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On the existence and nonexistence of positive entire large solutions for semilinear elliptic equations

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Abstract

In this paper we are studying the existence of positive entire large solutions for the problem $\Delta u = p_1(x)u^\alpha + p_2(x)u^\beta f(u)$ in $\mathbb{R}^N$, $N \geq 3$. Moreover, we are interested in the relation between the existence or nonexistence of such solutions for the above problem and the existence or nonexistence of such solutions for the problems $\Delta u = p_1(x)u^\alpha$ and $\Delta u = p_2(x)u^\beta f(u)$.

1 Introduction

In the present paper we are discussing the existence of large solutions for some semilinear elliptic equations.

Definition 1 *(i) A positive solution u of an elliptic equation on $\Omega \not\equiv \mathbb{R}^N$ satisfying the condition*

$$u(x) \to \infty, \qquad as \;\; dist(x, \partial\Omega) \to 0$$

is called a large (blow-up, explosive) solution of that equation.

(ii) A positive solution u of an elliptic equation on $\mathbb{R}^N$ satisfying the condition

$$u(x) \to \infty, \qquad as \;\; |x| \to \infty$$

is called a positive entire large solution of that equation.

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The study of large solutions has been initiated in 1916 by Bieberbach [2] who studied the equation $\Delta u = e^u$ in the plane. He showed that there exists a unique solution of this equation such that $u(x) - \log(dist(x, \partial\Omega)^{-2})$ is bounded as $dist(x, \partial\Omega) \to 0$. Problems of this type arise in Riemannian geometry: if a Riemannian metric of the form $|ds|^2 = e^{2u(x)}|dx|^2$ has constant Gaussian curvature $-c^2$, then $\Delta u = c^2 e^{2u}$. Motivated by a problem in mathematical physics, in 1943 Rademacher [15] continued the study of Bieberbach on smooth bounded domains in $\mathbb{R}^3$.

Later on, due to their applicability in a number of different areas, problems of this type were studied under the general form $\Delta u = f(u)$ in N-dimensional domains. A special attention was paid to the equations of the form

$$\Delta u = p(x)u^\gamma \tag{1}$$

which was studied in bounded and in unbounded domains. In [5], Cheng and Ni proved that (1) has a unique entire large solution in $\mathbb{R}^N$ provided that function p is positive and smooth, $\gamma > 1$ and that there exists $m > 2$ such that $|x|^m p(x)$ is bounded for large $|x|$. In [1], Bandle and Marcus showed the existence and uniqueness of a positive entire large solution for the more general equation

$$\Delta u = g(x, u),$$

which includes the case $g(x, u) = p(x)u^\gamma$ where $\gamma > 1$ and the function $p(x)$ is positive and continuous such that p and $1/p$ are bounded. The study of the existence of positive large solutions for (1) in the superlinear case is included in many studies, see for example [6]. Nonexistence results of large positive solutions for (1) with $\gamma > 1$ were given in [13], [12] and [4]. Although the sublinear case has not received as much attention as the superlinear case, we will recall some results concerning the existence of solution to equation (1) in both cases in Section 2.

For more information on problems with large solutions we refer to [16] or to the recent book [7]. In [16] Rădulescu was concerned with some recent results related to various singular phenomena arising in the study of nonlinear elliptic equations. He established qualitative results on the existence, nonexistence or the uniqueness of solutions concerning the following types of problems:

(i) blow-up boundary solutions of logistic equations;

(ii) Lane-Emden-Fowler equations with singular nonlinearities and sub-quadratic convection term.

Rădulescu studied the combined effects of various terms involved in these problems: sublinear or superlinear nonlinearities, singular nonlinear terms, convection nonlinearities, as well as sign-changing potentials, also taking into consideration bifurcation nonlinear problems. The precise rate decay of the solution was established in some concrete situations.

In the second chapter of [7], Rădulescu was concerned with a class of singular solutions for logistic - type equations. He focused on positive solutions with blow-up boundary behavior and he formulated several sufficient conditions for the existence of such solutions. In one of the main results it was established a necessary and sufficient condition for the existence of explosive boundary solutions in the singular case of a potential that vanishes on the boundary.

In this paper our interest goes to the intriguing situation when

$$\Delta u = q_1(x) f_1(u) \qquad \text{and} \qquad \Delta u = q_2(x) f_2(u)$$

both have positive entire large solutions while

$$\Delta u = q_1(x) f_1(u) + q_2(x) f_2(u)$$

has not. One of our main results present such a case. On the other hand, there are cases when if one of the problems $\Delta u = q_1(x)f_1(u)$ and $\Delta u = q_2(x)f_2(u)$ has no positive entire large solutions, then necessarily problem $\Delta u = q_1(x)f_1(u) + q_2(x)f_2(u)$ has none. Some interesting results of this kind were given by Lair in [10].

Here we consider the following class of semilinear elliptic equations

$$\begin{cases} \Delta u = p_1(x)u^\alpha + p_2(x)u^\beta f(u) & \text{in } \mathbb{R}^N, \\ u \geq 0, \ u \not\equiv 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{2}$$

where $N \geq 3$ and f is under the assumptions

$$f \in C^1([0, \infty)), \ \ f' \geq 0, \ \ f \geq 1. \tag{3}$$

Note that the conditions imposed on f are quite permissive, thus f can be chosen to be an appropriate polynomial function or a logarithmical function, an exponential function etc. Our purpose is to establish conditions on p_1, p_2, α and β so that situations like the ones described above would appear.

The rest of our paper has the following structure. In Section 2 we make some notation and we give some results that will be used later in the proof of our main theorems. In Section 3 we are studying the relation between the existence or nonexistence of positive entire large solutions for problem (2) and the existence or nonexistence of such solutions for the problems $\Delta u = p_1(x)u^\alpha$ and $\Delta u = p_2(x)u^\beta f(u)$.

In order to simplify the reading, throughout this paper, C will denote a universal positive constant, depending on different parameters, whose value may change from line to line.

2 Preliminary results

We consider the following semilinear elliptic problem:

$$\begin{cases} \Delta u = p(x)k(u) & \text{in } \Omega, \\ u \geq 0, \ u \not\equiv 0 & \text{in } \Omega. \end{cases} \tag{4}$$

Let us denote

$$M_p(r) \equiv \max_{|x|=r} p(x) \tag{5}$$

and

$$m_p(r) \equiv \min_{|x|=r} p(x). \tag{6}$$

Definition 2 *A nonnegative function p is said to be c-positive in Ω if for every $x_0 \in \Omega$ with $p(x_0) = 0$ there is a domain $\Omega_0 \ni x_0$ such that $\overline{\Omega}_0 \subset \Omega$ and $p > 0$ on $\partial\Omega_0$.*

We recall two theorems given by Cîrstea and Rădulescu in [6].

Theorem 1 *Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be an open, bounded, connected, smooth set with compact boundary. Assume $p \in C^{0,\mu}(\overline{\Omega})$ $(0 < \mu < 1)$ is a c-positive function in Ω and k satisfies*

$$k \in C^1([0, \infty)), \ \ k' \geq 0, \ \ k(0) = 0 \text{ and } k > 0 \text{ on } (0, \infty) \tag{7}$$

and the so-called Keller-Osserman condition (see [9] and [14])

$$\int_1^\infty [2K(t)]^{-1/2} dt < \infty, \tag{8}$$

where $K(t) = \int_0^t k(s)ds$. Then the problem (4) has a positive large solution.

Theorem 2 *Let us consider problem (4) with $\Omega = \mathbb{R}^N$, $N \geq 3$. Suppose there exists a sequence of smooth bounded connected sets $(\Omega_n)_{n\geq 1}$ such that $\overline{\Omega}_n \subset \Omega_{n+1}$, $\mathbb{R}^N = \bigcup_{n=1}^\infty \Omega_n$ and p is c-positive in Ω_n, for any $n \geq 1$. Also, suppose that the function $p \in C^{0,\mu}_{\text{loc}}(\mathbb{R}^N)$ $(0 < \mu < 1)$ verifies*

$$\int_0^\infty r M_p(r) dr < \infty \tag{9}$$

and the nonlinearity k verifies (7) and (8). Then (4) has a positive entire large solution.

Next, we consider the problems

$$\begin{cases} \Delta u = p_1(x)u^{\alpha} & \text{in } \mathbb{R}^N, \\ u \geq 0, \ u \not\equiv 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{10}$$

and

$$\begin{cases} \Delta u = p_2(x)u^{\beta}f(u) & \text{in } \mathbb{R}^N, \\ u \geq 0, \ u \not\equiv 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{11}$$

where $N \geq 3$.

Based on the already known results, a characterization of the existence and the nonexistence of positive entire large solutions for problem (10) may be formulated as follows. For the superlinear case where $\alpha > 1$ equation (10) has such solutions if p satisfies (9) and it will not generally have positive entire large solutions if

$$\int_0^{\infty} r m_p(r) dr = \infty. \tag{12}$$

The sublinear case, where $0 < \alpha \leq 1$, behaves generally in the opposite manner. In the case when $p(x)$ is radial, we have the following result, given by Lair and Wood [11].

Theorem 3 *Suppose* $0 < \alpha \leq 1$ *and* $p(x) = p(|x|) \in C(\mathbb{R}^N)$ *is nonnegative and nontrivial. Then the equation* $\Delta u = p(|x|)u^{\alpha}$ *has a positive entire large solution if and only if*

$$\int_0^{\infty} r p(r) dr = \infty \tag{13}$$

takes place.

In what concerns problem (11), with the aid of Theorem 2 we can prove the following lemma.

Lemma 1 *Assume that* $\beta > 1$*,* f *verifies* (3) *and* $p_2 \in C^{0,\mu}_{\text{loc}}(\mathbb{R}^N)$ *(*$N \geq 3$*,* $0 < \mu < 1$*) satisfies condition* (9)*. Suppose there exists a sequence of smooth bounded connected sets* $(\Omega_n)_{n\geq 1}$ *such that* $\overline{\Omega}_n \subset \Omega_{n+1}$*,* $\mathbb{R}^N = \bigcup_{n=1}^{\infty} \Omega_n$ *and* p_2 *is c-positive in* Ω_n*, for any* $n \geq 1$*. Then problem* (11) *has a positive entire large solution.*

Proof. We will show that we are under the hypotheses of Theorem 2.

First we consider the function k defined by $k(u) = u^{\beta}f(u)$. We will show that our k verifies conditions (7) and (8).

Due to the fact that f is satisfying (3), it is easy to see that k is satisfying (7). Since p_2 satisfies condition (9), it only remains to see if (8) is also fulfilled.

Since $f \geq 1$,

$$K(t) = \int_0^t k(s)ds = \int_0^t s^\beta f(s)ds \geq \int_0^t s^\beta ds = \frac{t^{\beta+1}}{\beta+1}.$$

There exists t_0 (e.g. $t_0 := \beta + 1$) such that for all $t > t_0 > 1$,

$$2K(t) \geq \frac{2}{\beta+1} t^{\beta+1} \geq 1.$$

Hence, for $t > t_0 > 1$,

$$[2K(t)]^{-1/2} \leq \left[\frac{2}{\beta+1} t^{\beta+1}\right]^{-1/2}$$

which implies

$$\int_{t_0}^{\infty} [2K(t)]^{-1/2} dt \leq \int_{t_0}^{\infty} \left[\frac{2}{\beta+1} t^{\beta+1}\right]^{-1/2} dt.$$

We obtain

$$\begin{aligned}\int_1^\infty [2K(t)]^{-1/2} dt &= \int_1^{t_0} [2K(t)]^{-1/2} dt + \int_{t_0}^\infty [2K(t)]^{-1/2} dt \leq C + C \int_{t_0}^\infty t^{-(\beta+1)/2} dt \\ &\leq C + C \cdot t^{(-\beta+1)/2} |_{t_0}^\infty .\end{aligned}$$

But $\alpha > 0$, therefore $\lim_{t\to\infty} t^{(-\beta+1)/2} = 0$ and consequently

$$\int_1^\infty [2K(t)]^{-1/2} dt < \infty.$$

Therefore we can conclude that problem (11) has a positive entire large solution.

□

3 Main results

In what follows we state our main results. The proofs are based on the maximum principle and on the results presented in the previous section.

Theorem 4 *Assume $\alpha > 1$ and $p_1 \in C^{0,\mu}_{\text{loc}}(\mathbb{R}^N)$ ($N \geq 3$, $0 < \mu < 1$) is c-positive in Ω_n, where by Ω_n we understand the ball $|x| < n$. If problem* (10) *has no positive entire large solutions, then problem* (2) *has no positive entire large solutions.*

Proof.

Firstly we show that the function $k(u) = u^\alpha$ fulfills conditions (7) and (8), for $\alpha > 1$. Obviously, $k \in C^1([0, \infty))$, $k' \geq 0$, $k(0) = 0$ and $k > 0$ on $(0, \infty)$, thus (7) is fulfilled. For the Keller-Osserman condition, we have

$$K(t) = \int_0^t k(s)ds = \int_0^t s^\alpha ds = \frac{t^{\alpha+1}}{\alpha+1}.$$

In the same manner as in the proof of Lemma 1 we obtain the fact that k satisfies (8). Therefore, by applying Theorem 1, we can consider v_n to be a positive solution of

$$\begin{cases} \Delta v_n = p_1(x) v_n^\alpha & \text{in } \Omega_n, \\ v_n(x) \to \infty & \text{as } x \to \partial\Omega_n. \end{cases}$$

Since $\overline{\Omega}_n \subset \Omega_{n+1}$ we can apply, for each $n \geq 1$, the maximum principle in order to find that $v_n \geq v_{n+1}$ in Ω_n. The positive sequence $(v_n)_n$ is monotonically decreasing and thus converges to a function v on $\mathbb{R}^N$.

Arguing by contradiction, we assume that problem (2) has a positive entire large solution and we denote it $\widetilde{u}$. Due to the fact that $\widetilde{u} \leq v_n$ on $\overline{\Omega}_n$ for all $n \in \mathbb{N}$ we have $\widetilde{u} \leq v$ on $\mathbb{R}^N$. Since $\widetilde{u}$ is positive and $\widetilde{u}(x) \to \infty$ as $|x| \to \infty$, the function v has the same properties. A standard regularity argument can be used to show that the function v is a positive entire large solution of (10), which, by hypothesis, has none. Hence we have obtained the desired contradiction.

□

Remark 1 *For problem* (10) *not to have positive entire large solutions under the hypotheses of Theorem 4,* p_1 *must not fulfill condition* (9)*, since function* $k(u) = u^\alpha$ *fulfills conditions* (7) *and* (8)*, for* $\alpha > 1$.

Moreover, the following corollary takes place.

Corollary 1 *Assume* $\alpha > 1$ *and* $p_1 \in C^{0,\mu}_{\text{loc}}(\mathbb{R}^N)$ *(*$N \geq 3$, $0 < \mu < 1$*) is c-positive in* Ω_n*, where by* Ω_n *we understand the ball* $|x| < n$*. Assume the function* p_1 *satisfies* (12)*,* $m_{p_1}(r)$ *is non-increasing for large* r *and*

$$\limsup_{r \to \infty} r^2 m_{p_1}(r) > 0,$$

where $m_{p_1}(r)$ *is given by formula* (6)*. Then problem* (2) *has no positive entire large solutions.*

Proof. Under these hypotheses, from [8, Theorem 3.1] we know that problem (10) has no positive entire large solutions. Thus by applying Theorem 4, problem (2) has no positive entire large solutions either. □

Theorem 5 *Assume $\beta > 1$, f satisfies* (3) *and $p_2 \in C^{0,\mu}_{\text{loc}}(\mathbb{R}^N)$ ($N \geq 3$, $0 < \mu < 1$) is c-positive in Ω_n, where by Ω_n we understand the ball $|x| < n$. If problem* (11) *has no positive entire large solutions, then problem* (2) *has no positive entire large solutions.*

Proof. As seen in the proof of Lemma 1, function $k(u) = u^\beta f(u)$ fulfills conditions (7) and (8), for $\beta > 1$. Using Theorem 1, we can consider v_n to be a nonnegative solution of

$$\begin{cases} \Delta v_n = p_2(x) v_n^\beta f(u) & \text{in } \Omega_n, \\ v_n(x) \to \infty & \text{as } x \to \partial\Omega_n. \end{cases}$$

By the maximum principle, the positive sequence $(v_n)_n$ is monotonically decreasing and thus converges to a function v on $\mathbb{R}^N$. Arguing by contradiction, we assume problem (2) has a positive entire large solution. With the same arguments as in the proof of the above theorem, we obtain that the function v is a positive entire large solution of (11). This contradicts the hypothesis. □

Remark 2 *For problem* (11) *not to have positive entire large solutions under the hypotheses of Theorem 5, p_2 must not fulfill condition* (9)*, since function $k(u) = u^\beta f(u)$ fulfills conditions* (7) *and* (8)*, for $\beta > 1$.*

We saw that under certain conditions, if one of the problems (10) and (11) has no positive entire large solutions, then problem (2) has no positive entire large solutions. What happens when both problems (10) and (11) have positive entire large solutions? We can not give a straight answer, because there are cases when (2) has positive entire large solutions, but there are also cases when (2) does not have such solutions. Let us give some examples in both directions.

For $f \equiv 1$ we consider the particular case of (2)

$$\begin{cases} \Delta u = p_1(x) u^\alpha + p_2(x) u^\beta & \text{in } \mathbb{R}^N, \\ u \geq 0, \ u \not\equiv 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{14}$$

where $N \geq 3$.

Theorem 6 *(Lair [10, Theorem 1]). Suppose the nonnegative functions p_1 and p_2 are locally Hölder continuous on $\mathbb{R}^N$ and have the property that $\min\{p_1(x), p_2(x)\}$ is c-positive in $\mathbb{R}^N$. Suppose $0 < \alpha \leq 1 < \beta$, p_1 satisfies* (9) *and p_2 satisfies*

$$\int_0^\infty r M_{p_2}(r) e^{(\beta-1)(N-2)\int_0^r s M_{p_1}(s) ds} dr < \infty,$$

where M_{p_1}, M_{p_2} are given as in formula (5)*. Then problem* (14) *has a positive entire large solution.*

This is an interesting and new result which complements some results that have already been established. Still, the situation when both problems (10) and (11) have positive entire large solutions while problem (2) does not have such solutions appears to be more intriguing. In [10], Lair showed, for $N \geq 3$ and $\alpha > 2$, that even though the problems

$$\begin{cases} \Delta u = u & \text{in } \mathbb{R}^N, \\ u \geq 0,\ u \not\equiv 0 & \text{in } \mathbb{R}^N. \end{cases} \tag{15}$$

and

$$\begin{cases} \Delta u = e^{-|x|} u^{\alpha} & \text{in } \mathbb{R}^N, \\ u \geq 0,\ u \not\equiv 0 & \text{in } \mathbb{R}^N. \end{cases}$$

have positive entire large solutions, the situation is completely different for the problem

$$\begin{cases} \Delta u = u + e^{-|x|} u^{\alpha} & \text{in } \mathbb{R}^N, \\ u \geq 0,\ u \not\equiv 0 & \text{in } \mathbb{R}^N. \end{cases}$$

In the recent work [3], for $N \geq 3$, f satisfying (3) and $\alpha > 2$, we improved the above results and obtained that (15) and

$$\begin{cases} \Delta u = e^{-|x|} u^{\alpha} f(u) & \text{in } \mathbb{R}^N, \\ u \geq 0,\ u \not\equiv 0 & \text{in } \mathbb{R}^N, \end{cases}$$

have positive entire large solutions while

$$\begin{cases} \Delta u = u + e^{-|x|} u^{\alpha} f(u) & \text{in } \mathbb{R}^N, \\ u \geq 0,\ u \not\equiv 0 & \text{in } \mathbb{R}^N. \end{cases}$$

has not. Here, by giving the next theorem, we make another improvement.

Theorem 7 *Assume $N \geq 3$, $\beta > 2$, $p_1 \in C(\mathbb{R}^N)$ satisfies $p_1(x) = p_1(|x|) \geq 1$ and relation* (13)*, $p_2 \in C^{0,\mu}_{\text{loc}}(\mathbb{R}^N)$ $(0 < \mu < 1)$ satisfies $p_2(x) \geq e^{-|x|}$ and relation* (9)*, f verifies* (3)*. Although both problems*

$$\begin{cases} \Delta u = p_1(|x|) u & \text{in } \mathbb{R}^N, \\ u \geq 0,\ u \not\equiv 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{16}$$

and (11) *have positive entire large solutions, problem*

$$\begin{cases} \Delta u = p_1(|x|) u + p_2(x) u^{\beta} f(u) & \text{in } \mathbb{R}^N, \\ u \geq 0,\ u \not\equiv 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{17}$$

has no such solutions.

Proof.

By Theorem 3 and Lemma 1, problems (16) and (11) have positive entire large solutions. Therefore we focus on showing that (17) does not have such solutions. The proof basically follows the same ideas as in [3] and [10]. Arguing by contradiction, we assume that there exists a positive entire large solution w of (17). Then, we can assume that there exists a radial solution u such that u satisfies the integral equation

$$u(r) = u_0 + \int_0^r t^{1-N} \int_0^t s^{N-1}[p_1(s)u(s) + p_2(s)u^\beta(s)f(u(s))]dsdt, \qquad (18)$$

where $0 < u_0 =: u(0) < w(0)$. We note that if (18) did not have a positive solution valid for all $r > 0$, its solution u, since it is an increasing function, would blow up at some $R > 0$ letting u to be a positive large solution of $\Delta u = p_1(|x|)u + p_2(x)u^\beta f(u)$ on the ball $|x| \leq R$, therefore $u \geq w$ on $|x| \leq R$ contradicting the fact $u_0 < w(0)$. Since we have established that u satisfies (18) and we also have $p_1(|x|) \geq 1$, $p_2(x)u^\beta f(u) \geq 0$, we come to the inequality

$$u(r) \geq u_0 + \int_0^r t^{1-N} \int_0^t s^{N-1}u(s)dsdt.$$

We substitute $u(r) \geq u_0$ into the right side and we obtain

$$u(r) \geq u_0 \left(1 + \frac{r^2}{1!2^1N}\right).$$

We substitute this new expression into the right side and we obtain

$$u(r) \geq u_0 \left(1 + \frac{r^2}{1!2^1N} + \frac{r^4}{2!2^2N(N+2)}\right).$$

Continuing to substitute every new expression obtained into the right side we arrive at

$$u(r) \geq u_0 \sum_{k=0}^{\infty} \frac{r^{2k}}{k!2^kN(N+2)\dots(N+2k-2)}.$$

Rewriting, we get

$$u(r) \geq u_0\Gamma(N/2) \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(\frac{N}{2}+k)} \left(\frac{r}{2}\right)^{2k}$$

hence

$$u(r) \geq Cr^{1-N/2}I_{N/2-1}(r),$$

where $I_{N/2-1}$ is the modified Bessel function with index $N/2-1$. It is a known fact that for r large enough there exists C such that

$$I_{N/2-1}(r) \geq Ce^r r^{-1/2}.$$

Combining the last two relations we obtain

$$u(r) \geq Ce^r r^{(1-N)/2} \qquad \text{for large } r,$$

thus there exists $\varepsilon > 0$ small enough such that

$$u(r) \geq \varepsilon[1 + e^{(1-\varepsilon)r}] \qquad \text{for all } r \geq 0. \tag{19}$$

We will choose $\varepsilon > 0$ small enough so that $\beta - \frac{1}{1-\varepsilon} > 1$.

For $\gamma := \beta - \frac{1}{1-\varepsilon} > 1$ and $c_0 := \varepsilon^{1/(1-\varepsilon)}$, let v_n be a nonnegative solution to the problem

$$\begin{cases} \Delta v_n = c_0 v_n^\gamma & \text{in } \Omega_n, \\ v_n(x) \to \infty & \text{as } x \to \partial\Omega_n, \end{cases}$$

where by Ω_n we understand the ball $|x| < n$. Since $\overline{\Omega}_n \subset \Omega_{n+1}$ we can apply, for each $n \geq 1$, the maximum principle in order to find that $v_n \geq v_{n+1}$ in Ω_n. The nonnegative sequence $(v_n)_n$ is monotonically decreasing and thus converges to a function v on $\mathbb{R}^N$ with

$$\Delta v = c_0 v^\gamma. \tag{20}$$

If we show that $u \leq v$, it follows that v is a positive entire large solution of (20) and we obtain the desired contradiction since (20) has no such solution (see [9] and [14]). Therefore, when we will show that $u \leq v$, the proof of our theorem will be complete.

To obtain $u \leq v$ we will show that $u \leq v_n$ in Ω_n, for all n, since $\mathbb{R}^N = \bigcup_{n=1}^\infty \Omega_n$. For that we will use again the method of reduction to absurdity. Suppose that there exists a n_0 such that $\max_{\overline{\Omega}_{n_0}}[u(x) - v_{n_0}(x)] > 0$. Since this maximum cannot occur on $\partial\Omega_{n_0}$, we deduce that there exists $x_0 \in \Omega_{n_0}$ where it does occur. Keeping in mind (19), the fact that $p_2(x) \geq e^{-|x|}$, $p_1(|x|) \geq 1$ and that $f \geq 1$, at this point x_0 we have

$$\begin{aligned} 0 &\geq \Delta(u - v_{n_0}) = p_1(|x_0|)u + p_2(x_0)u^\beta f(u) - c_0 v_{n_0}^\gamma \geq u + e^{-|x_0|}u^\beta - c_0 v_{n_0}^\gamma \\ &\geq u + e^{-|x_0|}u^{1/(1-\varepsilon)}u^\gamma - c_0 v_{n_0}^\gamma \geq u + e^{-|x_0|}\left[\varepsilon\left(1 + e^{(1-\varepsilon)|x_0|}\right)\right]^{1/(1-\varepsilon)} u^\gamma - c_0 v_{n_0}^\gamma \\ &\geq u + c_0 u^\gamma - c_0 v_{n_0}^\gamma > 0, \end{aligned}$$

which is a contradiction and our proof is complete.

□

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