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Domination in Circulant Graphs

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Abstract

A graph G with no isolated vertex is total domination vertex critical if for any vertex v of G that is not adjacent to a vertex of degree one, the total domination number of $G - v$ is less than the total domination number of G. We call these graphs γ_t-critical. In this paper, we determine the domination and the total domination number in the Circulant graphs $C_n\langle 1, 3\rangle$, and then study γ-criticality and γ_t-criticality in these graphs. Finally, we provide answers to some open questions.

1 Introduction

A vertex in a graph G *dominates* itself and its neighbors. A set of vertices S in a graph G is a *dominating set*, if each vertex of G is dominated by some vertex of S. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. A dominating set S is called a *total dominating set* if each vertex v of G is dominated by some vertex $u \neq v$ of S. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G.

We denote the *open neighborhood* of a vertex v of G by $N_G(v)$, or just $N(v)$, and its *closed neighborhood* by $N[v]$. For a vertex set $S \subseteq V(G)$, $N(S) = \cup_{v\in S} N(v)$ and $N[S] = \cup_{v\in S} N[v]$. So, a set of vertices S in G is a dominating set, if $N[S] = V(G)$. Also, S is a total dominating set, if $N(S) = V(G)$. For notation and graph theory terminology in general we follow [3].

An *end-vertex* in a graph G is a vertex of degree one and a support vertex is one that is adjacent to an end-vertex. We call a dominating set of cardinality

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$\gamma(G)$, a $\gamma_t(G)$-set, and a total dominating set of cardinality $\gamma_t(G)$, a $\gamma_t(G)$-set. We also let $S(G)$ be the set of all support vertices of G.

For many graph parameters, criticality is a fundamental question. A graph G is called *vertex domination critical* if $\gamma(G-v) < \gamma(G)$, for every vertex v in G. For references on vertex domination critical graphs see [1, 2, 3].

Goddard, et. al., [2], introduced total domination vertex critical graphs. A graph G is *total domination vertex critical*, or just γ_t-critical, if for every vertex $v \in V(G)\backslash S(G)$, $\gamma_t(G-v) < \gamma_t(G)$. If G is γ_t-critical, and $\gamma_t(G) = k$, then G is called *k-γ_t-critical*. They posed the following open question:

Question 1([2]): Which graphs are γ-critical and γ_t-critical or one but not the other?

Let $\Delta(G)$ be the maximum degree of vertices in a graph G. Mojdeh, et. al., [4], studied γ_t-critical graphs G of order $\Delta(G)(\gamma_t(G)-1)+1$ and posed the following question:

Question 2([4]): Does there exist a $k-\gamma_t$-critical graph of order $(\gamma_t(G)-1) \triangle (G)+1$ for all odd $k \geq 3$?

Let $n \geq 4$ be a positive integer. The Circulant graph $C_n\langle 1,3\rangle$ is the graph with vertex set $\{v_0, v_1, ..., v_{n-1}\}$, and edge set $\{\{v_i, v_{i+j}\} : i \in \{0,1,...,n-1\}$ and $j \in \{1,3\}\}$. All arithmetic on the indices is assumed to be modulo n.

In this paper, we first determine the domination number and the total domination number in the Circulant graphs $C_n\langle 1,3\rangle$ for any integer n, and then study γ-criticality and γ_t-criticality in these class of graphs. We then provide an answer to Question 1, and an answer to Question 2.

For a subset $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S. Also for two vertices x and y in a graph G we denote the distance between x and y by $d_G(x,y)$, or just $d(x,y)$.

2 Domination and total domination

Let $n \geq 4$ be a positive integer, and let $G = C_n\langle 1,3\rangle$. Let Cycle $C = C(G)$ be the subgraph with vertex set $\{v_0, v_1, ..., v_{n-1}\}$ and edge set $\{\{v_i, v_{i+1}\} : i \in \{0,1,...,n-1\}$. For a subset $S \subseteq V(G)$ with at least three vertices, we say that $x, y \in S$ are *consecutive* if there is no vertex $z \in S$ such that z lies between x and y in C. For two consecutive vertices x, y in a subset of vertices S, we define $|x-y| = d_C(x,y)$. So, $|x-y|$ equals to the number of edges in a shortest path between x and y in the cycle C.

In this section, we determine the domination number and the total domination number in the Circulant graphs $G = C_n\langle 1,3\rangle$, for any integer $n \geq 4$.

2.1 Domination number

It is obvious that $C_4\langle 1,3\rangle = C_4$ and $C_5\langle 1,3\rangle = K_5$. So $\gamma(C_4\langle 1,3\rangle) = 2$ and $\gamma(C_5\langle 1,3\rangle) = 1$. For $n \geq 6$ we have the following result.

Theorem 1 *For any integer* $n \geq 6, \gamma(G) = \begin{cases} \lceil \frac{n}{5} \rceil, & n \not\equiv 4(mod5) \\ \lceil \frac{n}{5} \rceil + 1, & n \equiv 4(mod5) \end{cases}$

Proof. Let $G = C_n\langle 1,3\rangle$ and let S be a $\gamma(G)$-set. Any vertex of S dominates five vertices of G including itself, so $|S| \geq \lceil \frac{n}{5} \rceil$. We proceed with the following fact.

Fact A. If $n \equiv 4$ (mod 5), then $|S| \geq \lceil \frac{n}{5} \rceil + 1$.

To see this, assume to the contrary that $n \equiv 4$ (mod 5), and $|S| = \lceil \frac{n}{5} \rceil$. There are two consecutive vertices $v_k, v_{k'} \in S$ such that $|k - k'| < 5$. Let $v_{k"} \neq v_k$ is a consecutive vertex of $v_{k'}$. Without loss of generality assume that $|k" - k| = 9$. Then there are eight possiblities for $v_{k'}$ to lies between v_k and $v_{k"}$. In each possiblity there exists a vertex between v_k and $v_{k"}$ which is not dominated by $\{v_k, v_{k'}, v_{k"}\}$, a contradiction. Hence, for $n \equiv 4$ (mod 5), $|S| \geq \lceil \frac{n}{5} \rceil + 1$.

Now, it is sufficient to define a dominating set of required cardinality. We consider the following cases:

Case 1. $n \equiv 0$ (mod 5). We define $S = \{v_{5k} : 0 \leq k < \frac{n}{5}\}$.

Case 2. $n \equiv 1$ (mod 5). For $n = 6$ we define $S = \{v_0, v_3\}$ and for $n > 6$ we define $S = \{v_{5k} : 0 \leq k < \lfloor \frac{n}{5} \rfloor\} \cup \{v_{n-1}\}$.

Case 3. $n \equiv 2$ (mod 5). For $n = 7$ we define $S = \{v_0, v_1\}$ and for $n > 7$ we define $S = \{v_{5k} : 0 \leq k < \lfloor \frac{n}{5} \rfloor\} \cup \{v_{n-2}\}$.

Case 4. $n \equiv 3$ (mod 5). For $n = 8$ we define $S = \{v_0, v_3\}$ and for $n > 8$ we define $S = \{v_{5k} : 0 \leq k < \lfloor \frac{n}{5} \rfloor\} \cup \{v_{n-3}\}$.

Case 5. $n \equiv 4$ (mod 5). For $n = 9$ we define $S = \{v_0, v_1, v_5\}$ and for $n > 9$ we define $S = \{v_{5k} : 0 \leq k \leq \lfloor \frac{n}{5} \rfloor\} \cup \{v_{n-2}\}$.

In each of the above cases S is a dominating set for $C_n\langle 1,3\rangle$ of cardinality $\lceil \frac{n}{5} \rceil + 1$ when $n \equiv 4$ (mod 5), and of cardinality $\lceil \frac{n}{5} \rceil$ when $n \not\equiv 4$ (mod 5). Hence, the result follows. ■

2.2 Total domination number

Here we study total domination numbers in $C_n\langle 1,3\rangle$ for $k \geq 4$. We need the following lemmas.

Lemma 2 *Let S be a subset of vertices of G and $G[S]$ has no isolated vertices. If $|S|$ is even, then S dominates at most $4\,|S|$ vertices of G.*

Proof. Let S be subset of vertices with $|S| = m$, where m is even. Any two adjacent vertices of S dominate eight vertices of G including themselves. So S dominates at most $8(\frac{|S|}{2}) = 4\,|S|$ vertices of G. ■

Lemma 3 *Let S be a subset of vertices of $G = C_n\langle 1,3\rangle$ and $G[S]$ has no isolated vertices. If $|S|$ is odd, then S dominates at most $4\,|S| - 1$ vertices of G.*

Proof. Let S be a subset of vertices with $|S| = m$, where m is odd. Without loss of generality we may assume that $G[S]$ has $k = \frac{|S|-3}{2} + 1$ components $G_1, G_2, ..., G_k$, where $|V(G_1| = 3$ and $|V(G_i| = 2$ for $i = 2, 3, ..., k$. Let $V(G_1) = \{a, b, c\}$, then $\{a, b, c\}$ dominates at most 11 vertices of G. So S dominates at most $8(\frac{|S|-3}{2}) + 11 = 4m - 1$ vertices of G. ■

Now, we determine the total domination numbers in G, by the following.

Theorem 4 *For any integer* $n \geq 4, \gamma_t(G) = \begin{cases} \lceil \frac{n}{4} \rceil + 1, & n \equiv 2, 4 (mod 8) \\ \lceil \frac{n}{4} \rceil, & Otherwise \end{cases}$.

Proof. The result is trivial for $n \leq 7$. So, we let $n \geq 8$. Let $G = C_n\langle 1,3\rangle$ and let S be a total dominating set for G. It follows from Lemma 2 and Lemma 3 that $|S| \geq \lceil \frac{n}{4} \rceil$.

Claim 1. If $n \equiv 2 \pmod 8$ and S is a total dominating set for G, then $|S| \geq \lceil \frac{n}{4} \rceil + 1$.

Proof of Claim 1. Let $n \equiv 2 \pmod 8$ and let S be a total dominating set for G. Assume to the contrary that $|S| = \lceil \frac{n}{4} \rceil$. Let $n = 8k + 2$, where k is a positive integer. Since $|S| = \lceil \frac{n}{4} \rceil$, then $|S| = 2k + 1$ is an odd number. So, the induced subgraph $G[S]$ has a component H with at least three vertices. We proceed with Fact B and Fact C.

Fact B. Any component of $G[S]$ has at most three vertices.

To see this, assume to the contrary that G_1 is a component of $G[S]$ and G_1 has at least 4 vertices. Without loss of generality assume that G_1 has 4 vertices. Then S dominates at most $14 + 8(\frac{|S|-4-3}{2}) + 11 = 8k + 1$ vertices of G, a contradiction.

Fact C. H is the only odd component of $G[S]$.

To see this, assume to the contrary that $H' \neq H$ is a component of $G[S]$ with $|V(H')|$ odd. It follows from Fact B that $|V(H')| = 3$. Since $|S|$ is odd, there is another component H" with three vertices. Now S dominates at most $8k+1$ vertices of G, a contradiction.

Let $V(H) = \{v_i, v_j, v_l\}$ where $i < j < l$. Let $v_{i'}$ be a consecutive vertex of v_i with $i' \neq j$ and $v_{l'}$ be a consecutive vertex of v_l with $l' \neq j$. Since S dominates $11 + 8(\frac{|S|-3}{2}) = 8k+3$ vertices of G, then there is a vertex x of G which has two neighbors in S. Now $\min\{|i-j|, |l-j|\} \neq 2$, and we can assume that x is adjacent to both v_l and $v_{l'}$. Let $x = v_t$ where $t < l'$. If $|l'-t| = 1$, then v_{t-1} is not dominated by S which is a contradiction. So $|l'-t| = 4$. But then v_{t+1} is not dominated by S, a contradiction. Hence $|S| \geq \lceil \frac{n}{4} \rceil + 1$. This completes the proof of Claim 1.□

With a similar manner as in the proof of Claim 1, the following Claim is verified, and we left the proof.

Claim 2. If $n \equiv 2$ *or* $4 \pmod 8$ and S is a total dominating set for G, then $|S| \geq \lceil \frac{n}{4} \rceil + 1$.

Now, it is sufficient to define a total dominating set S of required cardinality.

For $n \equiv 0 \pmod 8$, we define $S = \{v_{8k}, v_{8k+3} : 0 \leq k < \frac{n}{8}\}$.
For $n \equiv 1 \pmod 8$, we define $S = \{v_0, v_3, v_6\}$ if $n = 9$, and define $S = \{v_{8k}, v_{8k+3} : 0 \leq k < \frac{n-1}{8} - 1\} \cup \{v_{n-3}, v_{n-6}, v_{n-9}\}$ if $n > 9$.

For $n \equiv 2 \pmod 8$, we define $S = \{v_{8k}, v_{8k+3} : 0 \leq k < \lfloor \frac{n}{8} \rfloor\} \cup \{v_{n-3}, v_{n-4}\}$.

For $n \equiv 3 \pmod 8$, we define $S = \{v_{8k}, v_{8k+3} : 0 \leq k < \lfloor \frac{n}{8} \rfloor\} \cup \{v_{n-5}\}$.

For $n \equiv 4 \pmod 8$, we define $S = \{v_{8k}, v_{8k+3} : 0 \leq k < \lfloor \frac{n}{8} \rfloor\} \cup \{v_{n-3}, v_{n-4}\}$.

For $n \equiv 5 \pmod 8$, we define $S = \{v_{8k}, v_{8k+3} : 0 \leq k < \lfloor \frac{n}{8} \rfloor\} \cup \{v_{n-4}, v_{n-5}\}$.

For $n \equiv 6 \pmod 8$, we define $S = \{v_{8k}, v_{8k+3} : 0 \leq k < \lfloor \frac{n}{8} \rfloor\} \cup \{v_{n-5}, v_{n-6}\}$.

For $n \equiv 7 \pmod 8$, we define $S = \{v_{8k}, v_{8k+1} : 0 \leq k \leq \lfloor \frac{n}{8} \rfloor\}$.

Then S is a total dominating set of cardinality $\lceil \frac{n}{4} \rceil + 1$ when $n \equiv 2, 4 \pmod 8$, and of cardinality $\lceil \frac{n}{4} \rceil$ when $n \not\equiv 2, 4 \pmod 8$. ■

3 Criticality of domination and total domination

In this section, we study γ-criticality and γ_t-criticality in Circulant graphs $G = C_n\langle 1,3\rangle$, for any integer $n \geq 4$. This leads us to provide answers to Question 1 and Question 2.

Theorem 5 *For $n \geq 6$, the Circulant graph $G = C_n\langle 1,3\rangle$ is $\gamma-$critical if and only if $n \equiv 4$ (mod 5).*

Proof. First we show that G is γ-critical for $n \equiv 4$ (mod 5). Let x be a vertex of $G = C_{5n+4}\langle 1,3\rangle$ for some positive integer n. Since G is vertex transitive, we assume that $x = v_{n-2}$. It is easy to see that $S = \{v_{5k} : 0 \leq k \leq \lfloor \frac{n}{5} \rfloor\}$ is a dominating set for $G - x$, concluding that $\gamma(G-x) \leq \lceil \frac{n}{5} \rceil < \lceil \frac{n}{5} \rceil + 1 = \gamma(G)$. Hence, G is $\gamma-$critical.

Suppose now, that $n \not\equiv 4$ (mod 5). We Show that G is not γ-critical. By Theorem 1, $\gamma(G) = \lceil \frac{n}{5} \rceil$. We show that any k vertices of G with $k < \lceil \frac{n}{5} \rceil$ dominate at most $n-2$ vertices of G. Let T be a subset of vertices with $|T| < \lceil \frac{n}{5} \rceil$.

If $n \equiv 0$ (mod 5), then $n = 5i$ for some integer i. It follows that $\gamma(G) = i$. Now, T dominates at most $5i - 5 \leq n-2$ vertices of G. Similarly for $n \equiv 2, 3$ (mod 5), T dominates at most $5i-5 \leq n-2$ vertices of G. So we assume that $n \equiv 1$ (mod 5). There is an integer l such that $n = 5l+1$. Without loss of generality let $|T| = \lceil \frac{n}{5} \rceil - 1 = l$.

If there are two consecutive vertices x, y in T such that $|x-y| < 5$, then $N_G[x] \cap N_G[y] \neq \emptyset$. Furthermore, $\{x,y\}$ dominates at most nine vertices of G and $T \backslash \{x,y\}$ dominates at most $5(l-2)$ vertices of G. So, T dominates at most $n-2$ vertices of G. It remains to assume that for any two consecutive vertices a, b in T, $|a-b| \geq 5$. But then there are two consecutive vertices x, y in T such that $|x-y| > 5$. There exist two vertices u, v in G such that u, v lie between x and y in C, and T does not dominate $\{u,v\}$. So, T dominates at most $n-2$ vertices of G. Hence, G is not γ-critical for $n \not\equiv 4$ (mod 5). ■

Theorem 6 *For $n \geq 4$ the Circulant graph $G = C_n\langle 1,3\rangle$ is γ_t-critical if and only if $n \equiv 1$ (mod 8).*

Proof. Let $G = C_n\langle 1,3\rangle$ and $n \geq 4$. First assume that $n \not\equiv 1$ (mod 8). We prove that G is not γ_t-critical. Let $T \subset V(G)$ be a subset of at most $\gamma_t(G) - 1$ vertices, and $G[T]$ has no isolated vertex. We show that T totally dominates at most $n-2$ vertices of G. Without loss of generality assume that $|T| = \gamma_t(G) - 1$. For $n \not\equiv 4$ (mod 8), the result follows from applying Lemma 2 and Lemma 3. So, we let $n \equiv 4$ (mod 8). We proceed with the following fact.

Fact D. T totally dominates at most $n-2$ vertices of G.

Proof. Let $|T| = 2k + 1$ for some integer k. If two vertices in T have a common neighbor, then the result follows. So, suppose that no two vertices in T have a common neighbor. Without loss of generality we may assume that $G[S]$ has k components $G_1, G_2, ..., G_k$ where $|V(G_1| = 3$ and $|V(G_i| = 2$ for $i = 2, 3, ..., k$. Let $V(G_1) = \{a, b, c\}$, then $T\backslash\{a, b, c\}$ dominates at most $8k-8$ vertices of G, and we may assume that $T\backslash\{a, b, c\}$ dominates $8k-8$ vertices of G. Let G_1 be the subgraph of G induced by $V(G)\backslash N[T\backslash\{a, b, c\}]$. Then there are the following possiblities for G_1.

1) $V(G_1) = \{v_t, v_{t+1}, ..., v_{t+11}\}$ where t is an integer and the addition in $t + i$ is in modulo n, and, $E(G_1) = \{\{V_{t+i}, v_{t+j}\} : i \in \{0, 1, ..., 8\}, j \in \{1, 3\}\} \cup \{\{V_{t+9}, v_{t+10}\}, \{V_{t+10}, v_{t+11}\}\}$.

2) $V(G_1) = \{v_t, v_{t+1}, ..., v_{t+11}\}$ where t is an integer and the addition in $t + i$ is in modulo n, and,
$E(G_1) = \{\{V_{t+i}, v_{t+j}\} : i \in \{1, ..., 9\}, i \neq 8, j \in \{1, 3\}\}$

$$\cup\{\{V_t, v_{t+2}\}, \{V_{t+10}, v_{t+11}\}, \{V_{t+8}, v_{t+9}\}\}.$$

For any possiblity for $\{a, b, c\}$, it is easy to see that $\{a, b, c\}$ dominates at most 10 vertices of G_1. This completes the proof of Fact D.$\square$

Now, it is sufficient to prove that for $n \equiv 1 \pmod 8$, G is γ_t-critical. Let $n \equiv 1 \pmod 8$, and let x be a vertex of G. We show that $\gamma_t(G - x) < \gamma_t(G)$. Since G is vertex transitive, we can assume that $x = v_{n-4}$. Then $S = \{v_{8k}, v_{8k+1} : 0 \leq k < \lfloor \frac{n}{8} \rfloor\}$ is a total dominating set for $G - x$. Hence, G is γ_t-critical. ■

Now, we are ready to provide an answer to Question 1, and a positive answer to question 2. The following is an immediate result of Theorem 5 and Theorem 6, and provide an answer to Question 1.

Theorem 7 *(1) For any positive integer $n \geq 1$ with $8 \nmid 5n + 3$, the Circulant graph $C_{5n+4}\langle 1, 3\rangle$ is $\gamma-$critical and is not γ_t-critical.*

(2) For any positive integer $n \geq 1$ with $5 \nmid 8n - 3$, the Circulant graph $C_{8n+1}\langle 1, 3\rangle$ is γ_t-critical and is not $\gamma-$critical.

(3) For any positive integer $n \geq 1$ with $8 \mid 5n + 3$, the Circulant graph $C_{5n+4}\langle 1, 3\rangle$ is both γ_t-critical and $\gamma-$critical.

Since for any $n \geq 1$, $\gamma_t(C_{8n+1}\langle 1, 3\rangle) = 2n+1$ and $\Delta(C_{8n+1}\langle 1, 3\rangle) = 4$, then $C_{8n+1}\langle 1, 3\rangle$ is a γ_t-critical graph of order $(\gamma_t - 1) \triangle +1$. Hence, the following result provides a positive answer to question 2.

Theorem 8 *For any odd $k \geq 3$ there exists a $k - \gamma_t$-critical graph G of order $(\gamma_t(G) - 1) \triangle (G) + 1$.*

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