(\alpha)}}{\Re(\alpha)} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1 \qquad (z \in \mathbb{U}),$$

then, for any complex number β with $\Re(\beta) \geq \Re(\alpha)$, the integral operator

$$F_\beta(z) = \left\{ \beta \int_0^z t^{\beta - 1} f'(t) dt \right\}^{\frac{1}{\beta}}$$

is in the class $\mathcal{S}$.

Theorem C [12]**.** *Let β be a complex number with $\Re(\beta) > 0$, c a complex number with $|c| \leq 1$, $c \neq -1$, and $f(z)$ given by (1.1) an analytic function in $\mathbb{U}$. If*

$$\left| c\,|z|^{2\beta} + (1 - |z|^{2\beta}) \frac{zf''(z)}{\beta f'(z)} \right| \leq 1$$

for all $z \in \mathbb{U}$, then the function

$$F_\beta(z) = \left\{ \beta \int_0^z t^{\beta - 1} f'(t) dt \right\}^{\frac{1}{\beta}} = z + \cdots$$

is analytic and univalent in $\mathbb{U}$.

2 Main Results

Theorem 2.1 *Let $\alpha_1, \ldots, \alpha_m, \beta \in \mathbb{C}$ and each of the functions $f_i \in \mathcal{A}$ $(i \in \{1, \ldots, m\})$. If*

$$\left| \frac{z\left(D_\lambda^{n,\gamma} f_i(z)\right)'}{D_\lambda^{n,\gamma} f_i(z)} - 1 \right| \leq 1 \quad (z \in \mathbb{U}, n \in \mathbb{N}_0)$$

and

$$\Re(\beta) \geq \sum_{i=1}^{m} |\alpha_i| > 0,$$

then the integral operator $I_\beta^{n,\gamma}(f_1, \ldots, f_m)$ defined by (1.8) is in the univalent function class $\mathcal{S}$.

Proof. Since $f_i \in \mathcal{A}$ $(i \in \{1, \ldots, m\})$, by (1.6), we have

$$\frac{D_\lambda^{n,\gamma} f_i(z)}{z} = 1 + \sum_{k=2}^{\infty} \Psi_{k,n}(\gamma, \lambda)\, a_{k,i} z^{k-1} \quad (n \in \mathbb{N}_0)$$

and

$$\frac{D_\lambda^{n,\gamma} f_i(z)}{z} \neq 0$$

for all $z \in \mathbb{U}$.

Let us define

$$h(z) = \int_0^z \prod_{i=1}^{m} \left(\frac{D_\lambda^{n,\gamma} f_i(t)}{t} \right)^{\alpha_i} dt,$$

so that, obviously,

$$h'(z) = \left(\frac{D_\lambda^{n,\gamma} f_1(z)}{z} \right)^{\alpha_1} \cdots \left(\frac{D_\lambda^{n,\gamma} f_m(z)}{z} \right)^{\alpha_m}$$

for all $z \in \mathbb{U}$. This equality implies that

$$\ln h'(z) = \alpha_1 \ln \frac{D_\lambda^{n,\gamma} f_1(z)}{z} + \cdots + \alpha_m \ln \frac{D_\lambda^{n,\gamma} f_m(z)}{z}$$

or equivalently

$$\ln h'(z) = \alpha_1 \left[\ln D_\lambda^{n,\gamma} f_1(z) - \ln z \right] + \cdots + \alpha_m \left[\ln D_\lambda^{n,\gamma} f_m(z) - \ln z \right].$$

By differentiating above equality, we get

$$\frac{h''(z)}{h'(z)} = \sum_{i=1}^{m} \alpha_i \left[\frac{\left(D_\lambda^{n,\gamma} f_i(z)\right)'}{D_\lambda^{n,\gamma} f_i(z)} - \frac{1}{z} \right].$$

Hence, we obtain from this equality that

$$\left| \frac{zh''(z)}{h'(z)} \right| \leq \sum_{i=1}^{m} |\alpha_i| \left| \frac{z\left(D_\lambda^{n,\gamma} f_i(z)\right)'}{D_\lambda^{n,\gamma} f_i(z)} - 1 \right|.$$

So by the conditions of the Theorem 2.1, we find

$$\begin{aligned} \frac{1 - |z|^{2\Re(\beta)}}{\Re(\beta)} \left| \frac{zh''(z)}{h'(z)} \right| &\leq \frac{1 - |z|^{2\Re(\beta)}}{\Re(\beta)} \sum_{i=1}^{m} |\alpha_i| \left| \frac{z\left(D_\lambda^{n,\gamma} f_i(z)\right)'}{D_\lambda^{n,\gamma} f_i(z)} - 1 \right| \\ &\leq \frac{1}{\Re(\beta)} \sum_{i=1}^{m} |\alpha_i| \leq 1. \end{aligned}$$

Finally, applying Theorem A for the function $h(z)$, we prove that $I_\beta^{n,\gamma}(f_1,\ldots,f_m) \in \mathcal{S}$.

Remark 2.1. If we set $\beta = 1$ and $\gamma = 0$ in Theorem 2.1, then we have Theorem 2.3 in [5].

Corollary 2.2 *Let $\alpha_i > 0$, $\beta \in \mathbb{C}$ and each of the functions $f_i \in \mathcal{A}$ $(i \in \{1,\ldots,m\})$. If*

$$\left|\frac{z\left(D_\lambda^{n,\gamma} f_i(z)\right)'}{D_\lambda^{n,\gamma} f_i(z)} - 1\right| \leq 1 \quad (z \in \mathbb{U}, n \in \mathbb{N}_0)$$

and

$$\Re(\beta) \geq \sum_{i=1}^{m} \alpha_i,$$

then the integral operator $I_\beta^{n,\gamma}(f_1,\ldots,f_m)$ defined by (1.8) *is in the univalent function class $\mathcal{S}$.*

Remark 2.2. If we set $\beta = 1$ and $\gamma = 0$ in Corollary 2.2, then we have Corollary 2.5 in [5].

Theorem 2.3 *Let $M_i \geq 1$ and suppose that each of the functions $f_i \in \mathcal{A}$ $(i \in \{1,\ldots,m\}, m \in \mathbb{N})$ satisfies the inequality*

$$\left|\frac{z^2\left(D_\lambda^{n,\gamma} f_i(z)\right)'}{\left(D_\lambda^{n,\gamma} f_i(z)\right)^2} - 1\right| \leq 1 \quad (z \in \mathbb{U}, n \in \mathbb{N}_0).$$

Also let $\alpha_1,\ldots,\alpha_m, \alpha \in \mathbb{C}$ with

$$\Re(\alpha) \geq \sum_{i=1}^{m} |\alpha_i| \left(2M_i + 1\right) > 0.$$

If

$$|D_\lambda^{n,\gamma} f_i(z)| \leq M_i \ (z \in \mathbb{U};\ i \in \{1,\ldots,m\}),$$

then, for any complex number β with $\Re(\beta) \geq \Re(\alpha)$, the integral operator $I_\beta^{n,\gamma}(f_1,\ldots,f_m)$ defined by (1.8) *is in the univalent function class $\mathcal{S}$.*

Proof. We know from the proof of Theorem 2.1 that

$$\left|\frac{zh''(z)}{h'(z)}\right| \leq \sum_{i=1}^{m} |\alpha_i| \left|\frac{z\left(D_\lambda^{n,\gamma} f_i(z)\right)'}{D_\lambda^{n,\gamma} f_i(z)} - 1\right|.$$

So, by the imposed conditions, we find

$$\begin{aligned}
\frac{1-|z|^{2\Re(\alpha)}}{\Re(\alpha)}\left|\frac{zh''(z)}{h'(z)}\right| &\le \frac{1-|z|^{2\Re(\alpha)}}{\Re(\alpha)}\sum_{i=1}^{m}|\alpha_i|\left(\left|\frac{z\left(D_\lambda^{n,\gamma}f_i(z)\right)'}{D_\lambda^{n,\gamma}f_i(z)}\right|+1\right)\\
&\le \frac{1-|z|^{2\Re(\alpha)}}{\Re(\alpha)}\sum_{i=1}^{m}|\alpha_i|\left(\left|\frac{z^2\left(D_\lambda^{n,\gamma}f_i(z)\right)'}{\left(D_\lambda^{n,\gamma}f_i(z)\right)^2}\right|\left|\frac{D_\lambda^{n,\gamma}f_i(z)}{z}\right|+1\right)\\
&\le \frac{1-|z|^{2\Re(\alpha)}}{\Re(\alpha)}\sum_{i=1}^{m}|\alpha_i|\left(\left|\frac{z^2\left(D_\lambda^{n,\gamma}f_i(z)\right)'}{\left(D_\lambda^{n,\gamma}f_i(z)\right)^2}-1\right|M_i+M_i+1\right)\\
&\le \frac{1}{\Re(\alpha)}\sum_{i=1}^{m}|\alpha_i|\left(2M_i+1\right)\le 1
\end{aligned}$$

By applying Theorem B for the function $h(z)$, we prove that $I_\beta^{n,\gamma}(f_1,\ldots,f_m)\in\mathcal{S}$.

Corollary 2.4 *Let $M_i\ge 1$, $\alpha_i>0$ and suppose that each of the functions $f_i\in\mathcal{A}$ $(i\in\{1,\ldots,m\})$ satisfies the inequality*

$$\left|\frac{z^2\left(D_\lambda^{n,\gamma}f_i(z)\right)'}{\left(D_\lambda^{n,\gamma}f_i(z)\right)^2}-1\right|\le 1 \quad (z\in\mathbb{U}, n\in\mathbb{N}_0).$$

Also let $\alpha\in\mathbb{C}$ with

$$\Re(\alpha)\ge\sum_{i=1}^{m}\alpha_i\left(2M_i+1\right).$$

If

$$|D_\lambda^{n,\gamma}f_i(z)|\le M_i \ (z\in\mathbb{U};\ i\in\{1,\ldots,m\}),$$

then, for any complex number β with $\Re(\beta)\ge\Re(\alpha)$, the integral operator $I_\beta^{n,\gamma}(f_1,\ldots,f_m)$ defined by (1.8) is in the univalent function class $\mathcal{S}$.

Corollary 2.5 *Let $M\ge 1$ and suppose that each of the functions $f_i\in\mathcal{A}$ $(i\in\{1,\ldots,m\}, m\in\mathbb{N})$ satisfies the inequality*

$$\left|\frac{z^2\left(D^n f_i(z)\right)'}{\left(D^n f_i(z)\right)^2}-1\right|\le 1 \quad (z\in\mathbb{U}, n\in\mathbb{N}_0).$$

Also let $\alpha_1,\ldots,\alpha_m,\alpha\in\mathbb{C}$ with

$$\Re(\alpha)\ge(2M+1)\sum_{i=1}^{m}|\alpha_i|>0.$$

If

$$|D^n f_i(z)| \leq M \ (z \in \mathbb{U};\ i \in \{1, \ldots, m\}),$$

then, for any complex number β *with* $\Re(\beta) \geq \Re(\alpha)$, *the integral operator* $I_\beta^{n,\gamma}(f_1, \ldots, f_m)$ *defined by* (1.8) *is in the univalent function class* $\mathcal{S}$.

Proof. In Theorem 2.3, we consider $M_1 = \cdots = M_m = M$.

Corollary 2.6 *Suppose that each of the functions* $f_i \in \mathcal{A}$ $(i \in \{1, \ldots, m\})$ *satisfies the inequality*

$$\left| \frac{z^2 \left(D_\lambda^{n,\gamma} f_i(z)\right)'}{\left(D_\lambda^{n,\gamma} f_i(z)\right)^2} - 1 \right| \leq 1 \quad (z \in \mathbb{U}, n \in \mathbb{N}_0).$$

Also let $\alpha_1, \ldots, \alpha_m, \alpha \in \mathbb{C}$ *with*

$$\Re(\alpha) \geq 3 \sum_{i=1}^{m} |\alpha_i| > 0.$$

If

$$|D_\lambda^{n,\gamma} f_i(z)| \leq 1 \ (z \in \mathbb{U};\ i \in \{1, \ldots, m\}),$$

then, for any complex number β *with* $\Re(\beta) \geq \Re(\alpha)$, *the integral operator* $I_\beta^{n,\gamma}(f_1, \ldots, f_m)$ *defined by* (1.8) *is in the univalent function class* $\mathcal{S}$.

Proof. In Corollary 2.5, we consider $M = 1$.

Remark 2.3. In Corollary 2.6, if we set

(**i**) $\beta = 1$ and $\gamma = 0$, then we have Theorem 2.6,

(**ii**) $\beta = 1$, $\gamma = 0$ and $\alpha_i > 0$ $(i \in \{1, \ldots, m\})$, then we have Corollary 2.8 in [5].

Theorem 2.7 *Suppose that each of the functions* $f_i \in \mathcal{A}$ $(i \in \{1, \ldots, m\})$ *satisfies the inequality*

$$\left| \frac{z \left(D_\lambda^{n,\gamma} f_i(z)\right)'}{D_\lambda^{n,\gamma} f_i(z)} - 1 \right| \leq 1 \quad (z \in \mathbb{U}, n \in \mathbb{N}_0).$$

Also let $\alpha_1, \ldots, \alpha_m, \beta \in \mathbb{C}$ *with*

$$\Re(\beta) \geq \sum_{i=1}^{m} |\alpha_i| > 0$$

and let $c \in \mathbb{C}$ be such that

$$|c| \leq 1 - \frac{1}{\Re(\beta)} \sum_{i=1}^{m} |\alpha_i| \,.$$

Then the integral operator $I_{\beta}^{n,\gamma}(f_1, \ldots, f_m)$ defined by (1.8) *is in the univalent function class $\mathcal{S}$.*

Proof. We know from the proof of Theorem 2.1 that

$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^{m} \alpha_i \left[\frac{z\left(D_{\lambda}^{n,\gamma} f_i(z)\right)'}{D_{\lambda}^{n,\gamma} f_i(z)} - 1 \right].$$

So we obtain

$$\begin{aligned}
\left| c\,|z|^{2\beta} + (1-|z|^{2\beta}) \frac{zh''(z)}{\beta h'(z)} \right| &= \left| c\,|z|^{2\beta} + (1-|z|^{2\beta}) \frac{1}{\beta} \sum_{i=1}^{m} \alpha_i \left[\frac{z\left(D_{\lambda}^{n,\gamma} f_i(z)\right)'}{D_{\lambda}^{n,\gamma} f_i(z)} - 1 \right] \right| \\
&\leq |c| + \left| \frac{1-|z|^{2\beta}}{\beta} \right| \sum_{i=1}^{m} |\alpha_i| \left| \frac{z\left(D_{\lambda}^{n,\gamma} f_i(z)\right)'}{D_{\lambda}^{n,\gamma} f_i(z)} - 1 \right| \\
&\leq |c| + \frac{1}{|\beta|} \sum_{i=1}^{m} |\alpha_i| \\
&\leq |c| + \frac{1}{\Re(\beta)} \sum_{i=1}^{m} |\alpha_i| \leq 1.
\end{aligned}$$

Finally, applying Theorem C for the function $h(z)$, we prove that $I_{\beta}^{n,\gamma}(f_1, \ldots, f_m) \in \mathcal{S}$.

Corollary 2.8 *Suppose that each of the functions $f_i \in \mathcal{A}$ $(i \in \{1, \ldots, m\})$ satisfies the inequality*

$$\left| \frac{z\left(D_{\lambda}^{n,\gamma} f_i(z)\right)'}{D_{\lambda}^{n,\gamma} f_i(z)} - 1 \right| \leq 1 \quad (z \in \mathbb{U}, n \in \mathbb{N}_0).$$

Also let $\alpha_i > 0, \beta \in \mathbb{C}$ with

$$\Re(\beta) \geq \sum_{i=1}^{m} \alpha_i$$

and let $c \in \mathbb{C}$ be such that

$$|c| \leq 1 - \frac{1}{\Re(\beta)} \sum_{i=1}^{m} \alpha_i.$$

Then the integral operator $I_\beta^{n,\gamma}(f_1,\ldots,f_m)$ *defined by* (1.8) *is in the univalent function class* $\mathcal{S}$.

Theorem 2.9 *Let* $M_i \geq 1$ *and suppose that each of the functions* $f_i \in \mathcal{A}$ $(i \in \{1,\ldots,m\})$ *satisfies the inequality*

$$\left| \frac{z^2\left(D_\lambda^{n,\gamma} f_i(z)\right)'}{\left(D_\lambda^{n,\gamma} f_i(z)\right)^2} - 1 \right| \leq 1 \quad (z \in \mathbb{U}, n \in \mathbb{N}_0).$$

Also let $\alpha_1,\ldots,\alpha_m,\beta \in \mathbb{C}$ *with*

$$\Re(\beta) \geq \sum_{i=1}^{m} |\alpha_i|\,(2M_i+1) > 0,$$

$c \in \mathbb{C}$ *be such that*

$$|c| \leq 1 - \frac{1}{\Re(\beta)} \sum_{i=1}^{m} |\alpha_i|\,(2M_i+1)$$

and

$$|D_\lambda^{n,\gamma} f_i(z)| \leq M_i \ (z \in \mathbb{U};\ i \in \{1,\ldots,m\}).$$

Then the integral operator $I_\beta^{n,\gamma}(f_1,\ldots,f_m)$ *defined by* (1.8) *is in the univalent function class* $\mathcal{S}$.

Proof. We know from the proof of Theorem 2.1 that

$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^{m} \alpha_i \left[\frac{z\left(D_\lambda^{n,\gamma} f_i(z)\right)'}{D_\lambda^{n,\gamma} f_i(z)} - 1 \right].$$

So we obtain

$$\begin{aligned}
\left| c\,|z|^{2\beta} + (1-|z|^{2\beta}) \frac{zh''(z)}{\beta h'(z)} \right| &= \left| c\,|z|^{2\beta} + (1-|z|^{2\beta}) \frac{1}{\beta} \sum_{i=1}^{m} \alpha_i \left[\frac{z\left(D_\lambda^{n,\gamma} f_i(z)\right)'}{D_\lambda^{n,\gamma} f_i(z)} - 1 \right] \right| \\
&\leq |c| + \left| \frac{1-|z|^{2\beta}}{\beta} \right| \sum_{i=1}^{m} |\alpha_i| \left(\left| \frac{z\left(D_\lambda^{n,\gamma} f_i(z)\right)'}{D_\lambda^{n,\gamma} f_i(z)} \right| + 1 \right) \\
&\leq |c| + \frac{1}{|\beta|} \sum_{i=1}^{m} |\alpha_i| \left(\left| \frac{z^2\left(D_\lambda^{n,\gamma} f_i(z)\right)'}{\left(D_\lambda^{n,\gamma} f_i(z)\right)^2} \right| \left| \frac{D_\lambda^{n,\gamma} f_i(z)}{z} \right| + 1 \right) \\
&\leq |c| + \frac{1}{|\beta|} \sum_{i=1}^{m} |\alpha_i| \left(\left| \frac{z^2\left(D_\lambda^{n,\gamma} f_i(z)\right)'}{\left(D_\lambda^{n,\gamma} f_i(z)\right)^2} - 1 \right| M_i + M_i + 1 \right)
\end{aligned}$$

$$\leq \quad |c| + \frac{1}{\Re(\beta)} \sum_{i=1}^{m} |\alpha_i| \left(2M_i + 1\right) \leq 1.$$

Finally, applying Theorem C for the function $h(z)$, we prove that $I_{\beta}^{n,\gamma}(f_1, \ldots, f_m) \in \mathcal{S}$.

Corollary 2.10 *Let $M_i \geq 1$, $\alpha_i > 0$ and suppose that each of the functions $f_i \in \mathcal{A}$ $(i \in \{1, \ldots, m\})$ satisfies the inequality*

$$\left| \frac{z^2 \left(D_{\lambda}^{n,\gamma} f_i(z)\right)'}{\left(D_{\lambda}^{n,\gamma} f_i(z)\right)^2} - 1 \right| \leq 1 \quad (z \in \mathbb{U}, n \in \mathbb{N}_0).$$

Also let $\beta \in \mathbb{C}$ with

$$\Re(\beta) \geq \sum_{i=1}^{m} \alpha_i \left(2M_i + 1\right),$$

$c \in \mathbb{C}$ be such that

$$|c| \leq 1 - \frac{1}{\Re(\beta)} \sum_{i=1}^{m} \alpha_i \left(2M_i + 1\right)$$

and

$$|D_{\lambda}^{n,\gamma} f_i(z)| \leq M_i \ (z \in \mathbb{U};\ i \in \{1, \ldots, m\}).$$

Then the integral operator $I_{\beta}^{n,\gamma}(f_1, \ldots, f_m)$ defined by (1.8) *is in the univalent function class $\mathcal{S}$.*

Corollary 2.11 *Let $M \geq 1$ and suppose that each of the functions $f_i \in \mathcal{A}$ $(i \in \{1, \ldots, m\})$ satisfies the inequality*

$$\left| \frac{z^2 \left(D_{\lambda}^{n,\gamma} f_i(z)\right)'}{\left(D_{\lambda}^{n,\gamma} f_i(z)\right)^2} - 1 \right| \leq 1 \quad (z \in \mathbb{U}, n \in \mathbb{N}_0).$$

Also let $\alpha_1, \ldots, \alpha_m, \beta \in \mathbb{C}$ with

$$\Re(\beta) \geq (2M + 1) \sum_{i=1}^{m} |\alpha_i| > 0,$$

$c \in \mathbb{C}$ be such that

$$|c| \leq 1 - \frac{2M + 1}{\Re(\beta)} \sum_{i=1}^{m} |\alpha_i|$$

and

$$|D_\lambda^{n,\gamma} f_i(z)| \leq M \ (z \in \mathbb{U};\ i \in \{1,\ldots,m\}).$$

Then the integral operator $I_\beta^{n,\gamma}(f_1,\ldots,f_m)$ *defined by* (1.8) *is in the univalent function class* $\mathcal{S}$.

Proof. In Theorem 2.9, we consider $M_1 = \cdots = M_m = M$.

Corollary 2.12 *Suppose that each of the functions* $f_i \in \mathcal{A}$ $(i \in \{1,\ldots,m\})$ *satisfies the inequality*

$$\left| \frac{z^2 \left(D_\lambda^{n,\gamma} f_i(z)\right)'}{\left(D_\lambda^{n,\gamma} f_i(z)\right)^2} - 1 \right| \leq 1 \quad (z \in \mathbb{U}, n \in \mathbb{N}_0).$$

Also let $\alpha_1,\ldots,\alpha_m,\beta \in \mathbb{C}$ *with*

$$\Re(\beta) \geq 3\sum_{i=1}^{m} |\alpha_i| > 0,$$

$c \in \mathbb{C}$ *be such that*

$$|c| \leq 1 - \frac{3}{\Re(\beta)} \sum_{i=1}^{m} |\alpha_i|$$

and

$$|D_\lambda^{n,\gamma} f_i(z)| \leq 1 \ (z \in \mathbb{U};\ i \in \{1,\ldots,m\}).$$

Then the integral operator $I_\beta^{n,\gamma}(f_1,\ldots,f_m)$ *defined by* (1.8) *is in the univalent function class* $\mathcal{S}$.

Proof. In Corollary 2.11, we consider $M = 1$.

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