



## On the Cohomology with local coefficients of Pure Braid Groups

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### Abstract

The need to calculate local system cohomology of the complement of a hyperplane arrangement arises in various contexts. Nevertheless until now very few it's known on the direct computation of such cohomology. In this paper the author describes all generators of the first homology group of braid arrangement giving also a complete description of its first characteristic variety, fulfilling results obtained by D. Cohen and A. Suciu in [7]. Moreover she gives also a complete description of the first characteristic variety in the case of the generalized braid arrangement coming from the dihedral group.

### Introduction

Let  $\mathcal{A} = \{H_1, \dots, H_n\}$  be a hyperplane arrangement in  $\mathbb{C}^l$ , with complement

$$M = M(\mathcal{A}) = \mathbb{C}^l \setminus \bigcup_{j=1}^n H_j.$$

Let  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$  be a collection of weights. Associated to  $\lambda$ , we have a rank one representation  $\rho : \pi_1(M) \rightarrow \mathbb{C}^*$  given by  $c_j \mapsto t_j = \exp(-2\pi i \lambda_j)$  for any meridian loop  $c_j$  about the hyperplane  $H_j$  of  $\mathcal{A}$ , and an associated rank one local system  $\mathcal{L}$  on  $M$ . The need to calculate the local system cohomology  $H^*(M; \mathcal{L})$  arises in various contexts. For instance, such local systems may be

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used to study the Milnor fiber of the non-isolated hypersurface singularity at the origin obtained by coning the arrangement, see [6, 5]. In mathematical physics, local systems on complements of arrangements arise in the Aomoto-Gelfand theory of multivariable hypergeometric integrals [1, 11, 16] and the representation theory of Lie algebras and quantum groups. These considerations lead to solutions of the Knizhnik-Zamolodchikov differential equation from conformal field theory, see [20, 23].

In light of these applications, and others, the cohomology  $H^*(M(\mathcal{A}), \mathcal{L})$  has been the subject of considerable recent interest. Call the local system  $\mathcal{L}$  *nonresonant* if this cohomology is concentrated in dimension  $n$ , that is,  $H^k(M(\mathcal{A}), \mathcal{L}) = 0$  for  $k \neq n$ . Necessary conditions for vanishing, or non-resonance, have been determined by a number of authors, including Esnault, Schectman, and Viehweg [9], Kohno [12], and Schechtman, Terao, and Varchenko [21].

In 1980, Orlik and Solomon gave a simple combinatorial description of the  $\mathbb{k}$ -algebra  $H^*(X, \mathbb{k})$ , for any field  $\mathbb{k}$ : it is the quotient  $A = E/I$  of the exterior algebra  $E$  on classes dual to the meridians, modulo a certain ideal  $I$  determined by the intersection poset, see [14, 15].

For each  $a \in A^1 \cong \mathbb{k}^n$ , the Orlik-Solomon algebra can be turned into a cochain complex  $(A, a)$ , with  $i$ -th term the degree  $i$  graded piece of  $A$ , and with differential given by multiplication by  $a$ , cf. [24]. The *resonance varieties* of  $A$  were defined in [10] to be the jumping loci for the cohomology of this cochain complex:

$$R_d^i(A) = \{a \in A^1 \mid \dim_{\mathbb{k}} H^i(A, a) \geq d\}. \quad (1)$$

The *characteristic varieties* of a space  $X$  are the jumping loci for the cohomology of  $X$  with coefficients in rank 1 local systems:

$$V_d^i(X) = \{\mathfrak{t} \in \text{Hom}(\pi_1(X), \mathbb{C}^*) \mid \dim_{\mathbb{C}} H^i(X, \mathbb{C}_{\mathfrak{t}}) \geq d\}, \quad (2)$$

where  $\mathbb{C}_{\mathfrak{t}}$  denotes the abelian group  $\mathbb{C}$ , with  $\pi_1(X)$ -module structure given by the representation  $\mathfrak{t}: \pi_1(X) \rightarrow \mathbb{C}^*$ .

Now suppose  $X$  is the complement of an arrangement of  $n$  hyperplanes. By work of Arapura [2], the irreducible components of the characteristic varieties of  $X$  are algebraic subtori of the character torus  $\text{Hom}(\pi_1(X), \mathbb{C}^*) \cong (\mathbb{C}^*)^n$ , possibly translated by unitary characters. It turns out that the tangent cone at the origin to  $V_d^i(X)$  coincides with the resonance variety  $R_d^i(A)$ , see [7, 13, 4]. Consequently, the resonance varieties are unions of linear subspaces; moreover, the algebraic subtori in the characteristic varieties are determined by the intersection lattice.

Until now there aren't a lot of computational results about characteristic varieties. Author considers the particular case given by the braid arrange-

ments, i.e. arrangements  $\mathcal{A}(\mathbf{W})$  obtained complexifying the reflection hyperplanes related to a finite Coxeter system  $\mathbf{W}$ .

These arrangements has been studied in a lot of occasion and their local cohomologies are equivalent to the cohomologies of the *pure braid groups* or *pure Artin groups*.

In this paper the author gives a complete description of generators of the first homology group with local coefficients of the braid arrangement retrieving, as consequence, the first characteristic variety of this arrangement. These results agree with (partial) results obtained by D. Cohen and A. Suciu in [7].

The author concludes this first part with a conjecture about the characteristic varieties of braid arrangement.

Moreover she computes in an original way the homology with local coefficients (equivalently the first characteristic variety) of the complement of the arrangement  $\mathcal{A}(\mathbf{I}_2(m))$ . These results agree with the one in [7].

## 1. First characteristic variety of braid arrangement

### 1.1. Salvetti's Complex for reflection arrangements.

Let  $\mathbf{W}$  be a finite group generated by reflections in the affine space  $\mathbb{A}^n(\mathbb{R})$ . Let  $\overline{\mathcal{A}}(\mathbf{W}) = \{H_j\}_{j \in J}$  be the arrangement in  $\mathbb{A}^n$  defined by the reflection hyperplanes of  $\mathbf{W}$ . We need to recall briefly some notations and results from [17] for the particular case of Coxeter arrangements, i.e. arrangements coming from Coxeter Systems  $(\mathbf{W}, S)$  (see [3]).  $\overline{\mathcal{A}}(\mathbf{W})$  induces a stratification  $\mathcal{S} = \mathcal{S}(\mathbf{W})$  of  $\mathbb{A}^n$  into facets (see [3]). The set  $\mathcal{S}$  is partially ordered by  $F > F'$  iff  $F' \subset \text{cl}(F)$ . We shall indicate by  $\mathbf{Q} = \mathbf{Q}(\mathbf{W})$  the cellular complex which is *dual* to  $\mathcal{S}$ . In a standard way, this can be realized inside  $\mathbb{A}^n$  by baricentric subdivision of the facets: inside each codimension  $j$  facet  $F^j$  of  $\mathcal{S}$  choose one point  $v(F^j)$  and consider the simplexes

$$s(F^{i_0}, \dots, F^{i_j}) = \left\{ \sum_{k=0}^j \lambda_k v(F^{i_k}) : \sum_{k=0}^j \lambda_k = 1, \lambda_k \in [0, 1] \right\}$$

where  $F^{i_{k+1}} < F^{i_k}$ ,  $k = 0, \dots, j-1$ . The dimension  $j$  cell  $e^j(\overline{F}^j)$  which is dual to  $\overline{F}^j$  is obtained by taking the union

$$\cup_s(F^0, \dots, F^{j-1}, \overline{F}^j)$$

over all chains  $\overline{F}^j < F^{j_1} < \dots < F^0$ . So  $\mathbf{Q} = \cup e^j(F^j)$ , the union being over all facets of  $\mathcal{S}$ .

One can think of the 1 – *skeleton*  $\mathbf{Q}_1$  as a graph (with vertex-set the 0 – *skeleton*  $\mathbf{Q}_0$ ) and can define the combinatorial distance between two vertices  $v, v'$  as the minimum number of edges in an edge-path connecting  $v$  and  $v'$ .

For each cell  $e^j$  of  $\mathbf{Q}$  one indicates by  $V(e^j) = \mathbf{Q}_0 \cap e^j$  the 0-*skeleton* of  $e^j$ . One has that given a vertex  $v \in \mathbf{Q}_0$  and a cell  $e^i \in \mathbf{Q}$ , there is a unique vertex  $\underline{w}(v, e^i) \in V(e^i)$  with the lowest combinatorial distance from  $v$ , i.e.:

$$d(v, \underline{w}(v, e^i)) < d(v, v') \text{ if } v' \in V(e^i) \setminus \{\underline{w}(v, e^i)\}.$$

If  $e^j \subset e^i$  then  $\underline{w}(v, e^j) = \underline{w}(\underline{w}(v, e^i), e^j)$ .

Let now  $\mathcal{A}(\mathbf{W})$  denote the *complexification* of  $\overline{\mathcal{A}}(\mathbf{W})$ , and  $\mathbf{Y}(\mathbf{W}) = \mathbb{C}^n \setminus \cup_{j \in J} H_{j, \mathbb{C}}$  the complement of the complexified arrangement. Then  $\mathbf{Y}(\mathbf{W})$  is homotopy equivalent to the complex  $\mathbf{X}(\mathbf{W})$  which is constructed as follows (see [17]).

Take a cell  $e^j = e^j(F^j) = \cup_s(F^0, \dots, F^{j-1}, F^j)$  of  $\mathbf{Q}$  as defined above and let  $v \in V(e^j)$ . Embed each simplex  $s(F^0, \dots, F^j)$  into  $\mathbb{C}^n$  by the formula

$$\begin{aligned} \phi_{v, e^j} \left( \sum_{k=0}^j \lambda_k v(F^k) \right) = \\ \sum_{k=0}^j \lambda_k v(F^k) + i \sum_{k=0}^j \lambda_k (\underline{w}(v, e^k) - v(F^k)). \end{aligned} \tag{3}$$

It is shown in [17] (see also [18]):

- (i) the preceding formula defines an embedding of  $e^j$  into  $\mathbf{Y}(\mathbf{W})$ ;
- (ii) if  $E^j = E^j(v, e^j)$  is its image, then varying  $e^j$  and  $v$  one obtains a cellular complex

$$\mathbf{X}(\mathbf{W}) = \cup E^j$$

which is homotopy equivalent to  $\mathbf{Y}(\mathbf{W})$ .

The previous result allows us to make cohomological computations over  $\mathbf{Y}(\mathbf{W})$  by using the complex  $\mathbf{X}(\mathbf{W})$ .

In [18] (see also [8]) the authors give a new combinatorial description of the stratification  $\mathcal{S}$  where the action of  $\mathbf{W}$  is more explicit. They prove that if  $S$  is the set of reflections with respect to the walls of the fixed base chamber  $C_0$ , then a cell in  $\mathbf{X}(\mathbf{W})$  is of the form  $E = E(w, \Gamma)$  with  $\Gamma \subset S$  and  $w \in \mathbf{W}$ . The action of  $\mathbf{W}$  is written as

$$\sigma.E(w, \Gamma) = E(\sigma w, \Gamma), \tag{4}$$

where the factor  $\sigma.w$  is just multiplication in  $\mathbf{W}$ .

We prefer at the moment to deal with chain complexes and boundary operator coming from  $\mathbf{X}(\mathbf{W})$  instead of cochain and coboundary. Indeed we will see that, in our case, it is simple to deduce cohomological results from homological ones.

We define a rank-1 local system on  $\mathbf{Y}(\mathbf{W})$  with coefficients in an unitary ring  $A$  by assigning an unit  $\tau_j = \tau(H_j)$  (thought as a multiplicative operator) to each hyperplane  $H_j \in \mathcal{A}$ . Call  $\bar{\tau}$  the collection of  $\tau_j$  and  $\mathcal{L}_{\bar{\tau}}$  the corresponding local system. Let  $C(\mathbf{W}, \mathcal{L}_{\bar{\tau}})$  be the free graduated  $A$ -module with basis all  $E(w, \Gamma)$  (see [17]).

We use the natural identification between the elements of the group and the vertices of  $\mathbf{Q}_0$ , given by  $w \leftrightarrow w.v_0$ . Here  $v_0 \in \mathbf{Q}_0$  is contained in the fixed base chamber  $C_0$ .

Then  $u(w, w')$  will denote the “minimal positive path” joining the corresponding vertices  $v$  and  $v'$  in the 1-skeleton  $\mathbf{X}(\mathbf{W})_1$  of  $\mathbf{X}(\mathbf{W})$  (see [17]).

The local system  $\mathcal{L}_{\bar{\tau}}$  defines for each edge-path  $c$  in  $\mathbf{X}(\mathbf{W})_1$ ,  $c : w \rightarrow w'$  an isomorphism  $c_* : A \rightarrow A$  such that for all  $d : w \rightarrow w'$  homotopic to  $c$ ,  $c_* = d_*$  and for all  $f : w'' \rightarrow w$ ,  $(cf)_* = c_*f_*$ .

Then the set  $\{s_0(w).E(w, \Gamma)\}_{|\Gamma|=k}$ , where  $s_0(w) := u(1, w)_*(1)$ , is a linear basis of  $C_k(\mathbf{W}, \mathcal{L}_{\bar{\tau}})$ .

Let now  $T = \{wsw^{-1} | s \in S, w \in \mathbf{W}\}$ , the set of reflections in  $\mathbf{W}$  and

$$\overline{\mathbf{W}} = \{\mathbf{s}(w) = (s_{i_1}, \dots, s_{i_q}) | w = s_{i_1} \cdots s_{i_q} \in \mathbf{W}\},$$

then for each  $\mathbf{s}(w) \in \overline{\mathbf{W}}$  and  $t \in T$ , we set

- i)  $\Psi(\mathbf{s}(w)) = (t_{i_1}, \dots, t_{i_q})$  with  $t_{i_j} = (s_{i_1} \cdots s_{i_{j-1}})s_{i_j}(s_{i_1} \cdots s_{i_{j-1}})^{-1} \in T$
- ii)  $\overline{\Psi(\mathbf{s}(w))} = \{t_{i_1}, \dots, t_{i_q}\}$
- iii)  $\eta(w, t) = (-1)^{n(\mathbf{s}(w), t)}$  with  $n(\mathbf{s}(w), t) = \#\{j | 1 \leq j \leq q \text{ and } t_{i_j} = t\}$ .

Moreover if  $t \in T$  is the reflection relative to the hyperplane  $H$ , then we set  $\tau(t) = \tau(H)$ .

We define

$$\begin{aligned} \partial_k(s_0(w).E(w, \Gamma)) = \\ \sum_{\sigma \in \Gamma} \sum_{\beta \in \mathbf{W}_{\Gamma}^{\Gamma \setminus \{\sigma\}}} (-1)^{l(\beta) + \mu(\Gamma, \sigma)} \tau(w, \beta) s_0(w\beta).E(w\beta, \Gamma \setminus \{\sigma\}). \end{aligned} \quad (5)$$

where  $\tau(w, \beta) = \prod_{\substack{t \in \overline{\Psi(\mathbf{s}(w))} \\ \eta(w, t) = 1}} \tau(t)$ , and  $\mu(\Gamma, \sigma) = \#\{i \in \Gamma | i \leq \sigma\}$ .

Remark that the action (4) is extended to  $\tau(w, \beta)$  in the obvious way

$$\sigma.\tau(w, \beta) = \tau(\sigma w, \beta). \quad (6)$$

This boundary map computes  $H_*(\mathbf{X}(\mathbf{W}), \mathcal{L}_{\bar{\tau}})$ . A similar result holds for cohomology.

Moreover let  $\mathbf{W}_{\Gamma}$  be the parabolic subgroup of  $\mathbf{W}$  generated by  $\Gamma$  and  $\mathbf{W}^{\Gamma} = \{w \in \mathbf{W} \mid \ell(ws) > \ell(w) \text{ for all } s \in \Gamma\}$ . Each  $w \in \mathbf{W}$  can be write as a product  $w = w^{\Gamma}w_{\Gamma}$  with  $w^{\Gamma} \in \mathbf{W}^{\Gamma}$  and  $w_{\Gamma} \in \mathbf{W}_{\Gamma}$ . From (5) it follows:

$$\partial(E(w, \Gamma)) = w^{\Gamma}.\partial(E(w_{\Gamma}, \Gamma)) \quad (7)$$

where the action (4) is extended to  $C(\mathbf{W})$  by linearity.

### 1.2. Reduced boundary operator for Salvetti's complex

Given the  $n$ -dimensional Salvetti's complex  $C(\mathbf{W})$  associated to the complexified arrangement  $\mathcal{A}(\mathbf{W})$ , we have an homotopically equivalent complex  $\bar{C}(\mathbf{W})$  obtained by  $C(\mathbf{W})$  simply contracting to a point the  $n$ -dimensional cell  $E(1, S)$ , where  $1 \in \mathbf{W}$  is the identity and  $S$  is the system of generators of  $\mathbf{W}$ .

This contraction is equivalent to contract to a point all 0-cells of  $C(\mathbf{W})$  and all  $k$ -dimensional cells  $E(w, \Gamma)$  such that  $w \in \mathbf{W}^{\Gamma}$ .

Then, clearly, the new boundary operator  $\bar{\partial}$  on this contracted complex can be obtained by the old one contracting all  $k$ -dimensional cells  $E(w, \Gamma)$ , for  $k > 0$ , to 0, i.e. they disappear from the boundary, and all 0-cells coincide with the same 0-cell  $E(\emptyset)$ .

In order to simplify computations, from now on we will deal with this new contracted complex.

### 1.3. Generators of the first homology group with local coefficients of the braid arrangement.

Let  $\mathbf{W} = \mathbf{A}_n$  be the symmetric group, then  $\mathcal{A}(\mathbf{A}_n)$  is the braid arrangement. The set of generators of  $\mathbf{A}_n$  will be  $S = \{s_1, \dots, s_n\}$ .

In this paragraph we give a complete description of generators for the first homology group  $H_1(\mathbf{Y}(\mathbf{A}_n), \mathcal{L})$ .

Let us remark that the contracted 1-skeleton  $\bar{C}_1(\mathbf{A}_n)$  will be given by all 1-dimensional cells of the form  $E(ws_i, \{s_i\})$  for  $w \in W^{s_i}$  and  $s_i \in S$ . While the boundary of a 1-dimensional cell  $E(ws_i, \{s_i\})$  will be:

$$\bar{\partial}E(ws_i, \{s_i\}) = (1 - \tau_{ws_iw^{-1}})E(\emptyset) \quad (8)$$

where  $\tau_{ws_iw^{-1}}$  is the weight relative to the reflection hyperplane  $H_{ws_iw^{-1}}$  related to reflection  $ws_iw^{-1}$ .

Then the kernel of  $\bar{\partial}_0$  can be represented as follow:

Let us define

$$\begin{aligned} c_{w,w'}^{i,j} &= E(w, \{s_i\}) - E(w', \{s_j\}) \text{ if } \tau_{ws_i w^{-1}} = \tau_{w' s_j w'^{-1}} \\ d_{w,w'}^{i,j} &= (1 - \tau_{w' s_j w'^{-1}})E(w, \{s_i\}) - (1 - \tau_{ws_i w^{-1}})E(w', \{s_j\}) \text{ if } \tau_{ws_i w^{-1}} \neq \tau_{w' s_j w'^{-1}} \end{aligned} \quad (9)$$

for  $1 \leq i, j \leq n$ ,  $w, w' \in \mathbf{A}_n$  and  $l(ws_i) < l(w)$ ,  $l(w' s_j) < l(w')$ . They verify the following relations:

$$\begin{aligned} i) & c_{w,w'}^{i,j} + c_{w',w''}^{j,k} = c_{w,w''}^{i,k} \\ ii) & d_{w,w'}^{i,j} = -d_{w',w}^{j,i} \\ iii) & d_{w,w'}^{i,j} = (1 - \tau_{w' s_j w'^{-1}})c_{w,w''}^{i,k} + d_{w'',w'}^{k,j} \text{ if } \tau_{ws_i w^{-1}} = \tau_{w'' s_k w''^{-1}} \\ iv) & (1 - \tau_{w'' s_k w''^{-1}})d_{w,w'}^{i,j} + (1 - \tau_{ws_i w^{-1}})d_{w',w''}^{j,k} = (1 - \tau_{w' s_j w'^{-1}})d_{w,w''}^{i,k} \end{aligned} \quad (10)$$

which are a complete system of relations, i.e. all relations verified by  $c_{w,w'}^{i,j}$  and  $d_{w,w'}^{i,j}$ .

By (8) all element in the kernel of  $\bar{\partial}_0$  are a combination of  $c_{w,w'}^{i,j}$  and  $d_{w,w'}^{i,j}$ .

It follows that the kernel of  $\bar{\partial}_0$  is generated by  $c_{w,w'}^{i,j}$  and  $d_{w,w'}^{i,j}$  with relations (10).

Another interesting remark will simplify our computations: by (7) elements  $c_{w,w'}^{i,j}$  can be wrote as a sum of  $c_{\bar{w}t, \bar{w}'t'}$  for  $t, t' \in W_{\{s_i, s_j\}}$  and  $\bar{w}, \bar{w}' \in W_{\{s_i, s_j\}}$ . Moreover, by relation (10) *i*), we obtain that all  $c_{\bar{w}t, \bar{w}'t'}$  can be write as a sum of elements  $c_{\bar{w}t, \bar{w}'t'}^{h,k}$  such that  $\bar{w} = \bar{w}'$ .

Similarly, from (7) and (10) *ii*), *iii*),  $d_{w,w'}^{i,j}$  are equivalents to a sum of generators of the form  $d_{\bar{w}t, \bar{w}'t'}^{h,k}$  and  $c_{\bar{w}t, \bar{w}'t'}^{h,k}$ , with  $t, t' \in W_{\{s_h, s_k\}}$  and  $\bar{w} \in W_{\{s_h, s_k\}}$  for some  $s_h, s_k \in S_n$ .

Then we can choose as representatives generators of the form  $c_{\bar{w}t, \bar{w}'t'}^{i,j}$  and  $d_{\bar{w}t, \bar{w}'t'}^{i,j}$ .

Moreover, using (4), it is possible to write the image of  $\bar{\partial}_1$  in terms of these generators, computing generators of the first homology group and their relations.

Indeed, with the above notations, let  $E(w, \{s_i, s_j\})$  be  $E(\bar{w}t, \{s_i, s_j\}) \in \bar{C}(\mathbf{W})$  with  $j - i > 1$ , i.e.  $s_i$  and  $s_j$  commute. Then, by direct computations, we obtain:

$$\bar{\partial}_1(E(\bar{w}t, \{s_i, s_j\})) = \begin{cases} 0 & t = 1 \\ c_{\bar{w}s_i, \bar{w}s_j}^{i,i} & t = s_i \\ -c_{\bar{w}s_j, \bar{w}s_i}^{j,j} & t = s_j \\ d_{\bar{w}s_i, \bar{w}s_j}^{i,j} - c_{\bar{w}s_i, \bar{w}s_j}^{i,i} + c_{\bar{w}s_j, \bar{w}s_i}^{j,j} & t = s_i s_j \end{cases} \quad (11)$$

While if  $s_j = s_{i+1}$  then:

$$\bar{\partial}_1(E(\bar{w}t, \{s_i, s_{i+1}\})) = \begin{cases} 0, & t = 1 \\ c_{\bar{w}s_i, \bar{w}s_{i+1}s_i}^{i,i+1}, & t = s_i \\ -c_{\bar{w}s_{i+1}, \bar{w}s_{i+1}s_i}^{i+1,i}, & t = s_{i+1} \\ d_{\bar{w}s_i, \bar{w}s_{i+1}s_i}^{i,i} - c_{\bar{w}s_i, \bar{w}s_{i+1}s_i}^{i,i+1} - \\ c_{\bar{w}s_{i+1}, \bar{w}s_{i+1}s_i}^{i+1,i}, & t = s_i s_{i+1} \\ -d_{\bar{w}s_{i+1}, \bar{w}s_{i+1}s_i}^{i+1,i+1} + \\ c_{\bar{w}s_{i+1}, \bar{w}s_{i+1}s_i}^{i+1,i} - \\ c_{\bar{w}s_i s_{i+1}, \bar{w}s_{i+1}s_i}^{i+1,i}, & t = s_{i+1}s_i \\ T_{\bar{w}s_i s_{i+1}s_i} \bar{w}^{-1} \\ d_{\bar{w}s_{i+1}, \bar{w}s_i}^{i+1,i} - \\ d_{\bar{w}s_i, \bar{w}s_{i+1}s_i}^{j,i} + \\ c_{\bar{w}s_i, \bar{w}s_{i+1}s_i}^{i,i+1} + \\ c_{\bar{w}s_i s_{i+1}, \bar{w}s_{i+1}s_i}^{i+1,i} \\ + d_{\bar{w}s_{i+1}, \bar{w}s_{i+1}s_i}^{i+1,i+1} - \\ c_{\bar{w}s_{i+1}, \bar{w}s_{i+1}s_i}^{i+1,i}, & t = s_i s_{i+1} s_i \end{cases} \quad (12)$$

The boundaries (11) and (12) give rise to new relations on  $c_{w,w'}^{i,j}$  and  $d_{w,w'}^{i,j}$  in the first homology group.

We will call *commutative relations* those in (11), *non commutative relations* those in (12).

**Theorem 1** *With the above notations all generators of type  $d_{w,w'}^{i,j}$ ,  $d_{w,w'}^{i,i}$  and  $c_{w,w'}^{i,j}$ ,  $c_{w,w'}^{i,i}$  with  $|j - i| > 1$ , i.e. the commutative case, disappear in  $H_1(C(\mathbf{A}_n), \mathcal{L})$ .*

**Proof** The proof come from the fact that, in the commutative case, all generators  $d_{w,w'}^{i,j}$  which verify  $l(ws_i) < l(w)$ ,  $l(w's_j) < l(w')$  and  $ws_i w^{-1} \neq w's_j w'^{-1}$  are of the form  $d_{\bar{w}s_i, \bar{w}s_j}^{i,j}$  with  $i \neq j$ , for a fixed  $\bar{w} \in W^{\{s_i, s_j\}}$ .

Similarly all generators  $c_{w,w'}^{i,j}$  which verify  $l(ws_i) < l(w)$ ,  $l(w's_j) < l(w')$  and

$ws_iw^{-1} = w's_jw'^{-1}$  are of the form  $c_{\bar{w}s_i, \bar{w}s_j s_i}^{i,i}$  for a fixed  $\bar{w} \in W^{\{s_i, s_j\}}$ .

But all these generators are on the image of  $\bar{\partial}_1$  (see (11)) and then are 0 in homology. This conclude the proof  $\square$

**Theorem 2** *With the above notations all generators in  $H_1(C(\mathbf{A}_n), \mathcal{L})$  are of the form  $d_{\bar{w}s_{i+1}, \bar{w}s_i}^{i+1,i}$  for  $i \in \{1, \dots, n-1\}$  and  $\bar{w} \in W^{\{s_i, s_{i+1}\}}$ .*

**Proof** By theorem 1 we have to deal only with generators coming from the 1-boundary of the non commutative case (12). As in the proof of theorem 1 we have that, in the commutative case, all generators  $c_{w,w'}^{i,j}$  which verify  $l(ws_i) < l(w)$ ,  $l(w's_j) < l(w')$  and  $ws_iw^{-1} = w's_jw'^{-1}$  are of the form  $c_{\bar{w}s_i, \bar{w}s_{i+1}s_i s_{i+1}}^{i,i+1}$ ,  $c_{\bar{w}s_{i+1}, \bar{w}s_{i+1}s_i s_{i+1}}^{i+1,i}$  or  $c_{\bar{w}s_i s_{i+1}, \bar{w}s_{i+1}s_i}^{i+1,i}$ . The first two are in the image of  $\bar{\partial}_1$  (see (12)) and then are 0 in the homology.

Similarly all generators  $d_{w,w'}^{i,j}$  which verify  $l(ws_i) < l(w)$ ,  $l(w's_j) < l(w')$  and  $ws_iw^{-1} \neq w's_jw'^{-1}$  are of the form  $d_{\bar{w}s_i, \bar{w}s_{i+1}s_i}^{i,i}$ ,  $d_{\bar{w}s_{i+1}, \bar{w}s_i s_{i+1}}^{i+1,i+1}$  or  $d_{\bar{w}s_{i+1}, \bar{w}s_i}^{i+1,i}$ . By the above considerations on the  $c_{w,w'}^{i,j}$  and by (12) we have that the first two elements are in the image of  $\bar{\partial}_1$ , while the generator  $c_{\bar{w}s_i s_{i+1}, \bar{w}s_{i+1}s_i}^{i+1,i}$  is equal, in  $H_1(C(\mathbf{A}_n), \mathcal{L})$ , to  $\tau_{\bar{w}s_i s_{i+1}s_i} \bar{w}^{-1} d_{\bar{w}s_{i+1}, \bar{w}s_i}^{i+1,i}$ . This conclude the proof  $\square$

Clearly these generators are not free. Then we have to study all relations for the  $d_{\bar{w}s_{i+1}, \bar{w}s_i}^{i+1,i}$  coming from (10).

**Example 1** *In  $\mathbf{A}_2$ , with generators  $S = \{s_1, s_2\}$ , we obtain that  $H_1(\mathcal{A}(\mathbf{A}_2), \mathcal{L}_{\bar{\tau}})$  is generated by  $d_{s_1, s_2}^{1,2}$  with the only relation  $(1 - \tau_{s_1} \tau_{s_2} \tau_{s_1 s_2 s_1}) d_{s_1, s_2}^{1,2} = 0$  :*

*In this case the only generator  $d_{s_1, s_2}^{1,2}$  has to satisfies relation (10) iv), i.e.*

$$(1 - \tau_{s_1 s_2 s_1}) d_{s_2, s_1}^{2,1} + (1 - \tau_{s_2}) d_{s_1, s_2 s_1}^{1,1} - (1 - \tau_{s_1}) d_{s_2, s_2 s_1}^{2,1} = 0.$$

*We have that  $d_{s_1, s_2 s_1}^{1,1} = \tau_{s_1 s_2 s_1} d_{s_2, s_1}^{2,1}$  and, by relation (10) iii),  $d_{s_2, s_2 s_1}^{2,1} = -\tau_{s_2} \tau_{s_1 s_2 s_1} d_{s_2, s_1}^{2,1}$ . Then we obtain  $(1 - \tau_{s_1} \tau_{s_2} \tau_{s_1 s_2 s_1}) d_{s_1, s_2}^{1,2} = 0$ .*

The boundary of an element  $E(w^\Gamma w_\Gamma, \Gamma)$  of the Salvetti's complex strongly depends on  $w_\Gamma$ . Then the 2-boundary depends on  $W_{\{s_i, s_j\}}$ , i.e. on the multiplicity  $m(s_i, s_j)$  which is the minimum integer such that  $s_i s_j^{m(s_i, s_j)} = 1$  (see [3]).

Then in  $\mathbf{A}_n$  we have only two kind of relations, the ones coming from the

boundary of copies of  $\mathbf{A}_2$ , i.e. (12), and the others coming from the commutative case  $m(s_i, s_j) = 2$ , i.e. (11).

In order to better understand the computations below it is important to recall some basic facts on the Coxeter group  $\mathbf{A}_n$  which correspond to the symmetric group  $\mathcal{S}_{n+1}$  on  $n + 1$  integers.

The generators  $S = \{s_1, \dots, s_n\}$  of  $\mathbf{A}_n$  correspond to transpositions  $s_i = (i, i + 1)$  and reflection hyperplanes correspond to elements  $w \in \mathbf{A}_n$  such that  $w^2 = 1$ , i.e. to transpositions  $(i, j) = s_i s_{i+1} \dots s_{j-1} s_j s_{j-1} \dots s_{i+1} s_i$ .

Hyperplanes corresponding to commutative reflections are perpendicular.

In  $\mathbf{A}_n$  we have  $\binom{n+1}{k+1}$  copies of  $\mathbf{A}_k$  obtained considering all possible subsets  $\{i_1, \dots, i_{k+1}\} \subset \{1, \dots, n + 1\}$ . The generators will be elements of order 2 which correspond to the  $k$  transpositions  $(i_j, i_{j+1})$ .

Moreover each element of the symmetric group  $\mathcal{S}_{n+1}$  acts on  $\mathbf{A}_n$  in the obvious way and, by (4) and (6), this action can be extended to generators  $d_{\overline{w}s_{i+1}, \overline{w}s_i}^{i+1, i}$  and their relations. While (7) allows us to conclude that, if we have a relation  $R$  on generators in  $H_1(C(\mathbf{A}_n), \mathcal{L})$  then these generators will verify all relation coming from  $R$  by symmetries.

**Example 2** *By above considerations, we can notice that in  $\mathcal{A}(\mathbf{A}_3)$  we have  $\binom{4}{3}$  copies of  $\mathcal{A}(\mathbf{A}_2)$  which give rise to the same relations of  $\mathcal{A}(\mathbf{A}_2)$  and 16 new non commutative relations:*

*By theorem 2 we obtain four generators  $d_{ws_1, ws_2}^{1,2}$  for  $w \in \mathbf{A}_3^{\{s_1, s_2\}}$ . By direct computations we have two kind of relations. The first one:*

$$(1 - \tau_{s_3})(\tau_{s_1 s_2 s_1} - \tau_{s_2 s_3 s_2})d_{s_1, s_2}^{1,2} = 0 \quad (13)$$

*coming from (10) iv) applied to the commutative case, i.e. the case in which index  $i$  and  $k$  in (10) iv) verify  $|i - k| > 1$ . In this case by theorem 1 the second element of the equality  $d_{w, w'}^{i, k}$  is 0.*

*The second one:*

$$(1 - \tau_{s_1} \tau_{s_2} \tau_{s_1 s_2 s_1})d_{s_1, s_2}^{1,2} = 0. \quad (14)$$

*coming from (10) iv) applied to the non commutative case.*

*Applying symmetries to these relations we obtain that in  $H_1(C(\mathbf{A}_3), \mathcal{L})$  we have twelve relations of the form (13) and height of the form (14).*

The interesting case is  $\mathbf{A}_4$ . Indeed by theorem 1 we know that if  $|j - i| > 1$  then  $d_{wt, wt'}^{i, j} \in \text{Im} \partial_1$ , i.e. it is 0 in homology. It follows that (10) iv) applied to indices  $i = 1, j = 2, k = 4$  gives rise to  $\sharp \mathbf{A}_4^{\{s_1, s_2, s_4\}} = 10$  new relations of

the form:

$$(1 - \tau_{ws_4w^{-1}})d_{ws_1,ws_2}^{1,2} = 0 \tag{15}$$

for  $w \in \mathbf{A}_4^{\{s_1,s_2,s_4\}}$ .

Moreover generators of  $H_1(C(\mathbf{A}_4), \mathcal{L})$  have to satisfy relations of the form (13) and (14). By direct computations it follows that all the above relations are satisfied iff at list four weights  $\tau_{ws_4w^{-1}}$  relative to different hyperplanes are equal to 1 and the remaining weights satisfy (13) and (14).

By direct computations (using symmetries) we have that this is possible iff the remaining weights are related to a copy of  $\mathbf{A}_3$  in  $\mathbf{A}_4$  (i.e. they are weights of reflections hyperplanes in a copy of  $\mathbf{A}_3$ ) or to disjoint copies of  $\mathbf{A}_2$  in  $\mathbf{A}_4$ . Then all relations involving generators of  $H_1(C(\mathbf{A}_4), \mathcal{L})$  comes from relations involving generators inside copies of  $H_1(C(\mathbf{A}_3), \mathcal{L})$  and  $H_1(C(\mathbf{A}_2), \mathcal{L})$  in  $H_1(C(\mathbf{A}_4), \mathcal{L})$ . In order to better understand how to immerge copies of  $H_1(C(\mathbf{A}_k), \mathcal{L})$  inside  $H_1(C(\mathbf{A}_n), \mathcal{L})$  for  $k < n$  see [22].

**Theorem 3** *With the above notations, all relations for generators in  $H_1(C(\mathbf{A}_n), \mathcal{L})$  are obtained by symmetries from relations of the form (13), (14) and (15).*

**Proof** The proof follows noticing that in  $\mathcal{A}(\mathbf{A}_n)$  for  $n > 4$  all 3-uples of hyperplanes which are not in the same copy of  $\mathcal{A}(\mathbf{A}_4)$  are perpendiculars, i.e. related to commutative reflections.

Then there aren't new relations except the ones of the form (13), (14) and (15)  $\square$

*1.4. The first characteristic variety of braid arrangement*

As seen in the introduction, the *characteristic varieties* of a space  $X$  are the jumping loci for the cohomology of  $X$  with coefficients in rank 1 local systems:

$$V_d^i(X) = \{\mathbf{t} \in \text{Hom}(\pi_1(X), \mathbb{C}^*) \mid \dim_{\mathbb{C}} H^i(X, \mathbb{C}_{\mathbf{t}}) \geq d\}, \tag{16}$$

where  $\mathbb{C}_{\mathbf{t}}$  denotes the abelian group  $\mathbb{C}$ , with  $\pi_1(X)$ -module structure given by the representation  $\mathbf{t}: \pi_1(X) \rightarrow \mathbb{C}^*$ .

Here  $X$  is the complement  $\mathbf{Y}(\mathbf{A}_n)$  of the braid arrangement  $\mathbf{A}_n$  and  $\mathcal{L}$  is the rank one local system.

In [7] D. Cohen and A. Suciuc compute the first central characteristic subvariety of the braid arrangement, i.e. the subvariety consisting of all irreducible components of the characteristic variety  $V^1$  passing through 1. We can now

complete their work.

Let us remark that, with the notations used in (5), the coboundary operator on the Salvetti's complex for Coxeter arrangements is defined as:

$$d_k(s_0(w).E(w, \Gamma)) = \sum_{\sigma \in S \setminus \Gamma} \sum_{\beta \in \mathbf{W}_{\Gamma \cup \{\sigma\}}^\Gamma} (-1)^{l(\beta) + \mu(\Gamma, \sigma)} \tau(w, \beta) s_0(w\beta).E(w\beta, \Gamma \cup \{\sigma\}).$$

As simple consequence the coboundary matrix on  $C(\mathbf{W})$  is simply the transpose of the boundary matrix on  $C(\mathbf{W})$ . It follows that all relations between weights which change the dimension of the first homology group, will change also the dimension of the first cohomology one.

Let  $\mathbf{W} = \mathbf{A}_n$  be the symmetric group and  $\mathcal{A}(\mathbf{A}_n)$  the braid arrangement. Given  $\bar{\tau} \in (\mathbb{C}^*)^{n(n+1)/2}$  collection of unit  $\tau_j = \tau(H_j)$  for each  $H_j \in \mathcal{A}(\mathbf{A}_n)$ , a 3-uple  $(\tau_{i_1}, \tau_{i_2}, \tau_{i_3})$  is of type  $\mathbf{A}_2$  if the sub-arrangement  $\{H_{i_1}, H_{i_2}, H_{i_3}\} \subset \mathcal{A}(\mathbf{A}_n)$  is a copy of  $\mathcal{A}(\mathbf{A}_2)$ ; a 6-uple is of type  $\mathbf{A}_3$  if it is related to a sub-arrangement which is a copy of  $\mathbf{A}_3$ .

Let us define for all  $\sigma \in \mathbf{A}_n$

$$(1 - \sigma.\tau_{s_3})(\sigma.\tau_{s_1s_2s_1} - \sigma.\tau_{s_2s_3s_2}) = 0 \tag{17}$$

$$(1 - \sigma.\tau_{s_1}\sigma.\tau_{s_2}\sigma.\tau_{s_1s_2s_1}) = 0 \tag{18}$$

where the action of  $\sigma$  is defined in (6).

**Theorem 4** *A collection of unit  $\bar{\tau} \in (\mathbb{C}^*)^{n(n+1)/2}$  is in the first characteristic variety of the braid arrangement  $\mathbf{A}_n$  iff one of the following assertion is true:*

*i) there are one or more disjoint 3-uples of type  $\mathbf{A}_2$  which verify relations (18) and all other entries are 1;*

*ii) there are one or more disjoint 6-uples of type  $\mathbf{A}_3$  which verify relations (18), (17) and all other entries are 1;*

*iii) there are one or more disjoint 6-uples of type  $\mathbf{A}_3$  and 3-uples of type  $\mathbf{A}_2$  which verify, respectively, relations (18), (17) and relations (18) and all other entries are 1.*

**Proof** The proof comes directly from theorem 3 □

**Remark 1** *An interesting remark is that all computations in this section are related to the multiplicity  $m(s_i, s_j)$  of generators in the Coxeter group  $\mathbf{W}$ . Then they can be performed for all Coxeter group in a similar way. We leave such computations to the interested reader.*

**Conjecture:** In [22] the author considers the cohomology of the complement of braid arrangements with coefficients in the module  $R = \mathbb{Q}[\tau, \tau^{-1}]$  proving that these cohomologies stabilize, with respect to the natural inclusion, at some number of copies of the trivial  $R$ -module  $\mathbb{Q}$ . In particular:

Denote by  $\varphi_i$  the cyclotomic polynomial having as roots the primitive  $i$ -roots of 1 and let

$$\{\varphi_i\} := \mathbb{Q}[\tau, \tau^{-1}]/(\varphi_i) = \mathbb{Q}[\tau]/(\varphi_i)$$

be the cyclotomic field of  $i$ -roots of 1, thought as  $R$ -module, then

$$H^k(\mathbf{Y}(\mathbf{A}_n), R_\tau) \simeq \{\varphi_1\}^{a_{k,n}}$$

for all  $k \leq q + 1$  and  $n \geq 3q + 1$ .

Our conjecture is that there is an analogous of this theorem also for cohomologies in a generic rank 1 local system  $\mathcal{L}$ . More precisely the characteristic variety  $V^k(\mathbf{Y}(\mathbf{A}_n))$  is completely determinate by the characteristic varieties  $V^k(\mathbf{Y}(\mathbf{A}_m))$  for  $m < n$  when  $n > 3k$ . Theorem 4 proves that this conjecture is true at list for the first characteristic variety. This conjecture motivated also studied in [19].

## 2. CASE $\mathbf{I}_2(m)$ , $m \geq 2$

Let  $Z$  be  $\mathbb{Z}[\tau, \tau^{-1}]$ , we begin with the following

### Theorem 5

$$\begin{aligned} H^1(\mathbf{I}_2(m), Z_\tau) &= Z/(\varphi_1) \\ H^2(\mathbf{I}_2(m), Z_\tau) &= Z/(\varphi_1) \oplus_{i|m} [Z/(\varphi_i)]^{m-2} \end{aligned}$$

*All other cohomologies are 0*

In order to give a proof of our claim we find more convenient (as we did in the previous chapters) to use the boundary operator instead of co-boundary. Of course, these are given by the transposed matrices.

Recall that  $rkC_0(\mathbf{I}_2(m)) = rkC_2(\mathbf{I}_2(m)) = 2m$  and  $rkC_1(\mathbf{I}_2(m)) = 4m$ .

It is very easy to see that

$$H^0(\mathbf{I}_2(m), Z_\tau) = 0 \text{ and } H_0(\mathbf{I}_2(m), Z_\tau) = Z/(\varphi_1).$$

It follows in particular that the 1-boundary  $\partial_1$  of  $C_*(\mathbf{I}_2(m))$  is equivalent, up to integral base-changes, to the diagonal matrix

$$D_1 = \text{diag}([1]^{2m-1}, \varphi_1) \tag{19}$$

where  $[a]^n = \overbrace{a, \dots, a}^{n\text{-times}}$ .

Theorem 5 clearly follows from 19 and next proposition 1.

**Proposition 1** *There are bases for  $C_2(\mathbf{I}_2(m))$  and  $C_1(\mathbf{I}_2(m))$  such that  $\partial_2$  is equivalent to the diagonal matrix*

$$D_2 = \text{diag}([1]^{m+1}, \varphi_1, [\prod_{i|m} \varphi_i]^{m-2})$$

We start by introducing some useful notations:

- $w_j = \text{prod}_j(s_1, s_2) = \overbrace{s_1 s_2 s_1 \dots s_i}^{j\text{-times}}$ ,  $i=1$  if  $j$  is odd  $i=2$  otherwise.
- $\{f_j = E(w_j, \Gamma)\}_{0 \leq j \leq 2m-1}$  with  $|\Gamma| = 2$  a basis for  $C_2(\mathbf{I}_2(m))$
- $\{e_j^i = E(w_j, \Gamma_i)\}_{\substack{i=1, 2 \\ 0 \leq j \leq 2m-1}}$  with  $\Gamma_i = \{s_i\}$  a basis for  $C_1(\mathbf{I}_2(m))$ .

Then we define, for  $0 \leq j \leq 2m-1$

$$a_j = \begin{cases} e_j^1 & \text{if } j \text{ is odd or } 0 \\ e_j^2 & \text{if } j \text{ is even and } j \neq 0 \end{cases} \quad b_j = \begin{cases} e_j^2 & \text{if } j \text{ is odd or } 0 \\ e_j^1 & \text{if } j \text{ is even and } j \neq 0 \end{cases}$$

Notice that the  $a_j$ 's and  $b_j$ 's take each 1 cell of the complex once.

**Example 3** *In the case of  $\mathbf{I}_2(3) = \mathbf{A}_2$  we have the following picture:*

We construct a new basis for  $C_1(\mathbf{I}_2(m))$  as follows:

$$\varepsilon_j = \begin{cases} a_j, & \text{if } 0 \leq j \leq m \\ (b_j - b_{j-1}\tau) - (a_{j-m} - a_{j-m+1}\tau)\tau^{j-m}, & \text{if } m+1 \leq j \leq 2m-1 \\ (a_{j-m+3} - a_{j-m+4}) - (a_{j-2m+2} - a_{j-2m+3}\tau)\tau^{j-2m+2}, & \text{if } 2m \leq j \leq 3m-3 \\ (b_{j-2m+3}\tau - b_{j-2m+4}) - (b_{j-3m+4} - b_{j-3m+3}\tau)\tau^{j-3m+4}, & \text{if } 3m-2 \leq j \leq 4m-5 \\ a_{m+1} + a_0, & \text{if } j = 4m-4 \\ b_m + \sum_{i=1}^m a_i - \tau \sum_{i=2}^m a_i, & \text{if } j = 4m-3 \\ b_0 + [a_0 - \sum_{i=m+1}^{2m-1} b_i + \tau \sum_{i=m}^{2m-2} b_i], & \text{if } j = 4m-2 \\ b_{m-1} - [a_0 - \sum_{i=1}^{m-1} a_i + \tau \sum_{i=2}^m a_i], & \text{if } j = 4m-1. \end{cases} \tag{20}$$

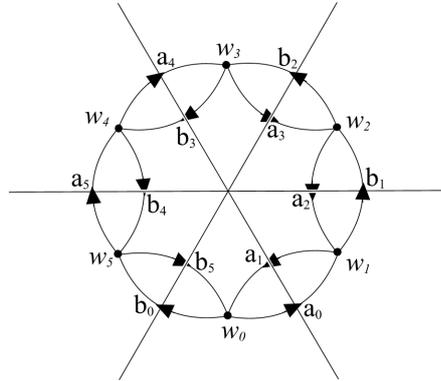


Figure 1: case  $\mathbf{I}_2(3)$

Our aim is to prove that:

**Lemma 1** *With respect to the basis 20 the matrix of  $\partial_2(\mathbf{I}_2(m))$  becomes:*

$$\overline{D}_2 = \begin{bmatrix} A_{m+1} & B_{(m+1,m-1)} \\ 0_{(m-1,m+1)} & D_{m-1}(1-\tau^m) \\ 0_{(m-2,m+1)} & 0_{(m-2,1)}D_{m-2}(1-\tau^m) \\ 0_{(m-2,m+1)} & 0_{(m-2,1)}D_{m-2}(1-\tau^m) \\ 0_{(1,m+1)} & (1-\tau^{m-1}) \cdots (1-\tau) \\ 0_{(1,m+1)} & (1-\tau^{m-1}) \cdots (1-\tau) \\ 0_{(1,m+1)} & (1-\tau^{m-1}) \cdots (1-\tau) \\ 0_{(1,m+1)} & 0_{(1,m-1)} \end{bmatrix}$$

where  $D_h(1-\tau^m)$  is a  $h \times h$  diagonal matrix with entries  $(1-\tau^m)$  and  $A_{m+1}$  is triangular with pivots  $-1$ .

In order to prove this lemma we will use induction. We need the following:

**Lemma 2** *The matrix  $\partial_2^{(m+1)} := \partial_2(\mathbf{I}_2(m+1))$  is obtained by adding to the matrix  $\partial_2^{(m)} := \partial_2(\mathbf{I}_2(m))$  two columns and four rows and multiplying by  $\tau$  the entries  $(\partial_2^{(m)})_{i,j}$  for  $m \leq j \leq 2m-1$  and  $1 \leq i \leq 2m-2, i = 4m-1, 4m$ .*

**Proof.** Geometrically,  $\mathcal{A}(\mathbf{I}_2(m+1))$  is obtained by adding an hyperplane to  $\mathcal{A}(\mathbf{I}_2(m))$ . This is equivalent to add the columns relative to the 2-cells  $f_{m-1}$  and  $f_{2m}$  at places  $m$  and  $2m+1$ , and the rows relative to the 1-cells  $b_{m-1}, a_m, b_{2m}$  and  $a_{2m+1}$  at palces  $2m-1, 2m, 4m+1$  and  $4m+2$ .

By an easy computation we can see that

$$\begin{aligned}
(\partial_2^{(m+1)})_{2m-1} &= [\overbrace{1 \cdots 1}^m \overbrace{0 \cdots 0}^{m+1} \tau] \\
(\partial_2^{(m+1)})_{4m+2} &= [-1 - \tau - \tau^2 \cdots - \tau^{m-1} \overbrace{0 \cdots 0}^{m+1} - 1] \\
(\partial_2^{(m+1)})_{2m} &= [\overbrace{0 \cdots 0}^m - 1 \overbrace{-\tau \cdots -\tau}^m 0] \\
(\partial_2^{(m+1)})_{4m+1} &= [\overbrace{0 \cdots 0}^m \tau^{m-1} \tau^{m-1} \tau^{m-2} \tau^{m-3} \cdots \tau 10].
\end{aligned} \tag{21}$$

and the  $m$ -th and  $2m + 1$ -th columns of  $\partial_2^{(m+1)}$  equal respectively the  $m$ -th and  $\tau$  times  $2m$ -th columns of  $\partial_2^{(m)}$  in the positions not involving the new rows added.

**Proof of lemma 1.** It is a simple computation to verify the first step of induction, i.e. the case  $m = 2$ .

From lemma 2 and its proof it follows by induction that the rows relative to  $[\varepsilon_j(\mathbf{I}_2(m+1))]_{0 \leq j \leq m+1}$  give rise to a triangular matrix with pivots  $-1$ .

In order to compute the rows relative to  $[\varepsilon_j(\mathbf{I}_2(m+1))]_{m+2 \leq j \leq 2m+1}$  we notice that:

$$\begin{aligned}
(b_j - b_{j-1}\tau) &: [\overbrace{0 \cdots 0}^{j-m-1} \tau^{j-m-1} \overbrace{0 \cdots 0}^{2m+2-j} \overbrace{0 \cdots 0}^{j-m-2} 1 \overbrace{0 \cdots 0}^{2m+1-j}] \\
(a_{j-m} - a_{j-m+1}\tau) &: [\overbrace{0 \cdots 0}^{j-m-1} 1 \overbrace{0 \cdots 0}^{2m+2-j} \overbrace{0 \cdots 0}^{j-m-2} -\tau^{2m+2-j} \overbrace{0 \cdots 0}^{2m+1-j}].
\end{aligned}$$

Then we have:

$$\varepsilon_j(\mathbf{I}_2(m+1)) : [\overbrace{0 \cdots 0}^j 1 - \tau^{m+1} \overbrace{0 \cdots 0}^{2m+1-j}]$$

for  $m+2 \leq j \leq 2m+1$ .

In the same way one computes the rows relative to  $[\varepsilon_j(\mathbf{I}_2(m+1))]_{2m+2 \leq j \leq 4m+3}$ .

This completes the proof of lemma 1.

**Remark 2** *Let us consider any rank-1 local system over  $\mathbf{Y}(\mathbf{I}_2(m))$  with weights  $\tau_1, \dots, \tau_m$  or, more generally, a local system defined over  $\mathbb{Z}[\tau_1, \tau_1^{-1}, \dots, \tau_m, \tau_m^{-1}]$ . Then the previous discussion can be easily extended to this case. One obtains a matrix analog to  $\overline{D}_2$ , containing the parameters  $\tau_1, \dots, \tau_m$ . By further elementary transformations the matrix of  $\partial_2^{(m)}$  becomes*

$$\begin{bmatrix} I_{m+1} & 0_{m+1,m-1} \\ 0_{(1,m+1)} & (1 - \tau_1 \tau_3 \cdots \tau_m)(1 - \tau_1 \tau_4 \cdots \tau_m) \cdots (1 - \tau_1 \tau_m)(1 - \tau_m) \\ 0_{(m-1,m+1)} & D_{m-1}(1 - \prod_{i=1}^m \tau_i) \end{bmatrix}$$

where  $I_{m+1}$  is the identity.

**Proof of proposition 1.** We consider the independent rows of the matrix  $\overline{D}_2$ : the first  $2m$  rows and the  $(4m - 3)$ -th.

With elementary transformations for the columns, we can reduce the first  $m + 1$  rows in a diagonal form with entries 1.

The proof ends if we observe that  $(1 - \tau)$  divides  $(1 - \tau^k)$  for  $k \geq 1$ , then with further elementary transformations the matrix

$$\begin{bmatrix} D_{m-1}(1 - \tau^m) \\ (1 - \tau^{m-1}) \cdots (1 - \tau) \end{bmatrix}$$

becomes

$$\begin{pmatrix} 1 - \tau & O & \cdots & 0 \\ 0 & (1 - \tau^m) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (1 - \tau^m) \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

that is the claim

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