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Strong convergence of the Modified Halpern-type iterative algorithms in Banach spaces

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Abstract

The purpose of this paper is to introduce a modified Halpern-type iteration algorithm and prove strong convergence of the algorithm for quasi-ϕ-asymptotically non-expansive mappings. Our results improve and extend the corresponding results announced by many others.

1. Introduction

Let E be a real Banach space, C a nonempty subset of E and $T : C \to C$ a nonlinear mapping. A point $x \in C$ is said to be a *fixed point* of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T.

Recall that the mapping T is said to be *non-expansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

T is said to be *asymptotically non-expansive* if there exists a sequence $\{k_n\}$ of real numbers with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \quad \forall x, y \in C.$$

The class of asymptotically non-expansive mappings was introduced by Goebel and Kirk [7] in 1972. They proved that, if C is a nonempty bounded

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closed convex subset of a uniformly convex Banach space E, then every asymptotically non-expansive self-mapping T of C has a fixed point in C. Further, the set $F(T)$ of fixed points of T is closed and convex. Since 1972, many authors have studied the weak and strong convergence problems of the iterative algorithms for such a class of mappings.

In 1976, Halpern [8] introduced the following explicit iteration for a single non-expansive mapping:

$$\begin{cases} x_0 \in C, \ \textit{chosen arbitrarily}, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \quad \forall n \geq 0. \end{cases} \tag{1.1}$$

He pointed out that the conditions

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

are necessary in the sense that, if the iteration scheme (1.1) converges to a fixed point of T, then these conditions must be satisfied. It is well know that the process (1.1) is widely believed to have slow convergence because the restriction of condition ($C2$). To improve the rate of convergence of process (1.1), one cannot rely only on the process itself.

Recently, hybrid projection algorithm has been applied to approximate fixed points of non-expansive mappings and its extensions (see [1,9,10,12-21,23-27,29,30] and the references therein).

Martinez-Yanes and Xu [13] proposed the following modification of the Halpern iteration for a single non-expansive mapping T in a Hilbert space. To be more precise, they proved the following theorem:

Theorem MX. *Let H be a real Hilbert space, C a closed convex subset of H and $T : C \to C$ a non-expansive mapping such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\} \subset (0,1)$ is such that $\lim_{n\to\infty} \alpha_n = 0$. Then the sequence $\{x_n\}$ defined by*

$$\begin{cases} x_0 \in C, \ \textit{chosen arbitrarily}, \\ y_n = \alpha_n x_0 + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \alpha_n(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle)\}, \\ Q_n = \{z \in C : \langle x_0 - x_n, x_n - z\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad \forall n \geq 0, \end{cases}$$

converges strongly to $P_{F(T)}x_0$.

Subsequently, Qin et al. [18] improved Theorem 3.1 of Martinez-Yanes and Xu [13] from non-expansive mappings to asymptotically non-expansive mappings still in the framework of Hilbert spaces. Recently, Qin and Su [17] further improved the result of Martinez-Yanes and Xu [13] from Hilbert spaces to Banach spaces. To be more precise, they proved the following theorem:

Theorem QS. *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let $T : C \to C$ be a relatively non-expansive mapping. Assume that $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in C \quad \textit{chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)JTx_n), \\ C_n = \{v \in C : \phi(v, y_n) \le \alpha_n \phi(v, x_0) + (1-\alpha_n)\phi(v, x_n), \\ Q_n = \{v \in C : \langle Jx_0 - Jx_n, x_n - v\rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad \forall n \ge 0, \end{cases}$$

where J is the single-valued duality mapping on E. If $F(T)$ is nonempty, then $\{x_n\}$ converges to $\Pi_{F(T)}x_0$.

Very recently, Plubtieng and Ungchittrakool [15] also considered the hybrid projection algorithm to modify the Halpern iteration (1.1) and obtained a strong convergence theorem for a pair of relatively non-expansive mappings in the framework of Banach spaces, see [15] for more details.

Motivated and inspired by the research going on in this direction, we modify the iterative process (1.1) for closed quasi-ϕ-asymptotically non-expansive mappings (see below) in the framework of Banach spaces. Our results improve and extend the corresponding result announced by many others.

2. Preliminaries

Let E be a Banach space with the dual space E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{f^* \in E^* : \langle x, f^*\rangle = \|x\|^2 = \|f^*\|^2\}, \quad \forall x \in E,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that, if E^* is strictly convex, then J is single-valued and, if E^* is uniformly convex, then J is uniformly continuous on bounded subsets of E.

Also, it is well known that, if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is non-expansive. This fact actually characterizes Hilbert spaces and, consequently, it is not available in more general Banach spaces. In this connection, Alber [3] recently introduced a generalized projection operator Π_C

in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Let E be a smooth Banach space. Consider the functional defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E. \tag{2.1}$$

Observe that, in a Hilbert space H, (2.1) reduces to $\phi(x,y) = \|x-y\|^2$ for all $x, y \in H$. The generalized projection $\Pi_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x,y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the following minimization problem:

$$\phi(\bar{x}, x) = \inf_{y \in C} \phi(y, x). \tag{2.2}$$

The existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(x,y)$ and the strict monotonicity of the mapping J (see, for example, [2,3,6,11]). In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of the function ϕ that

$$(\|y\| - \|x\|)^2 \le \phi(y,x) \le (\|y\| + \|x\|)^2, \quad \forall x, y \in E. \tag{2.3}$$

Remark 2.1. If E is a reflexive, strictly convex and smooth Banach space, then, for all $x, y \in E$, $\phi(x,y) = 0$ if and only if $x = y$. It is sufficient to show that, if $\phi(x,y) = 0$, then $x = y$. From (2.3), we have $\|x\| = \|y\|$. This implies $\langle x, Jy\rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, one has $Jx = Jy$. Therefore, we have $x = y$ (see [6,28] for more details).

Now, we give some definitions for our main results in this paper.

Let C be a nonempty, closed and convex subset of a smooth Banach E and T a mapping from C into itself.

(1) A point p in C is said to be an *asymptotic fixed point* [22] of T if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{F(T)}$.

(2) A mapping T from C into itself is said to be *relatively non-expansive* [4,5,11,12,17] if

$$\widetilde{F(T)} = F(T) \neq \emptyset, \quad \phi(p, Tx) \le \phi(p, x), \quad \forall x \in C,\, p \in F(T).$$

The asymptotic behavior of a relatively non-expansive mapping was studied in [4,5,22].

(3) The mapping T is said to be *relatively asymptotically non-expansive* [1,19,21,23] if

$$\widetilde{F(T)} = F(T) \neq \emptyset, \quad \phi(p, T^n x) \le k_n \phi(p, x), \quad \forall x \in C,\, p \in F(T),$$

where $k_n \geq 1$ is a sequence such that $k_n \to 1$ as $n \to \infty$.

(4) The mapping T is said to be *ϕ-nonexpansive* [16,20] if

$$\phi(Tx, Ty) \leq \phi(x, y), \quad \forall x, y \in C.$$

(5) The mapping T is said to be *quasi-ϕ-non-expansive* [16,20] if

$$F(T) \neq \emptyset, \quad \phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C, \, p \in F(T).$$

(6) The mapping T is said to be *ϕ-asymptotically non-expansive* if there exists some real sequence $\{k_n\}$ with $k_n \geq 1$ and $k_n \to \infty$ as $n \to \infty$ such that

$$\phi(T^n x, T^n y) \leq k_n \phi(x, y), \quad \forall x, y \in C.$$

(7) The mapping T is said to be *quasi-ϕ-asymptotically non-expansive* if

$$F(T) \neq \emptyset, \quad \phi(p, T^n x) \leq k_n \phi(p, x), \quad \forall x \in C, \, p \in F(T).$$

(8) The mapping T is said to be *asymptotically regular* on C if, for any bounded subset K of C,

$$\limsup_{n \to \infty} \{\|T^{n+1}x - T^n x\| : x \in K\} = 0.$$

(9) The mapping T is said to be *closed* on C if, for any sequence $\{x_n\}$ such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} Tx_n = y_0$, then $Tx_0 = y_0$.

Remark 2.2. The class of quasi-ϕ-nonexpansive mappings and quasi-ϕ-asymptotically non-expansive mappings are more general than the class of relatively non-expansive mappings and relatively asymptotically non-expansive mappings, respectively. The quasi-ϕ-nonexpansive mappings and quasi-ϕ-asymptotically non-expansive mappings do not require $F(T) = \widetilde{F(T)}$, where $\widetilde{F(T)}$ denotes the asymptotic fixed point set of T (see [4-6] for more details).

Remark 2.3. A ϕ-asymptotically non-expansive mapping with $F(T) \neq \emptyset$ is a quasi-ϕ-asymptotically non-expansive mapping, but the converse may be not true.

Next, we give some examples which are closed quasi-ϕ-asymptotically non-expansive mappings.

Example 2.4 (Qin et al. [16]). Let E be a uniformly smooth and strictly convex Banach space and $A \subset E \times E^*$ be a maximal monotone mapping such that its zero set $A^{-1}0$ is nonempty. Then $J_r = (J + rA)^{-1}J$ is a closed quasi-ϕ-asymptotically non-expansive mapping from E onto $D(A)$ and $F(J_r) = A^{-1}0$.

Example 2.5 (Qin et al. [16]). Let Π_C be the generalized projection from a smooth, strictly convex and reflexive Banach space E onto a nonempty closed convex subset C of E. Then Π_C is a closed quasi-ϕ-asymptotically non-expansive mapping from E onto C with $F(\Pi_C) = C$.

A Banach space E is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be *uniformly convex* if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$.

Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be *smooth* provided

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.4}$$

exists for each $x, y \in U$. It is also said to be *uniformly smooth* if the limit (2.4) is attained uniformly for $x, y \in E$. It is well known that, if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.

In order to the main results of this paper, we need the following lemmas.

Lemma 2.1 ([11]). *Let E be a uniformly convex and smooth Banach space and $\{x_n\}$, $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n - y_n \to 0$.*

Lemma 2.2 ([3]). *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0\rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.3 ([3]). *Let E be a reflexive, strictly convex and smooth Banach space and C a nonempty closed convex subset of E. Let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x). \quad \forall y \in C.$$

Lemma 2.4. *Let E be a uniformly convex and uniformly smooth Banach space, C a nonempty, closed and convex subset of E and T a closed quasi-ϕ-asymptotically non-expansive mapping from C into itself. Then $F(T)$ is a closed convex subset of C.*

Proof. The closedness of $F(T)$ can be deduced by the closedness of T. Next, we show that $F(T)$ is convex. for $x, y \in F(T)$ and $t \in (0, 1)$, put $p = tx + (1-$

$t)y$. It is sufficient to show $Tp = p$. In fact, we have

$$
\begin{aligned}
&\phi(p, T^n p) \\
&= \|p\|^2 - 2\langle p, JT^n p\rangle + \|T^n p\|^2 \\
&= \|p\|^2 - 2\langle tx + (1-t)y, JT^n p\rangle + \|T^n p\|^2 \\
&= \|p\|^2 - 2t\langle x, JT^n p\rangle - 2(1-t)\langle y, JT^n p\rangle + \|T^n p\|^2 \\
&= \|p\|^2 + t\phi(x, T^n p) + (1-t)\phi(y, T^n p) - t\|x\|^2 - (1-t)\|y\|^2 \\
&\le \|p\|^2 + k_n t\phi(x, p) + k_n(1-t)\phi(y, p) - t\|x\|^2 - (1-t)\|y\|^2 \\
&= (k_n - 1)(t\|x\|^2 + (1-t)\|y\|^2 - \|p\|^2).
\end{aligned}
\tag{2.5}
$$

Let $n \to \infty$ in (2.5) yields that $\lim_{n\to\infty} \phi(p, T^n p) = 0$. We, therefore, apply Lemma 2.1 to see that $T^n p \to p$ as $n \to \infty$. Hence

$$TT^n p = T^{n+1} p \to p$$

as $n \to \infty$. By the closed-ness of T, it follows that $p \in F(T)$. This completes the proof.

3. Main results

Now, we are ready to give our main results in this paper.

Theorem 3.1. *Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space E and $T : C \to C$ a closed quasi-ϕ-asymptotically non-expansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ such that $k_n \to 1$ as $n \to \infty$. Assume that T is asymptotically regular on C, $F(T) \neq \emptyset$ and $F(T)$ is bounded. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$
\begin{cases}
x_0 \in E \text{ chosen arbitrarily}, \\
C_1 = C, \\
x_1 = \Pi_{C_1} x_0, \\
y_n = J^{-1}[\alpha_n Jx_1 + (1-\alpha_n)JT^n x_n], \\
C_{n+1} = \{z \in C_n : \phi(z, y_n) \le \phi(z, x_n) + \alpha_n M\}, \\
x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \ge 0,
\end{cases}
\tag{3.1}
$$

where M is an appropriate constant such that $M \ge \phi(w, x_1)$ for all $w \in F(T)$. Assume that the control sequence $\{\alpha_n\}$ in $(0, 1)$ satisfies the following restrictions:

(a) $\lim_{n\to\infty} \alpha_n = 0$,

(b) $(1-\alpha_n)k_n \le 1$ *for all* $n \ge 0$.

Then $\{x_n\}$ *converges strongly to* $\Pi_{F(T)}x_1$.

Proof. First, we show that C_n is closed and convex for all $n \ge 1$. It is obvious that $C_1 = C$ is closed and convex. Suppose that C_h is closed and convex for some $h \in \mathbb{N}$. For any $z \in C_h$ such that

$$\phi(z, y_h) \le \phi(z, x_h) + \alpha_h M.$$

This inequality is equivalent to

$$2\langle z, Jx_h\rangle - 2\langle z, Jy_h\rangle \le \|x_h\|^2 - \|y_h\|^2 + \alpha_h M.$$

It is to see that C_{h+1} is closed and convex. Then, for all $n \ge 1$, C_n is closed and convex.

Next, we prove that $F(T) \subset C_n$ for all $n \ge 1$. $F(T) \subset C_1 = C$ is obvious. Suppose that $F(T) \subset C_h$ for some $h \in \mathbb{N}$. Then, for all $w \in F(T) \subset C_h$, one has

$$\begin{aligned}\phi(w, y_h) &= \phi(w, J^{-1}[\alpha_h Jx_1 + (1-\alpha_h)JT^h x_h]) \\ &= \|w\|^2 - 2\langle w, \alpha_h Jx_1 + (1-\alpha_h)JT^h x_h\rangle \\ &\quad + \|\alpha_h Jx_1 + (1-\alpha_h)JT^h x_h\|^2 \\ &\le \|w\|^2 - 2\alpha_h\langle w, Jx_1\rangle - 2(1-\alpha_h)\langle w, JT^h x_h\rangle \\ &\quad + \alpha_h\|x_1\|^2 + (1-\alpha_h)\|T^h x_h\|^2 \\ &= \alpha_h\phi(w, x_1) + (1-\alpha_h)\phi(w, T^h x_h) \\ &\le \alpha_h\phi(w, x_1) + (1-\alpha_h)k_h\phi(w, x_h) \\ &= \phi(w, x_h) - [1-(1-\alpha_h)k_h]\phi(w, x_h) + \alpha_h\phi(w, x_1) \\ &\le \phi(w, x_h) + \alpha_h M,\end{aligned}$$

which shows $w \in C_{h+1}$. This implies that $F(T) \subset C_n$ for all $n \ge 1$. From $x_n = \Pi_{C_n}x_1$, one sees

$$\langle x_n - z, Jx_1 - Jx_n\rangle \ge 0, \quad \forall z \in C_n. \tag{3.2}$$

Since $F(T) \subset C_n$ for all $n \ge 1$, we arrive at

$$\langle x_n - w, Jx_1 - Jx_n\rangle \ge 0, \quad \forall w \in F(T). \tag{3.3}$$

From Lemma 2.3, one has

$$\phi(x_n, x_1) = \phi(\Pi_{C_n}x_1, x_1) \le \phi(w, x_1) - \phi(w, x_n) \le \phi(w, x_1)$$

for all $w \in F(T) \subset C_n$ and $n \ge 1$. The sequence $\phi(x_n, x_1)$ is, therefore, bounded.

On the other hand, noticing that $x_n = \Pi_{C_n} x_1$ and $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$, one has

$$\phi(x_n, x_1) \le \phi(x_{n+1}, x_1), \quad \forall n \ge 1.$$

Therefore, $\{\phi(x_n, x_1)\}$ is nondecreasing and so the limit of $\{\phi(x_n, x_1)\}$ exists. By the construction of C_n, one knows that $C_m \subset C_n$ and $x_m = P_{C_m} x_1 \in C_n$ for any positive integer $m \ge n$. It follows that

$$\begin{aligned} \phi(x_m, x_n) &= \phi(x_m, \Pi_{C_n} x_1) \\ &\le \phi(x_m, x_1) - \phi(\Pi_{C_n} x_1, x_1) \\ &= \phi(x_m, x_1) - \phi(x_n, x_1). \end{aligned} \tag{3.4}$$

Letting $m, n \to \infty$ in (3.4), one has $\phi(x_m, x_n) \to 0$. It follows from Lemma 2.1 that $x_m - x_n \to 0$ as $m, n \to \infty$ Hence $\{x_n\}$ is a Cauchy sequence in C. Since E is a Banach space and C is closed and convex, one can assume that

$$x_n \to p \in C \quad (n \to \infty).$$

Finally, we show that $p = \Pi_{F(T)} x_1$. To end this, we first show that $p \in F(T)$. By taking $m = n + 1$ in (3.4), one arrives at

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{3.5}$$

From Lemma 2.1, it follows that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.6}$$

Noticing that $x_{n+1} \in C_{n+1}$, one obtains

$$\phi(x_{n+1}, y_n) \le \phi(x_{n+1}, x_n) + \alpha_n M.$$

It follows from (3.5) and the assumption (a) that

$$\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0.$$

Thus, from Lemma 2.1, one has

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0. \tag{3.7}$$

Notice that

$$\|x_n - y_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|.$$

It follows from (3.6) and (3.7) that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.8}$$

Since J is uniformly norm-to-norm continuous on any bounded sets, we have

$$\lim_{n\to\infty} \|Jx_n - Jy_n\| = 0. \tag{3.9}$$

On the other hand, we have

$$\|Jy_n - JT^n x_n\| = \alpha_n \|JTx_1 - JT^n x_n\|.$$

By the assumption (a), one sees that

$$\lim_{n\to\infty} \|Jy_n - JT^n x_n\| = 0.$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{n\to\infty} \|y_n - T^n x_n\| = 0. \tag{3.10}$$

On the other hand, one has

$$\|x_n - T^n x_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \|y_n - T^n x_n\|.$$

From (3.6), (3.7) and (3.10), it follows that $\lim_{n\to\infty} \|T^n x_n - x_n\| = 0$. Noting that $x_n \to p$ as $n \to \infty$, one has

$$T^n x_n \to p \quad (n \to \infty). \tag{3.11}$$

On the other hand, one has

$$\|T^{n+1} x_n - p\| \le \|T^{n+1} x_n - T^n x_n\| + \|T^n x_n - p\|.$$

Thus it follows from the asymptotic regularity of T and (3.11) that

$$T^{n+1} x_n \to p \quad (n \to \infty).$$

That is, $TT^n x_n \to p$. From the closedness of T, one gets $p = Tp$.

Finally, we show that $p = \Pi_{F(T)} x_1$. From $x_n = \Pi_{C_n} x_1$, one has

$$\langle x_n - w, Jx_1 - Jx_n \rangle \ge 0, \quad \forall w \in F(T) \subset C_n. \tag{3.12}$$

Taking the limit as $n \to \infty$ in (3.12), we obtain

$$\langle p - w, Jx_1 - Jp \rangle \ge 0, \quad \forall w \in F(T),$$

and hence $p = \Pi_{F(T)}x_1$ by Lemma 2.2. This completes the proof.

Remark 3.2. Theorem 3.1 improves the corresponding results of Martinez-Yanes and Xu [13] and Qin et al. [18] from Hilbert spaces to Banach spaces. Theorem 3.1 also improves Qin et al. [20] from quasi-ϕ-nonexpansive mapping to quasi-ϕ-asymptotically nonexpansive mappings.

In Hilbert spaces, Theorem 3.1 is reduced to the following result.

Theorem 3.3. *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $T : C \to C$ be a closed asymptotically quasi-nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ such that $k_n \to 1$ as $n \to \infty$. Assume that T is asymptotically regular on C, $F(T) \neq \emptyset$ and $F(T)$ is bounded. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{cases} x_0 \in H \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = P_{C_1}x_0, \\ y_n = \alpha_n x_1 + (1-\alpha_n)T^n x_n, \\ C_{n+1} = \{z \in C_n : \|z - y_n\|^2 \leq \|z - x_n\|^2 + \alpha_n M\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad \forall n \geq 0, \end{cases}$$

where M is an appropriate constant such that $M \geq \|w - x_1\|^2$ for all $w \in F(T)$. Assume that the control sequence $\{\alpha_n\}$ in $(0, 1)$ satisfies the restrictions:

(a) $\lim_{n\to\infty} \alpha_n = 0$,

(b) $(1 - \alpha_n)k_n \leq 1$ *for all* $n \geq 0$.

Then $\{x_n\}$ converges strongly to $P_{F(T)}x_1$.

Remark 3.4. Theorem 3.3 improves Theorem 3.1 of Martinez-Yanes and Xu [13] in the following senses:

(1) from non-expansive mappings to asymptotically quasi-nonexpansive mappings.

(2) from computation point of view, the hybrid projection algorithm in Theorem 3.2 is also more simple and convenient to compute than the one given by Martinez-Yanes and Xu. To be more precise, we remove the set "Q_n" in [13].

Next, we give a strong convergence theorem for an infinite family of quasi-ϕ-asymptotically non-expansive mappings.

Theorem 3.5. *Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space E and $\{T_i\}_{i\in I} : C \to C$*

a family of closed quasi-ϕ-asymptotically non-expansive mappings such that $F = \bigcap_{i\in I} F(T_i) \neq \emptyset$. Assume that T_i is asymptotically regular on C for each $i \in I$ and F is bounded. For each $i \in I$, let $\{\alpha_{n,i}\}$ be a sequence in $(0, 1)$ such that

(a) $\lim_{n\to\infty} \alpha_{n,i} = 0$,

(b) $(1 - \alpha_{n,i})k_{n,i} \leq 1$ *for each* $i \in I$.

Define a sequence $\{x_n\}$ in C in the following manner:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_{n,i} = J^{-1}[\alpha_{n,i}Jx_0 + (1-\alpha_{n,i})JT_i^n x_n], \\ C_{n,i} = \{z \in C : \phi(z, y_{n,i}) \leq \phi(z, x_n) + \alpha_{n,i}Q\}, \\ C_n = \bigcap_{i\in I} C_{n,i}, \\ Q_0 = C, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n\rangle\}, \\ x_{n+1} = \Pi_{C_n\cap Q_n} x_0, \quad \forall n \geq 0, \end{cases} \tag{3.13}$$

where Q is an appropriate constant such that $Q \geq \phi(w, x_0)$ for all $w \in F$. Then $\{x_n\}$ converges strongly to $\Pi_F x_0$.

Proof. We first show that C_n and Q_n are closed and convex for each $n \geq 0$. From the definition of C_n and Q_n, it is obvious that C_n is closed and Q_n is closed and convex for each $n \geq 0$. We show that C_n is convex for each $n \geq 0$. Indeed,

$$C_{n,i} = \{z \in C : \phi(z, y_{n,i}) \leq \phi(z, x_n) + \alpha_{n,i}Q\}$$

is equivalent to

$$C_{n,i} = \{z \in C : 2\langle z, Jx_n\rangle - 2\langle z, Jy_{n,i}\rangle \leq \|x_n\|^2 - \|y_{n,i}\|^2 + \alpha_{n,i}Q\}.$$

This shows that $C_{n,i}$ is closed convex for each $n \geq 0$ and $i \in I$. Therefore, one has $C_n = \bigcap_{i\in I} C_{n,i}$ is closed convex for each $n \geq 0$.

Next, we show that $F \subset C_n$ for all $n \geq 0$. For all $w \in F \subset C$ and $i \in I$, one has

$$\begin{aligned}
\phi(w, y_{n,i}) &= \phi(w, J^{-1}[\alpha_{n,i}Jx_0 + (1-\alpha_{n,i})JT_i^n x_n]) \\
&= \|w\|^2 - 2\langle w, \alpha_n Jx_0 + (1-\alpha_{n,i})JT_i^n x_n\rangle \\
&\quad + \|\alpha_{n,i}Jx_0 + (1-\alpha_{n,i})JT_i^n x_n\|^2 \\
&\le \|w\|^2 - 2\alpha_{n,i}\langle w, Jx_0\rangle - 2(1-\alpha_{n,i})\langle w, JT_i^n x_n\rangle \\
&\quad + \alpha_{n,i}\|x_0\|^2 + (1-\alpha_{n,i})\|T_i^n x_n\|^2 \\
&\le \alpha_{n,i}\phi(w, x_0) + (1-\alpha_{n,i})\phi(w, T_i^n x_n) \\
&\le \alpha_{n,i}\phi(w, x_0) + (1-\alpha_{n,i})k_{n,i}\phi(w, x_n), \\
&= \phi(w, x_n) - [1-(1-\alpha_{n,i})k_{n,i}]\phi(w, x_n) + \alpha_{n,i}\phi(w, x_0) \\
&\le \phi(w, x_n) + \alpha_{n,i}Q,
\end{aligned}$$

which yields that $w \in C_{n,i}$ for all $n \ge 0$ and $i \in I$. It follows that $w \in C_n = \bigcap_{i\in I} C_{n,i}$. This proves that $F \subset C_n$ for all $n \ge 0$.

Next, we prove that $F \subset Q_n$ for all $n \ge 0$ by induction. For $n = 0$, we have $F \subset C = Q_0$. Assume that $F \subset Q_{n-1}$ for some $n \ge 1$, we show that $F \subset Q_n$ for the same $n \ge 1$. Since x_n is the projection of x_0 onto $C_{n-1} \cap Q_{n-1}$, we arrive at

$$\langle x_n - z, Jx_0 - Jx_n\rangle \ge 0, \quad \forall z \in C_{n-1} \cap Q_{n-1}. \tag{3.14}$$

Since $F \subset C_{n-1} \cap Q_{n-1}$ by the induction assumptions, (3.14) holds, in particular, for all $w \in F$. This together with the definition of Q_n implies that $F \subset Q_n$ for all $n \ge 0$. Noticing that $x_{n+1} = \Pi_{C_n \cap Q_n} x_0 \in Q_n$ and $x_n = \Pi_{Q_n} x_0$, one sees

$$\phi(x_n, x_0) \le \phi(x_{n+1}, x_0), \quad \forall n \ge 0. \tag{3.15}$$

We, therefore, obtain that $\{\phi(x_n, x_0)\}$ is nondecreasing. From Lemma 2.3, it follows that

$$\phi(x_n, x_0) = \phi(\Pi_{Q_n} x_0, x_0) \le \phi(w, x_0) - \phi(w, x_n) \le \phi(w, x_0)$$

for all $w \in F \subset C_n$ and $n \ge 0$. This shows that $\{\phi(x_n, x_0)\}$ is bounded. It follows that the limit of $\{\phi(x_n, x_0)\}$ exists. By the construction of Q_n, one knows that $Q_m \subset Q_n$ and $x_m = \Pi_{Q_m} x_0 \in Q_n$ for any positive integer $m \ge n$. Notice that

$$\begin{aligned}
\phi(x_m, x_n) &= \phi(x_m, \Pi_{Q_n} x_0) \\
&\le \phi(x_m, x_0) - \phi(\Pi_{Q_n} x_0), x_0) \\
&= \phi(x_m, x_0) - \phi(x_n, x_0).
\end{aligned} \tag{3.16}$$

Taking the limit as $m, n \to \infty$ in (3.16), one gets $\phi(x_m, x_n) \to 0$. From Lemma 2.1, it follows that $x_m - x_n \to 0$ as $m, n \to \infty$ and so $\{x_n\}$ is a Cauchy sequence. Since E is a Banach space and C is closed and convex, one can assume that

$$x_n \to q \in C \quad (n \to \infty).$$

Finally, we show that $q = \Pi_F x_0$. To end this, we first show that $q \in F$. By taking $m = n + 1$ in (3.16), one arrives at

$$\phi(x_{n+1}, x_n) \to 0 \quad (n \to \infty). \tag{3.17}$$

From Lemma 2.1, one has

$$x_{n+1} - x_n \to 0 \quad (n \to \infty). \tag{3.18}$$

Noticing that $x_{n+1} \in C_{n+1}$, one obtains

$$\phi(x_{n+1}, y_{n,i}) \leq \phi(x_{n+1}, x_n) + \alpha_{n,i} Q.$$

It follows from the assumption on $\{\alpha_{n,i}\}$ and (3.17) that

$$\lim_{n\to\infty} \phi(x_{n+1}, y_{n,i}) = 0, \quad \forall i \in I.$$

Thus, from Lemma 2.1, one obtains

$$\lim_{n\to\infty} \|x_{n+1} - y_{n,i}\| = 0, \quad \forall i \in I. \tag{3.19}$$

On the other hand, we have $\|Jy_{n,i} - JT_i x_n\| = \alpha_{n,i}\|Jx_0 - JT_i^n x_n\|$. By the assumption (a), one sees

$$\lim_{n\to\infty} \|Jy_{n,i} - JT_i^n x_n\| = 0, \quad \forall i \in I.$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{n\to\infty} \|y_{n,i} - T_i^n x_n\| = 0. \tag{3.20}$$

On the other hand, one has

$$\|x_n - T_i^n x_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_{n,i}\| + \|y_{n,i} - T_i^n x_n\|.$$

From (3.18)-(3.20), one sees that $\lim_{n\to\infty} \|T_i^n x_n - x_n\| = 0$. Noting that $x_n \to q$ as $n \to \infty$, one has

$$T_i^n x_n \to q \quad (n \to \infty). \tag{3.21}$$

On the other hand, one has

$$\|T_i^{n+1}x_n - q\| \le \|T_i^{n+1}x_n - T_i^n x_n\| + \|T_i^n x_n - q\|.$$

It follows from the asymptotic regularity of T_i and (3.21) that $T_i^{n+1}x_n = T_i T_i^n x_n \to q$ as $n \to \infty$. From the closed-ness of T_i, one gets $q = T_i q$ for each $i \in I$, that is, $q \in F$.

Finally, we show that $q = \Pi_F x_0$. From $x_n = \Pi_{Q_n} x_0$, it follows that

$$\langle x_n - w, Jx_0 - Jx_n\rangle \ge 0, \quad \forall w \in F. \tag{3.22}$$

Taking the limit as $n \to \infty$ in (3.22), we obtain

$$\langle q - w, Jx_0 - Jq\rangle \ge 0, \quad \forall w \in F,$$

and hence $q = \Pi_F x_0$ by Lemma 2.2. This completes the proof.

In Hilbert spaces, Theorem 3.5 reduces to the following theorem.

Theorem 3.6. *Let C be a nonempty, closed and convex subset of a Hilbert space H and $\{T_i\}_{i\in I} : C \to C$ a family of closed asymptotically quasi-nonexpansive mappings such that $F = \bigcap_{i\in I} F(T_i) \ne \emptyset$. Assume that T_i is asymptotically regular on C for each $i \in I$ and F is bounded. For each $i \in I$, let $\{\alpha_{n,i}\}$ be a sequence in $(0,1)$ such that*

(a) $\lim_{n\to\infty} \alpha_{n,i} = 0$,

(b) $(1 - \alpha_{n,i})k_{n,i} \le 1$ *for each* $i \in I$.

Define a sequence $\{x_n\}$ in C in the following manner:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_{n,i} = \alpha_{n,i}x_0 + (1-\alpha_{n,i})T_i^n x_n, \\ C_{n,i} = \{z \in C : \|z - y_{n,i}\|^2 \le \|z - x_n\|^2 + \alpha_{n,i}Q\}, \\ C = \bigcap_{i\in I} C_i, \\ Q_0 = C, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle\}, \\ x_{n+1} = P_{C_n\cap Q_n} x_0, \quad \forall n \ge 0, \end{cases}$$

where Q is an appropriate constant such that $Q \ge \|w - x_0\|^2$ for all $w \in F$. Then $\{x_n\}$ converges strongly to $P_F x_0$.

Remark 3.7. Theorem 3.6 improves Theorem 3.1 of Martinez-Yanes and Xu [13] from a single non-expansive mapping to an infinite family asymptotically non-expansive mappings. Theorem 2.2 of Qin et al. [18] is also a special case of Theorem 3.6.

References

[1] R. P. Agarwal, Y. J. Cho and X. Qin, *Generalized projection algorithms for nonlinear operators,* Numer. Funct. Anal. Optim. **28** (2007), 1197–1215.

[2] Ya. I. Alber and S. Reich, *An iterative method for solving a class of nonlinear operatorequations in Banach spaces,* Panamer. Math. J. **4** (1994), 39–54.

[3] Ya. I. Alber, *Metric and generalized projection operators in Banach spaces: properties and applications,* in: A. G. Kartsatos (Ed.), Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Marcel Dekker, New York, 1996, pp. 15–50.

[4] D. Butnariu, S. Reich and A.J. Zaslavski, *Asymptotic behavior of relatively nonexpansive operators in Banach spaces,* J. Appl. Anal. **7** (2001), 151–174.

[5] D. Butnariu, S. Reich and A.J. Zaslavski, *Weak convergence of orbits of nonlinear operators in reflexive Banach spaces,* Numer. Funct. Anal. Optim. **24** (2003), 489–508.

[6] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems,* Kluwer, Dordrecht, 1990.

[7] K. Goebel and W.A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings,* Proc. Amer. Math. Soc. **35** (1972), 171–174.

[8] B. Halpern, *Fixed points of nonexpanding maps,* Bull. Amer. Math. Soc. **73** (1967), 957–961.

[9] T. H. Kim and H. K. Xu, *Strong convergence of modified Mann iterations for asymptotically nonexpansive mappings and semigroups,* Nonlinear Anal. **64** (2006), 1140–1152.

[10] T. H. Kim and H. K. Xu, *Convergence of the modified Mann's iteration method for asymptotically strict pseudo-contractions,* Nonlinear Anal. **68** (2008), 2828–2836.

[11] S. Kamimura and W. Takahashi, *Strong convergence of a proximal-type algorithm in a Banach space,* SIAM J. Optim. **13** (2002), 938–945.

[12] S. Matsushita and W. Takahashi, *A strong convergence theorem for relatively nonexpansive mappings in a Banach space,* J. Approx. Theory **134** (2005), 257–266.

[13] C. Martinez-Yanes and H. K. Xu, *Strong convergence of the CQ method for fixed point iteration processes,* Nonlinear Anal. **64** (2006), 2400–2411.

[14] K. Nakajo and W. Takahashi, *Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups,* J. Math. Anal. Appl. **279** (2003), 372–379.

[15] S. Plubtieng and K. Ungchittrakool, *Strong convergence theorems for a common fixed point of two relatively nonexpansive mappings in a Banach space,* J. Approx. Theory **149** (2007), 103–115.

[16] X. Qin, Y. J. Cho and S. M. Kang, *Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces,* J. Comput. Appl. Math. **225** (2009), 20–30.

[17] X. Qin and Y. Su, *Strong convergence theorems for relatively nonexpansive mappings in a Banach space,* Nonlinear Anal. **67** (2007), 1958–1965.

[18] X. Qin, Y. Su and M. Shang, *Strong convergence theorems for asymptotically nonexpansive mappings by hybrid methods,* Kyungpook Math. J. **48** (2008), 133–142.

[19] X. Qin, S. M. Kang and S. Y. Cho, *Strong convergence theorems of fixed points for relatively asymptotically nonexpansive mappings,* J. Chungcheong Math. Soc. **21** (2008), 1–9.

[20] X. Qin, Y. J. Cho, S. M. Kang and H. Zhou, *Convergence of a modified Halpern-type iteration algorithm for quasi-ϕ-nonexpansive mappings,* Appl. Math. Lett. doi:10.1016/j.aml.2009.01.015.

[21] X. Qin, Y. Su, C. Wu and K. Liu, *Strong convergence theorems for nonlinear operators in Banach spaces,* Commun. Appl. Nonlinear Anal. **21** (2008), 1–9.

[22] S. Reich, *A weak convergence theorem for the alternating method with Bregman distance,* in: A.G. Kartsatos (Ed.), Theory and Applicationsof Nonlinear Operatorsof Accretive and Monotone Type, Marcel Dekker, New York, 1996.

[23] Y. Su and X. Qin, *Strong convergence of modified Ishikawa iterations for nonlinear mappings,* Proc. Indian Acad. Sci. (Math. Sci.) **117** (2007), 97–107.

[24] Y. Su and X. Qin, *Strong convergence theorems for asymptotically nonexpansive mappings and asymptotically nonexpansive semigroups,* Fixed Point Theory Appl. 2006 (2006) Art. ID 96215.

[25] Y. Su and X. Qin, *Monotone CQ iteration processes for nonexpansive semigroups and maximal monotone operators,* Nonlinear Anal. **68** (2008), 3657–3664.

[26] Y. Su, D. Wang and M. Shang, *Strong convergence of monotone hybrid algorithm for hemi-relatively nonexpansive mappings,* Fixed Point Theory Appl. **2008** (2008), 284613.

[27] W. Takahashi, Y. Takeuchi and R. Kubota, *Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces,* J. Math. Anal. Appl. **341** (2008), 276–286.

[28] W. Takahashi, *Nonlinear Functional Analysis,* Yokohama-Publishers, 2000.

[29] Y. Yao, R. Chen and H. Zhou, *Strong convergence and control condition of modified Halpern iterations in Banach spaces*, Int. J. Math. Math. Sci. **2006** (2006), Art. ID 29728.

[30] H. Zhou, *Convergence theorems of fixed points for Lipschitz pseudo-contractions in Hilbert spaces,* J. Math. Anal. Appl. **343** (2008), 546–556.

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