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Endpoint boundedness for multilinear integral operators of some sublinear operators on Herz and Herz type Hardy spaces

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Abstract

The purpose of this paper is to study the endpoint boundedness properties of some multilinear operators related to certain integral operators on Herz and Herz type Hardy Spaces. The operators include Littlewood-Paley operator and Marcinkiewicz operator.

1. Introduction and Theorems

In this paper, we will study some multilinear operators related to some integral operators, whose definition are the following ones.

Fix $\delta > 0$, suppose that m is a positive integer and A be a function on R^n. We denote that that $\Gamma(x) = \{(y,t) \in R_+^{n+1} : |x-y| < t\}$ and the characteristic function of $\Gamma(x)$ by $\chi_{\Gamma(x)}$. Let

$$R_{m+1}(A;x,y) = A(x) - \sum_{|\beta|\leq m} \frac{1}{\beta!} D^\beta A(y)(x-y)^\beta$$

and

$$Q_{m+1}(A;x,y) = R_m(A;x,y) - \sum_{|\beta|=m} \frac{1}{\beta!} D^\beta A(x)(x-y)^\beta.$$

Definition 1. Let $\varepsilon > 0$ and ψ be a fixed function which satisfies the following properties:

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(1) $\int_{R^n} \psi(x)dx = 0$,
(2) $|\psi(x)| \le C(1+|x|)^{-(n+1-\delta)}$,
(3) $|\psi(x+y)-\psi(x)| \le C|y|^{\varepsilon}(1+|x|)^{-(n+1+\varepsilon-\delta)}$ when $2|y|<|x|$.

The multilinear Littlewood-Paley operator is defined by

$$S_\psi^A(f)(x) = \left[\int\int_{\Gamma(x)} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2},$$

where

$$F_t^A(f)(x,y) = \int_{R^n} \frac{R_{m+1}(A;x,z)}{|x-z|^m} f(z)\psi_t(y-z)dz$$

and $\psi_t(x) = t^{-n+\delta}\psi(x/t)$ for $t>0$. Set $F_t(f)(y) = f * \psi_t(y)$. We also define

$$S_\psi(f)(x) = \left(\int\int_{\Gamma(x)} |F_t(f)(y)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2}$$

which is the Littlewood-Paley operator (see [15]).

Let H be the Hilbert space

$$H = \left\{ h : ||h|| = \left(\int\int_{R_+^{n+1}} |h(y,t)|^2 dydt/t^{n+1}\right)^{1/2} < \infty \right\}.$$

Then for each fixed $x \in R^n$, $F_t^A(f)(x,y)$ may be viewed as a mapping from $(0,+\infty)$ to H, and it is clear that

$$S_\psi^A(f)(x) = \left|\left|\chi_{\Gamma(x)} F_t^A(f)(x,y)\right|\right|, \quad S_\psi(f)(x) = \left|\left|\chi_{\Gamma(x)} F_t(f)(y)\right|\right|.$$

We also consider the variant of S_ψ^A, which is defined by

$$\tilde{S}_\psi^A(f)(x) = \left(\int\int_{\Gamma(x)} |\tilde{F}_t^A(f)(x)|^2 \frac{dt}{t^{n+1}}\right)^{1/2},$$

where

$$\tilde{F}_t^A(f)(x) = \int_{R^n} \frac{Q_{m+1}(A;x,y)}{|x-y|^m} \psi_t(x-y)f(y)dy.$$

Definition 2. Let $0<\gamma\le 1$ and Ω be homogeneous of degree zero on R^n such that $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$. Assume that $\Omega \in Lip_\gamma(S^{n-1})$, that is there exists a constant $M > 0$ such that for any $x, y \in S^{n-1}$, $|\Omega(x) - \Omega(y)| \leq$ $\leq M|x-y|^\gamma$. The multilinear Marcinkiewicz integral operator is defined by

$$\mu_S^A(f)(x) = \left[\int \int_{\Gamma(x)} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+3}} \right]^{1/2},$$

where

$$F_t^A(f)(x,y) = \int_{|y-z| \leq t} \frac{\Omega(y-z)}{|y-z|^{n-1-\delta}} \frac{R_{m+1}(A;x,z)}{|x-z|^m} f(z)dz.$$

Set

$$F_t(f)(y) = \int_{|y-z| \leq t} \frac{\Omega(y-z)}{|y-z|^{n-1-\delta}} f(z)dz.$$

We also define

$$\mu_S(f)(x) = \left(\int \int_{\Gamma(x)} |F_t(f)(y)|^2 \frac{dydt}{t^{n+3}} \right)^{1/2},$$

which is the Marcinkiewicz integral operator (see [16]).

Let H be the Hilbert space

$$H = \left\{ h : ||h|| = \left(\int \int_{R_+^{n+1}} |h(y,t)|^2 dydt/t^{n+3} \right)^{1/2} < \infty \right\},$$

then for each fixed $x \in R^n$, $F_t^A(f)(x,y)$ may be viewed as a mapping from $(0, +\infty)$ to H, and it is clear that

$$\mu_S^A(f)(x) = \left|\left|\chi_{\Gamma(x)} F_t^A(f)(x,y)\right|\right|, \ \mu_S(f)(x) = \left|\left|\chi_{\Gamma(x)} F_t(f)(y)\right|\right|.$$

The variant of μ_S^A is defined by

$$\tilde{\mu}_S^A(f)(x) = \left(\int \int_{\Gamma(x)} |\tilde{F}_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+3}} \right)^{1/2},$$

where

$$\tilde{F}_t^A(f)(x,y) = \int_{|y-z| \leq t} \frac{\Omega(y-z)}{|y-z|^{n-1-\delta}} \frac{Q_{m+1}(A;x,z)}{|x-z|^m} f(z)dz.$$

More generally, we consider the following multilinear operators related to certain convolution operators.

Definition 3. For $F(x,t)$ defined on $R^n \times [0,+\infty)$, we denote

$$F_t(f)(x) = \int_{R^n} F(x-y,t)f(y)dy$$

and

$$F_t^A(f)(x) = \int_{R^n} \frac{R_{m+1}(A;x,y)}{|x-y|^m} F(x-y,t)f(y)dy.$$

Let H be the normed space $H = \{h : ||h|| < \infty\}$. For each fixed $x \in R^n$, we view $F_t(f)(x)$ and $F_t^A(f)(x)$ as a mapping from $[0,+\infty)$ to H. Then, the multilinear operators related to F_t is defined by

$$T^A(f)(x) = ||F_t^A(f)(x)||;$$

We also define $T(f)(x) = ||F_t(f)(x)||$.

It is clear that Definition 1 and 2 are particular cases of Definition 3. Note that when $m = 0$, T^A is just the commutator of T and A(see [11][16]). Let T be the Calderon-Zygmund singular integral operator. A classical result of Coifman, Rochberg and Weiss (see [6]) states that the commutator $[b,T] = T(bf) - bTf$ (where $b \in BMO(R^n)$) is bounded on $L^p(R^n)$ for $1 < p < \infty$. Chanillo (see [1]) proves a similar result when T is replaced by the fractional integral operator. In [11], the boundedness properties of the commutators for the extreme values of p are obtained. It is well-known that the multilinear operator, as a non-trivial extension of the commutator, is of great interest in harmonic analysis and has been widely studied by many authors (see [3-5]). In [7], the weighted $L^p(p > 1)$-boundedness of the multilinear operator related to some singular integral operator is obtained. In [2], the weak (H^1, L^1)-boundedness of the multilinear operator related to some singular integral operator is obtained. In recent years, the theory of Herz spaces and Herz type Hardy spaces, as a local version of Lebesgue spaces and Hardy spaces, has been developed (see [8][9][12][13]).

In this paper, we establish the endpoint continuity properties of the multilinear operators S_ψ^A and $\tilde{S}_\psi^A$, μ_S^A and $\tilde{\mu}_S^A$ on Herz and Herz type Hardy spaces.

First, let us introduce some notations (see [8][9][12][13][14]).Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For a cube Q and a locally integrable function f, let $f_Q = |Q|^{-1}\int_Q f(x)dx$ and $f^\#(x) = \sup_{x\in Q} |Q|^{-1}\int_Q |f(y) - f_Q|dy$. Moreover, f is said to belong to $BMO(R^n)$ if $f^\# \in L^\infty$ and define $||f||_{BMO} = ||f^\#||_{L^\infty}$. We also define the central BMO space by $CMO(R^n)$, which is the space of those functions $f \in L_{loc}(R^n)$ such that

$$||f||_{CMO} = \sup_{r>1} |Q(0,r)|^{-1} \int_Q |f(y) - f_Q|dy < \infty.$$

It is well-known that (see [9][14])

$$||f||_{CMO} \approx \sup_{r>1} \inf_{c\in C} |Q(0,r)|^{-1} \int_Q |f(x)-c|dx.$$

Definition 4. Let $0<\delta<n$ and $1<p<n/\delta$. We call $B_p^\delta(R^n)$ the space of those functions f on R^n for which

$$||f||_{B_p^\delta} = \sup_{r>1} r^{-n(1/p-\delta/n)} ||f\chi_{Q(0,r)}||_{L^p} < \infty.$$

For $k\in Z$, define $B_k = \{x\in R^n : |x|\le 2^k\}$ and $C_k = B_k \setminus B_{k-1}$. Denote by χ_k the characteristic function of C_k and $\tilde{\chi}_0$ the characteristic function of B_0.

Definition 5. Let $0<p<\infty$ and $\alpha\in R$.

(1) The homogeneous Herz space $\dot{K}_p^\alpha(R^n)$ is defined by

$$\dot{K}_p^\alpha(R^n) = \{f\in L^p_{loc}(R^n\setminus\{0\}) : ||f||_{\dot{K}_p^\alpha} < \infty\},$$

where

$$||f||_{\dot{K}_p^\alpha} = \sum_{k=-\infty}^{\infty} 2^{k\alpha} ||f\chi_k||_{L^p}.$$

(2) The nonhomogeneous Herz space $K_p^\alpha(R^n)$ is defined by

$$K_p^\alpha(R^n) = \{f\in L^p_{loc}(R^n) : ||f||_{K_p^\alpha} < \infty\},$$

where

$$||f||_{K_p^\alpha} = \sum_{k=0}^{\infty} 2^{k\alpha} ||f\chi_k||_{L^p}.$$

(3) The homogeneous Herz type Hardy space $H\dot{K}_p(R^n)$ is defined by

$$H\dot{K}_p(R^n) = \{f\in S'(R^n) : G(f)\in \dot{K}_p(R^n)\},$$

where

$$||f||_{H\dot{K}_p} = ||G(f)||_{\dot{K}_p}.$$

(4) The nonhomogeneous Herz type Hardy space $HK_p(R^n)$ is defined by

$$HK_p(R^n) = \{f\in S'(R^n) : G(f)\in K_p(R^n)\},$$

where

$$||f||_{HK_p} = ||G(f)||_{K_p}$$

and $G(f)$ is the maximal function of f.

If $\alpha = n(1-1/p)$, we denote $\dot{K}_p^\alpha(R^n) = \dot{K}_p(R^n)$, $K_p^\alpha(R^n) = K_p(R^n)$.

The Herz type Hardy spaces have the atomic decomposition characterization.

Definition 6. Let $1 < p < \infty$. A function $a(x)$ on R^n is called a central $(n(1-1/p),p)$-atom (or a central $(n(1-1/p),p)$-atom of restricted type), if

1) $\text{Supp} a \subset B(0,r)$ for some $r > 0$ (or for some $r \geq 1$),

2) $||a||_{L^p} \leq |B(0,r)|^{1/p-1}$,

3) $\int a(x)dx = 0$.

Lemma 1.(see[9][13]) Let $1 < p < \infty$. A temperate distribution f belongs to $H\dot{K}_p(R^n)$(or $HK_p(R^n)$) if and only if there exist central $(n(1-1/p),p)$-atoms (or central $(n(1-1/p),p)$-atoms of restricted type) a_j supported on $B_j = B(0,2^j)$ and constants λ_j, $\sum_j |\lambda_j| < \infty$ such that $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$ (or $f = \sum_{j=0}^{\infty} \lambda_j a_j$)in the $S'(R^n)$ sense, and

$$||f||_{H\dot{K}_p}(\text{ or } \ ||f||_{HK_p}) \approx \sum_j |\lambda_j|.$$

Now, we are in position to state our theorems.

Theorem 1. *Let $0 < \delta < n$, $1 < p < n/\delta$ and $D^\beta A \in BMO(R^n)$ for all β with $|\beta| = m$. Then both S_ψ^A and μ_S^A map $B_p^\delta(R^n)$ continuously into $CMO(R^n)$.*

Theorem 2. *Let $0 < \delta < n$, $1 < p < n/\delta$, $1/q = 1/p - \delta/n$ and $D^\beta A \in BMO(R^n)$ for all β with $|\beta| = m$. Then both $\tilde{S}_\psi^A$ and $\tilde{\mu}_S^A$ map $H\dot{K}_p(R^n)$(or $HK_p(R^n)$) continuously into $\dot{K}_q^\alpha(R^n)$ (or $K_q^\alpha(R^n)$) with $\alpha = n(1-1/p)$.*

Theorem 3. *Let $0 < \delta < n$, $1 < p < n/\delta$ and $D^\beta A \in BMO(R^n)$ for all β with $|\beta| = m$.*

(i) If for any cube Q and $u \in 3Q \setminus 2Q$, there is

$$\frac{1}{|Q|}\int_Q \left\| \chi_{\Gamma(x)} \sum_{|\alpha|=m} \frac{1}{\alpha!}|D^\alpha A(x) - (D^\alpha A)_Q| \int_{(4Q)^c} \frac{(u-z)^\alpha}{|u-z|^m} \psi_t(y-z)f(z)dz \right\| dx \leq$$

$$\leq C||f||_{B_p^\delta},$$

then $\tilde{S}_\psi^A$ maps $B_p^\delta(R^n)$ continuously into $CMO(R^n)$.

(ii)If for any cube Q and $u \in 3Q \setminus 2Q$, there is

$$\frac{1}{|Q|}\int_Q \left\| \chi_{\Gamma(x)} \sum_{|\alpha|=m} \frac{1}{\alpha!}|D^\alpha A(x) - (D^\alpha A)_Q| \int_{(4Q)^c} \frac{(u-z)^\alpha}{|u-z|^m} \frac{\Omega(y-z)\chi_{\Gamma(z)}(y,t)}{|y-z|^{n-1-\delta}} f(z)dz \right\| dx \leq$$

$$\leq C||f||_{B_p^\delta},$$

then $\tilde{\mu}_S^A$ *maps* $B_p^\delta(R^n)$ *continuously into* $CMO(R^n)$.

2. Proofs of Theorems

We begin with the following

Main Theorem. *Let* $0 < \delta < n$, $1 < p < n/\delta$ *and* $D^\beta A \in BMO(R^n)$ *for all* β *with* $|\beta| = m$. *Suppose that* T^A *is the same as in Definition 3 such that* T *is bounded from* $L^p(R^n)$ *to* $L^q(R^n)$ *for any* $p, q \in (1, +\infty)$ *with* $1/q = 1/p - \delta/n$. *If* T *satisfies the size condition*

$$||F_t^A(f)(x) - F_t^A(f)(0)|| \leq C||f||_{B_p^\delta},$$

for any cube $Q = Q(0, d)$ *with* $d > 1$, *suppf* $\subset (2Q)^c$ *and* $x \in Q$, *then* T^A *maps* $B_p^\delta(R^n)$ *continuously into* $CMO(R^n)$.

To prove the theorem, we need the following lemma.

Lemma 2(see [5]). *Let* A *be a function on* R^n *and* $D^\beta A \in L^q(R^n)$ *for* $|\beta| = m$ *and some* $q > n$. *Then*

$$|R_m(A; x, y)| \leq C|x - y|^m \sum_{|\beta|=m} \left(\frac{1}{|\tilde{Q}(x, y)|} \int_{\tilde{Q}(x,y)} |D^\beta A(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}(x, y)$ *is the cube centered at* x *and having side length* $5\sqrt{n}|x - y|$.

Proof of Main Theorem. It is only to prove that there is a constant C_Q such that

$$\frac{1}{|Q|} \int_Q |T^A(f)(x) - C_Q| dx \leq C||f||_{B_p^\delta}$$

holds for any cube $Q = Q(0, d)$ with $d > 1$. Fix a cube $Q = Q(0, d)$ with $d > 1$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}(x) = A(x) - \sum_{|\beta|=m} \frac{1}{\beta!}(D^\beta A)_{\tilde{Q}} x^\beta$, then $R_m(A; x, y) = R_m(\tilde{A}; x, y)$ and $D^\beta \tilde{A} = D^\beta A - (D^\beta A)_{\tilde{Q}}$ for all β with $|\beta| = m$. We write, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{R^n \setminus \tilde{Q}}$,

$$F_t^A(f)(x) = \int_{R^n} \frac{R_{m+1}(\tilde{A}; x, y)}{|x - y|^m} F(x - y, t) f_2(y) dy +$$

$$+ \int_{R^n} \frac{R_m(\tilde{A}; x, y)}{|x - y|^m} F(x - y, t) f_1(y) dy$$

$$-\sum_{|\beta|=m}\frac{1}{\beta!}\int_{R^n}\frac{F(x-y,t)(x-y)^\beta}{|x-y|^m}D^\beta\tilde{A}(y)f_1(y)dy,$$

then

$$\left|T^A(f)(x)-T^{\tilde{A}}(f_2)(0)\right|=\left|||F_t^A(f)(x)||-||F_t^{\tilde{A}}(f)(0)||\right|$$

$$\leq \quad \left|\left|F_t\left(\frac{R_m(\tilde{A};x,\cdot)}{|x-\cdot|^m}f_1\right)(x)\right|\right|+\sum_{|\beta|=m}\frac{1}{\beta!}\left|\left|F_t\left(\frac{(x-\cdot)^\beta}{|x-\cdot|^m}D^\beta\tilde{A}f_1\right)(x)\right|\right|$$

$$+||F_t^{\tilde{A}}(f_2)(x)-F_t^{\tilde{A}}(f_2)(0)||=I(x)+II(x)+III(x),$$

thus

$$\frac{1}{|Q|}\int_Q\left|T^A(f)(x)-T^{\tilde{A}}(f_2)(0)\right|dx\leq$$

$$\leq\frac{1}{|Q|}\int_Q I(x)dx+\frac{1}{|Q|}\int_Q II(x)dx+\frac{1}{|Q|}\int_Q III(x)dx=I+II+III.$$

Now, let us estimate I, II and III. First, for $x\in Q$ and $y\in\tilde{Q}$, using Lemma 2, we get

$$R_m(\tilde{A};x,y)\leq C|x-y|^m\sum_{|\beta|=m}||D^\beta A||_{BMO},$$

thus, by the $L^p(R^n)$ to $L^q(R^n)$ boundedness of T for $1<p,q<\infty$ with $1/q=1/p-\delta/n$, we get

$$I\leq\frac{C}{|Q|}\int_Q|T(\sum_{|\beta|=m}||D^\beta A||_{BMO}f_1)(x)|dx\leq$$

$$\leq C\sum_{|\beta|=m}||D^\beta A||_{BMO}\left(\frac{1}{|Q|}\int_Q|T(f_1)(x)|^q dx\right)^{1/q}\leq$$

$$\leq C|Q|^{-1/q}||f_1||_{L^p}\leq Cd^{-n(1/p-\delta/n)}||f\chi_{\tilde{Q}}||_{L^p}\leq C||f||_{B_p^\delta}.$$

Secondly, taking $q,r,s>1$ such that $1/r=1/s-\delta/n$, $qs<p$, then by the (L^s,L^r)-boundedness of T and Hölder's inequality, denoting that $1/q+1/q'=1$, we gain

$$II\leq\frac{C}{|Q|}\int_Q|T(\sum_{|\beta|=m}(D^\beta A-(D^\beta A)_{\tilde{Q}})f_1)(x)|dx\leq$$

$$\leq C \sum_{|\beta|=m} \left(\frac{1}{|Q|} \int_Q |T((D^\beta A - (D^\beta A)_{\tilde{Q}}) f_1)(x)|^r dx \right)^{1/r} \leq$$

$$\leq C \sum_{|\beta|=m} |Q|^{-1/r} \left(\int |(D^\beta A(x) - (D^\beta A)_{\tilde{Q}}) f_1(x)|^s dx \right)^{1/s} \leq$$

$$\leq C \sum_{|\beta|=m} |Q|^{-1/r} \left(\int_{\tilde{Q}} |D^\beta A(x) - (D^\beta A)_{\tilde{Q}}|^{sq'} dx \right)^{1/(sq')} \left(\int_{\tilde{Q}} |f_1(x)|^{qs} dx \right)^{1/(qs)} \leq$$

$$\leq C \sum_{|\beta|=m} ||D^\beta A||_{BMO} |Q|^{1/(sq')} |Q|^{-1/r} \left(\int_{\tilde{Q}} |f_1(x)|^p dx \right)^{1/p} |Q|^{(p-qs)/(pqs)} \leq$$

$$\leq C \sum_{|\beta|=m} ||D^\beta A||_{BMO} |\tilde{Q}|^{\delta/n-1/p} ||f \chi_{\tilde{Q}}||_{L^p} \leq C ||f||_{B_p^\delta}.$$

For III, using the size condition of T, we have

$$III \leq C ||f||_{B_p^\delta}.$$

This completes the proof of Main Theorem.

To prove Theorems 1, 2 and 3, we need the following lemma.

Lemma 3. *Let $0 \leq \delta < n$, $1 < p < n/\delta$, $1/q = 1/p - \delta/n$ and $D^\beta A \in BMO(R^n)$ for all β with $|\beta| = m$. Then both S_ψ^A and μ_S^A map $L^p(R^n)$ continuously into $L^q(R^n)$.*

Proof. For S_ψ^A, by Minkowski's inequality and the conditions of ψ, we have

$$S_\psi^A(f)(x) \leq \int_{R^n} \frac{|f(z)||R_{m+1}(A;x,z)|}{|x-z|^m} \left(\int_{\Gamma(x)} |\psi_t(y-z)|^2 \frac{dydt}{t^{1+n}} \right)^{1/2} dz \leq$$

$$\leq C \int_{R^n} \frac{|f(z)||R_{m+1}(A;x,z)|}{|x-z|^m} \left(\int_0^\infty \int_{|x-y|\leq t} \frac{t^{-2n+2\delta}}{(1+|y-z|/t)^{2n+2-2\delta}} \frac{dydt}{t^{1+n}} \right)^{1/2} dz \leq$$

$$\leq C \int_{R^n} \frac{|f(z)||R_{m+1}(A;x,z)|}{|x-z|^m} \left(\int_0^\infty \int_{|x-y|\leq t} \frac{2^{2n+2-2\delta} t^{1-n}}{(2t+|y-z|)^{2n+2}} dydt \right)^{1/2} dz,$$

noting that $2t + |y-z| \geq 2t + |x-z| - |x-y| \geq t + |x-z|$ when $|x-y| \leq t$ and

$$\int_0^\infty \frac{tdt}{(t+|x-z|)^{2n+2-2\delta}} = C|x-z|^{-2n+2\delta},$$

we obtain

$$S_\psi^A(f)(x) \le C \int\limits_{R^n} \frac{|f(z)||R_{m+1}(A;x,z)|}{|x-z|^m} \left(\int_0^\infty \frac{tdt}{(t+|x-z|)^{2n+2-2\delta}}\right)^{1/2} dz =$$

$$= C \int_{R^n} \frac{|f(z)||R_{m+1}(A;x,z)|}{|x-z|^{m+n-\delta}} dz.$$

For μ_S^A, notice that $|x-z| \le 2t$, $|y-z| \ge |x-z|-t \ge |x-z|-3t$ when $|x-y| \le t$, $|y-z| \le t$ and we have

$$\mu_S^A(f)(x) \le \int\limits_{R^n} \left[\int \int\limits_{|x-y|\le t} \left(\frac{|\Omega(y-z)||R_{m+1}(A;x,z)||f(z)|}{|y-z|^{n-1-\delta}|x-z|^m}\right)^2 \chi_{\Gamma(z)}(y,t)\frac{dydt}{t^{n+3}}\right]^{1/2} dz \le$$

$$\le C \int_{R^n} \frac{|R_{m+1}(A;x,z)||f(z)|}{|x-z|^m} \left[\int \int\limits_{|x-y|\le t} \frac{\chi_{\Gamma(z)}(y,t)t^{-n-3}}{(|x-z|-3t)^{2n-2-2\delta}} dydt\right]^{1/2} dz \le$$

$$\le C \int_{R^n} \frac{|R_{m+1}(A;x,z)||f(z)|}{|x-z|^{m+3/2}} \left[\int_{|x-z|/2}^\infty \frac{dt}{(|x-z|-3t)^{2n-2-\delta}}\right]^{1/2} dz$$

$$\le C \int_{R^n} \frac{|R_{m+1}(A;x,z)|}{|x-z|^{m+n-\delta}} |f(z)| dz,$$

thus, the lemma follows from [7].

Proof of Theorem 1. From Lemma 3, we know that S_ψ and μ_S are bounded from $L^p(R^n)$ to $L^q(R^n)$ for $1 < p < n/\delta$ and $1/q = 1/p - \delta/n$. Now, it suffices to verify that S_ψ^A and μ_S^A satisfy the size condition in Main Theorem, that is

$$||\chi_{\Gamma(x)} F_t^A(f)(x,y) - \chi_{\Gamma(0)} F_t^A(f)(0,y)|| \le C||f||_{B_p^\delta}.$$

Let supp$f \subset (2Q(0,d))^c$ and $\tilde{A}(x) = A(x) - \sum\limits_{|\beta|=m} \frac{1}{\beta!}(D^\beta A)_Q x^\beta$.

For S_ψ^A, we write, for $x \in Q$,

$$\chi_{\Gamma(x)}(y,t)F_t^{\tilde{A}}(f)(x,y) - \chi_{\Gamma(0)}(y,t)F_t^{\tilde{A}}(f)(0,y) =$$

$$= \int \left[\frac{1}{|x-z|^m} - \frac{1}{|z|^m}\right] \chi_{\Gamma(x)}(y,t)\psi_t(y-z)R_m(\tilde{A};x,z)f(z)dz +$$

$$+ \int \frac{\chi_{\Gamma(x)}(y,t)\psi_t(y-z)f(z)}{|z|^m}[R_m(\tilde{A};x,z) - R_m(\tilde{A};0,z)]dz +$$

$$+ \int (\chi_{\Gamma(x)}(y,t) - \chi_{\Gamma(0)}(y,t)) \frac{\psi_t(y-z)R_m(\tilde{A};0,z)f(z)}{|z|^m} dz -$$

$$-\sum_{|\beta|=m}\frac{1}{\beta!}\int\left[\frac{\chi_{\Gamma(x)}(y,t)(x-z)^\beta}{|x-z|^m}-\frac{\chi_{\Gamma(0)}(y,t)(-z)^\beta}{|z|^m}\right]\psi_t(y-z)D^\beta\tilde{A}(z)f(z)dz:=$$

$$:=I_1^t(x)+I_2^t(x)+I_3^t(x)+I_4^t(x).$$

Note that $|x-z|\sim|z|$ for $x\in Q$ and $z\in R^n\setminus 2Q$, by Lemma 2 and the following inequality(see[14])

$$|b_{Q_1}-b_{Q_2}|\le C\log(|Q_2|/|Q_1|)||b||_{BMO},\ \text{ for }\ Q_1\subset Q_2,$$

we know that, for $x\in Q$ and $y\in 2^{k+1}Q\setminus 2^kQ$,

$$|R_m(\tilde{A};x,y)|\le C|x-y|^m\sum_{|\beta|=m}(||D^\beta A||_{BMO}+|(D^\beta A)_{Q(x,y)}-(D^\beta A)_Q|)\le$$

$$\le Ck|x-y|^m\sum_{|\beta|=m}||D^\beta A||_{BMO}.$$

Thus, we obtain, similarly to the proof of Lemma 3,

$$\begin{aligned}||I_1^t(x)|| &\le C\int_{R^n\setminus 2Q}\frac{|x|}{|z|^{m+n+1-\delta}}|R_m(\tilde{A};x,z)||f(z)|dz\le\\ &\le C\sum_{|\beta|=m}||D^\beta A||_{BMO}\sum_{k=1}^\infty\int_{2^{k+1}Q\setminus 2^kQ}k\frac{|x|}{|z|^{n+1-\delta}}|f(z)|dz\le\\ &\le C\sum_{|\beta|=m}||D^\beta A||_{BMO}\sum_{k=1}^\infty k2^{-k}(2^kd)^{-n(1/p-\delta/n)}||f\chi_{2^{k+1}Q}||_{L^p}\le\\ &\le C\sum_{|\beta|=m}||D^\beta A||_{BMO}\sum_{k=1}^\infty k2^{-k}||f||_{B_p^\delta}\le C\sum_{|\beta|=m}||D^\beta A||_{BMO}||f||_{B_p^\delta}.\end{aligned}$$

For $I_2^t(x)$, by the formula (see [5]):

$$R_m(\tilde{A};x,y)-R_m(\tilde{A};x_0,y)=\sum_{|\gamma|<m}\frac{1}{\gamma!}R_{m-|\gamma|}(D^\gamma\tilde{A};x,x_0)(x-y)^\gamma$$

and by Lemma 2, we have

$$|R_m(\tilde{A};x,y)-R_m(\tilde{A};x_0,y)|\le C\sum_{|\gamma|<m}\sum_{|\beta|=m}|x-x_0|^{m-|\gamma|}|x-y|^{|\gamma|}||D^\beta A||_{BMO},$$

thus, similarly to by to the estimates of $I_1^t(x)$, we get

$$||I_2^t(x)|| \le C \sum_{|\beta|=m} ||D^\beta A||_{BMO} \sum_{k=1}^\infty \int_{2^{k+1}Q\setminus 2^kQ} \frac{|x|}{|z|^{n+1-\delta}} |f(z)|dz \le$$

$$\le C \sum_{|\beta|=m} ||D^\beta A||_{BMO} ||f||_{B_p^\delta}.$$

For $I_3^t(x)$, note that $|x+y-z| \sim |y-z|$ for $x \in Q$ and $z \in R^n \setminus 2Q$, similarly to the estimates of $I_1^t(x)$, we get

$$||I_3^t(x)|| \le$$

$$\le C \int_{R^n\setminus 2Q} \left(\int\int_{R_+^{n+1}} \left[\frac{|\psi_t(y-z)||f(z)||R_m(\tilde{A};0,z)|}{|z|^m} |\chi_{\Gamma(x)}(y,t) - \chi_{\Gamma(0)}(y,t)| \right]^2 \frac{dydt}{t^{n+1}} \right)^{1/2} dz \le$$

$$\le C \int_{R^n\setminus 2Q} \frac{|f(z)||R_m(\tilde{A};0,z)|}{|z|^m} \left| \int\int_{\Gamma(x)} \frac{t^{1-n}dydt}{(t+|y-z|)^{2n+2-2\delta}} - \int\int_{\Gamma(0)} \frac{t^{1-n}dydt}{(t+|y-z|)^{2n+2-2\delta}} \right|^{1/2} dz \le$$

$$\le C \int_{R^n\setminus 2Q} \frac{|f(z)||R_m(\tilde{A};0,z)|}{|z|^m} \left(\int\int_{|y|\le t} \left| \frac{1}{(t+|x+y-z|)^{2n+2-2\delta}} - \frac{1}{(t+|y-z|)^{2n+2-2\delta}} \right| \frac{dydt}{t^{n-1}} \right)^{1/2} dz$$

$$\le C \int_{R^n\setminus 2Q} \frac{|f(z)||R_m(\tilde{A};0,z)|}{|z|^m} \left(\int\int_{|y|\le t} \frac{|x|t^{1-n}dydt}{(t+|x+y-z|)^{2n+3-2\delta}} \right)^{1/2} dz$$

$$\le C \int_{R^n\setminus 2Q} \frac{|f(z)||x|^{1/2}|R_m(\tilde{A};0,z)|}{|z|^{m+n+1/2-\delta}} dz$$

$$\le C \sum_{|\beta|=m} ||D^\beta A||_{BMO} \sum_{k=1}^\infty k 2^{-k/2} (2^k d)^{-n(1/p-\delta/n)} ||f\chi_{2^{k+1}Q}||_{L^p} \le$$

$$\le C \sum_{|\beta|=m} ||D^\beta A||_{BMO} ||f||_{B_p^\delta}.$$

For $I_4^t(x)$, by Hölder's inequality, similar to the estimates of $I_1^t(x)$, we get

$$||I_4^t(x)|| \le C \sum_{|\beta|=m} \sum_{k=1}^\infty \int_{2^{k+1}Q\setminus 2^kQ} \left(\frac{|x|}{|z|^{n+1-\delta}} + \frac{|x|^{1/2}}{|z|^{n+1/2-\delta}} \right) |D^\beta \tilde{A}(z)||f(z)|dz \le$$

$$\le C \sum_{|\beta|=m} \sum_{k=1}^\infty (2^{-k} + 2^{-k/2})(2^k d)^{\delta - n/p} \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |D^\beta A(z) - (D^\beta A)_Q|^{p'} dy \right)^{1/p'} ||f\chi_{2^{k+1}Q}||_{L^p} \le$$

$$\le C \sum_{|\beta|=m} ||D^\beta A||_{BMO} \sum_{k=1}^\infty (2^{-k} + 2^{-k/2})(2^k d)^{-n(1/p-\delta/n)} ||f\chi_{2^{k+1}Q}||_{L^p} \le$$

$$\leq C\sum_{|\beta|=m}||D^\beta A||_{BMO}||f||_{B_p^\delta}.$$

For μ_S^A, we write, for $x\in Q$,

$$\chi_{\Gamma(x)}(y,t)F_t^{\tilde{A}}(f)(x,y)-\chi_{\Gamma(x)}(y,t)F_t^{\tilde{A}}(f)(0,y)=$$

$$=\int_{|y-z|\leq t}\left[\frac{1}{|x-z|^m}-\frac{1}{|z|^m}\right]\frac{\chi_{\Gamma(x)}(y,t)\Omega(y-z)R_m(\tilde{A};x,z)f(z)}{|y-z|^{n-1}}dz+$$

$$+\int_{|y-z|\leq t}\frac{\chi_{\Gamma(x)}(y,t)\Omega(y-z)f(z)}{|y-z|^{n-1}|z|^m}[R_m(\tilde{A};x,z)-R_m(\tilde{A};0,z)]dz+$$

$$+\int_{|y-z|\leq t}(\chi_{\Gamma(x)}(y,t)-\chi_{\Gamma(0)}(y,t))\frac{\Omega(y-z)R_m(\tilde{A};0,z)f(z)}{|y-z|^{n-1}|z|^m}dz-$$

$$-\sum_{|\beta|=m}\frac{1}{\beta!}\int_{|y-z|\leq t}\left[\frac{\chi_{\Gamma(x)}(y,t)(x-z)^\beta}{|x-z|^m}-\frac{\chi_{\Gamma(0)}(y,t)(-z)^\beta}{|z|^m}\right]\frac{\Omega(y-z)D^\beta\tilde{A}(z)f(z)}{|y-z|^{n-1}}dz:=$$

$$:=J_1^t(x)+J_2^t(x)+J_3^t(x)+J_4^t(x).$$

Similarly to the proof of Lemma 3 and S_ψ^A, we obtain

$$||J_1^t(x)||\leq C\int_{R^n\setminus 2Q}\frac{|x||f(z)|}{|z|^{n+m+1-\delta}}|R_m(\tilde{A};x,z)|dz\leq C\sum_{|\beta|=m}||D^\beta A||_{BMO}||f||_{B_p^\delta}$$

and

$$||J_2^t(x)||\leq C\sum_{|\beta|=m}||D^\beta A||_{BMO}\sum_{k=1}^\infty\int_{2^{k+1}Q\setminus 2^kQ}\frac{|x|}{|z|^{n+1-\delta}}|f(z)|dz\leq C\sum_{|\beta|=m}||D^\beta A||_{BMO}||f||_{B_p^\delta}.$$

For $J_3^t(x)$, similarly to the estimates of Lemma 3 and $I_3^t(x)$, we obtain

$$\begin{aligned}
&||J_3^t(x)||\leq\\
\leq\ & C\int\limits_{R^n\setminus 2Q}\left(\int\limits_{R_+^{n+1}}\left[\frac{|f(z)||\Omega(y-z)|\chi_{\Gamma(z)}(y,t)|R_m(\tilde{A};0,z)|}{|y-z|^{n-1-\delta}|z|^m}(\chi_{\Gamma(x)}(y,t)-\chi_{\Gamma(0)}(y,t))\right]^2\frac{dydt}{t^{n+3}}\right)^{1/2}dz\leq\\
\leq\ & C\int\limits_{R^n\setminus 2Q}\frac{|f(z)||R_m(\tilde{A};0,z)|}{|z|^m}\left|\int\int\limits_{|x-y|\leq t}\frac{t^{-n-3}\chi_{\Gamma(z)}(y,t)}{|y-z|^{2n-2-2\delta}}dydt-\int\int\limits_{|y|\leq t}\frac{t^{-n-3}\chi_{\Gamma(z)}(y,t)}{|y-z|^{2n-2-2\delta}}dydt\right|^{1/2}dz\leq\\
\leq\ & C\int\limits_{R^n\setminus 2Q}\frac{|f(z)||R_m(\tilde{A};0,z)|}{|z|^m}\left(\int\int\limits_{|y|\leq t,|x+y-z|\leq t}\left|\frac{1}{|x+y-z|^{2n-2-2\delta}}-\frac{1}{|y-z|^{2n-2-2\delta}}\right|\frac{dydt}{t^{n+3}}\right)^{1/2}dz\leq\\
\leq\ & C\int\limits_{R^n\setminus 2Q}\frac{|f(z)||R_m(\tilde{A};0,z)|}{|z|^m}\left(\int\int\limits_{|y|\leq t,|x+y-z|\leq t}\frac{|x|}{|x+y-z|^{2n+2-2\delta}}t^{-n}dydt\right)^{1/2}dz\leq\\
\leq\ & C\int\limits_{R^n\setminus 2Q}\frac{|f(z)||x|^{1/2}|R_m(\tilde{A};0,z)|}{|z|^{m+n+1/2-\delta}}dz\leq\\
\leq\ & C\sum_{|\beta|=m}||D^\beta A||_{BMO}||f||_{B_p^\delta}.
\end{aligned}$$

For $J_4^t(x)$, similarly to the estimates of $J_1^t(x)$, $J_3^t(x)$ and $I_4^t(x)$, we get

$$||J_4^t(x)|| \le C\int_{R^n\setminus 2Q}\left(\frac{|x|}{|z|^{n+1-\delta}}+\frac{|x|^{1/2}}{|z|^{n+1/2-\delta}}\right)\sum_{|\beta|=m}|D^\beta\tilde{A}(z)||f(z)|dz$$
$$\le C\sum_{|\beta|=m}||D^\beta A||_{BMO}||f||_{B_p^\delta}.$$

Thus

$$||\chi_{\Gamma(x)}F_t^A(f)(x,y)-\chi_{\Gamma(0)}F_t^A(f)(0,y)|| \le C||f||_{B_p^\delta}.$$

These yield the desired results and complete the proof of Theorem 1.

Proof of Theorem 2. We only give the proof on homogeneous weighted Herz and Herz type Hardy spaces. To simplifiy, we denote $\tilde{T}^A = \tilde{S}_\psi^A$ or $\tilde{\mu}_S^A$. Let $f\in H\dot{K}_p(R^n)$. By Lemma 1, $f=\sum_{j=-\infty}^{\infty}\lambda_j a_j$, where $a_j's$ are the central $(n(1-1/p),p)$-atoms with supp$a_j\subset B_j=B(0,2^j)$ and $||f||_{H\dot{K}_p}\sim\sum_j|\lambda_j|$. We write

$$||\tilde{T}^A(f)||_{\dot{K}_p^\delta} = \sum_{k=-\infty}^{\infty}2^{k\alpha}||\chi_k\tilde{T}^A(f)||_{L^q}$$
$$\le \sum_{k=-\infty}^{\infty}2^{k\alpha}\sum_{j=-\infty}^{k-1}|\lambda_j|||\chi_k\tilde{T}^A(a_j)||_{L^q}+\sum_{k=-\infty}^{\infty}2^{k\alpha}\sum_{j=k}^{\infty}|\lambda_j|||\chi_k\tilde{T}^A(a_j)||_{L^q}=L+LL.$$

For LL, by the equality

$$Q_{m+1}(A;x,y)=R_{m+1}(A;x,y)+\sum_{|\beta|=m}\frac{1}{\beta!}(x-y)^\beta(D^\beta A(x)-D^\beta A(y)),$$

we have, similarly to the proof of Lemma 3,

$$\tilde{T}^A(f)(x)\le T^A(f)(x)+C\sum_{|\beta|=m}\int_{R^n}\frac{|D^\beta A(x)-D^\beta A(y)|}{|x-y|^{n-\delta}}|f(y)|dy,$$

thus, $\tilde{T}^A$ is bounded from $L^p(R^n)$ to $L^q(R^n)$ for $1<p<n/\delta$ with $1/q=1/p-\delta/n$ by Lemma 3 and [1]. We see that

$$LL \le C\sum_{k=-\infty}^{\infty}2^{k\alpha}\sum_{j=k}^{\infty}|\lambda_j|||a_j||_{L^p}\le C\sum_{k=-\infty}^{\infty}2^{kn(1-1/p)}\sum_{j=k}^{\infty}|\lambda_j|2^{-jn(1-1/p)}$$
$$\le C\sum_{j=-\infty}^{\infty}|\lambda_j|\sum_{k=-\infty}^{j}2^{(k-j)n(1-1/p)}\le C\sum_{j=-\infty}^{\infty}|\lambda_j|\le C||f||_{H\dot{K}_p};$$

To obtain the estimate of L, we denote that $\tilde{A}(x)=A(x)-\sum_{|\beta|=m}\frac{1}{\beta!}(D^\beta A)_{2B_j}x^\beta$. Then $Q_m(A;x,y)=Q_m(\tilde{A};x,y)$ and $Q_{m+1}(A;x,y)=R_m(A;x,y)-\sum_{|\beta|=m}\frac{1}{\beta!}(x-y)^\beta D^\beta A(x)$.

For $\tilde{S}_\psi^A$, we write, by the vanishing moment of a and for $x\in B_k$ with $k\geq j+1$,

$$\tilde{F}_t^A(a_j)(x,y)=$$

$$=\int\frac{\psi_t(y-z)R_m(\tilde{A};x,z)}{|x-z|^m}a_j(z)dz-\sum_{|\beta|=m}\frac{1}{\beta!}\int\frac{\psi_t(y-z)D^\beta\tilde{A}(x)(x-z)^\beta}{|x-z|^m}a_j(z)dz=$$

$$=\int\left[\frac{\psi_t(y-z)R_m(\tilde{A};x,z)}{|x-z|^m}-\frac{\psi_t(y)R_m(\tilde{A};x,0)}{|x|^m}\right]a_j(z)dz-$$

$$-\sum_{|\beta|=m}\frac{1}{\beta!}\int\left[\frac{\psi_t(y-z)(x-z)^\beta}{|x-z|^m}-\frac{\psi_t(y)x^\beta}{|x|^m}\right]D^\beta\tilde{A}(x)a_j(z)dz,$$

similarly to the proof of Lemma 3 and Theorem 1, we obtain

$$||\tilde{F}_t^A(a_j)(x,y)||\leq C\int_{R^n}\left[\frac{|z|}{|x|^{m+n+1-\delta}}+\frac{|z|^{1/2}}{|x|^{m+n+1/2-\delta}}\right]|R_m(\tilde{A};x,z)||a_j(z)|dz+$$

$$+C\sum_{|\alpha|=m}\int_{R^n}\left[\frac{|z|}{|x|^{n+1-\delta}}+\frac{|z|^{1/2}}{|x|^{n+1/2-\delta}}\right]|D^\alpha\tilde{A}(x)||a_j(z)|dz\leq$$

$$\leq C\sum_{|\beta|=m}||D^\beta A||_{BMO}\left[\frac{2^j}{2^{k(n+1-\delta)}}+\frac{2^{j/2}}{2^{k(n+1/2-\delta)}}\right]+$$

$$+C\sum_{|\beta|=m}\left[\frac{2^j}{2^{k(n+1-\delta)}}+\frac{2^{j/2}}{2^{k(n+1/2-\delta)}}\right]|D^\beta\tilde{A}(x)|,$$

thus

$$L\leq C\sum_{|\beta|=m}||D^\beta A||_{BMO}\sum_{k=-\infty}^{\infty}2^{kn(1-1/p)}\sum_{j=-\infty}^{k-1}|\lambda_j|\left[\frac{2^j}{2^{k(n+1-\delta)}}+\frac{2^{j/2}}{2^{k(n+1/2-\delta)}}\right]2^{kn/q}+$$

$$+C\sum_{|\beta|=m}\sum_{k=-\infty}^{\infty}2^{kn(1-1/p)}\sum_{j=-\infty}^{k-1}|\lambda_j|\left[\frac{2^j}{2^{k(n+1-\delta)}}+\frac{2^{j/2}}{2^{k(n+1/2-\delta)}}\right]\left(\int_{B_k}|D^\beta\tilde{A}(x)|^q dx\right)^{1/q}\leq$$

$$\leq C\sum_{|\beta|=m}||D^\beta A||_{BMO}\sum_{k=-\infty}^{\infty}2^{kn(1-\delta/n)}\sum_{j=-\infty}^{k-1}|\lambda_j|\left[\frac{2^j}{2^{k(n+1-\delta)}}+\frac{2^{j/2}}{2^{k(n+1/2-\delta)}}\right]\leq$$

$$\leq C\sum_{|\beta|=m}||D^\beta A||_{BMO}\sum_{j=-\infty}^{\infty}|\lambda_j|\sum_{k=j+1}^{\infty}[2^{j-k}+2^{(j-k)/2}]\leq$$

$$\leq C\sum_{|\beta|=m}||D^\beta A||_{BMO}\sum_{j=-\infty}^{\infty}|\lambda_j|\leq C\sum_{|\beta|=m}||D^\beta A||_{BMO}||f||_{H\dot{K}_p}.$$

A similar argument as in the proof of Theorem 1 gives the proof for $\tilde{\mu}_S^A$ and we omit here the details. This completes the proof of Theorem 2.

Proof of Theorem 3. We only give the proof of $\tilde{S}_\psi^A$. For any cube $Q = Q(0,d)$ with $d > 1$, let $f \in B_p(w)$ and $\tilde{A}(x) = A(x) - \sum_{|\beta|=m} \frac{1}{\beta!}(D^\beta A)_{\tilde{Q}} x^\beta$. We write, for $f = f\chi_{4Q} + f\chi_{(4Q)^c} = f_1 + f_2$ and $u \in 3Q \setminus 2Q$,

$$\begin{aligned}\tilde{F}_t^A(f)(x,y) &= \tilde{F}_t^A(f_1)(x,y) + \int_{R^n} \frac{R_m(\tilde{A};x,z)}{|x-z|^m}\psi_t(y-z)f_2(z)dz - \\ &- \sum_{|\beta|=m} \frac{1}{\beta!}(D^\beta A(x) - (D^\beta A)_Q) \int_{R^n} \left[\frac{(x-z)^\beta}{|x-z|^m} - \frac{(u-z)^\beta}{|u-z|^m}\right]\psi_t(y-z)f_2(z)dz - \\ &- \sum_{|\beta|=m} \frac{1}{\beta!}(D^\beta A(x) - (D^\beta A)_Q) \int_{R^n} \frac{(u-z)^\beta}{|u-z|^m}\psi_t(y-z)f_2(z)dz,\end{aligned}$$

then

$$\begin{aligned}&\left|\tilde{S}_\psi^A(f)(x) - S_\psi\left(\frac{R_m(\tilde{A};0,\cdot)}{|\cdot|^m}f_2\right)(0)\right| = \\ &= \left|||\chi_{\Gamma(x)}\tilde{F}_t^A(f)(x,y)|| - \left\|\chi_{\Gamma(0)}F_t\left(\frac{R_m(\tilde{A};0,\cdot)}{|\cdot|^m}f_2\right)(0)\right\|\right| \le \\ \le\ &\left\|\chi_{\Gamma(x)}\tilde{F}_t^A(f)(x,y) - \chi_{\Gamma(0)}F_t\left(\frac{R_m(\tilde{A};0,\cdot)}{|\cdot|^m}f_2\right)(0)\right\| \le \\ \le\ &||\chi_{\Gamma(x)}\tilde{F}_t^A(f_1)(x,y)|| + \\ &+ \left\|\left[\chi_{\Gamma(x)}\int_{R^n}\frac{R_m(\tilde{A};x,z)}{|x-z|^m}\psi_t(y-z) - \chi_{\Gamma(0)}\int_{R^n}\frac{R_m(\tilde{A};0,z)}{|z|^m}\psi_t(-z)\right]f_2(z)dz\right\| + \\ &+ \left\|\chi_{\Gamma(x)}\sum_{|\beta|=m}\frac{1}{\beta!}(D^\beta A(x) - (D^\beta A)_Q)\int_{R^n}\left[\frac{(y-z)^\beta}{|y-z|^m} - \frac{(u-z)^\beta}{|u-z|^m}\right]\psi_t(y-z)f_2(z)dz\right\| + \\ &+ \left\|\chi_{\Gamma(x)}\sum_{|\beta|=m}\frac{1}{\beta!}(D^\beta A(x) - (D^\beta A)_Q)\int_{R^n}\frac{(u-z)^\beta}{|u-z|^m}\psi_t(y-z)f_2(z)dz\right\| = \\ =\ &I_1(x) + I_2(x) + I_3(x,u) + I_4(x,u).\end{aligned}$$

By the $L^p(R^n)$ to $L^q(R^n)$-boundedness of $\tilde{S}_\psi^A$ for $1 < p < n/\delta$, with $1/q = 1/p - \delta/n$, we get

$$\frac{1}{|Q|}\int_Q I_1(x)dx \le C\left(\frac{1}{|Q|}\int_Q |\tilde{S}_\psi^A(f_1)(x)|^q dx\right)^{1/q} \le C|Q|^{-1/q}||f_1||_{L^p} \le C||f||_{B_p^\delta}.$$

Similarly to the proof of Theorem 1, we obtain

$$\frac{1}{|Q|}\int_Q I_2(x)dx \le C||f||_{B_p^\delta}\frac{1}{|Q|}\int_Q I_3(x,u)dx \le C||f||_{B_p^\delta}.$$

Thus, using the estimates of $I_4(x,u)$, we obtain

$$\frac{1}{|Q|}\int_Q\left|\tilde{S}_\psi(x) - S_\psi\left(\frac{R_m(\tilde{A};0,\cdot)}{|\cdot|^m}f_2\right)(0)\right|dx \le C||f||_{B_p^\delta}.$$

This completes the proof of Theorem 3.

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