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Some results on an iterative method for nonexpansive mappings in different spaces

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Abstract

In this paper, we introduce a modified three-step iterative scheme for approximating a fixed point of nonexpansive mappings in the framework of uniformly smooth Banach spaces and the reflexive Banach spaces which have a weakly continuous duality map, respectively. we establish the strong convergence of the modified three-step iterative scheme. The results improve and extend recent ones given by other authors.

1. Introduction and preliminaries

Throughout this paper, we assume that E is a real Banach space, C is a nonempty closed convex subset of E, and $T : C \to C$ is a nonlinear mapping.

Recall that T is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\| \quad \forall x, y \in C.$$

A point $x \in C$ is a *fixed point* of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$. It is assumed throughout the paper that T is a nonexpansive mapping such that $F(T) \neq \emptyset$.

One classical way to study nonexpansive mappings is to use contractions to approximate a nonexpansive mapping (Browder [2], Reich [19]). More precisely, take $t \in (0, 1)$ and define a contraction $T_t : C \to C$ by

$$T_t x = tu + (1-t)Tx \; x \in C, \tag{1.1}$$

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where $u \in C$ is a fixed point. Banach's Contraction Mapping Principle guarantees that T_t has a unique fixed point x_t in C. It is unclear, in general, what is the behavior of x_t as $t \to 0$, even if T has a fixed point. However, in the case of T having a fixed point, Browder [2] proved that if E is a Hilbert space, then x_t does converge strongly to a fixed point of T that is nearest to u. Reich [19] extended Browder' result to the setting of Banach spaces and proved the if E is a uniformly smooth Banach space, then x_t converges strongly to a fixed point of T and the limit defines the (unique) sunny nonexpansive retraction from C onto $F(T)$.

In 1967, Halpern [7] first introduced the following iteration scheme (see also Browder [3])

$$\begin{cases} x_0 = x \in C \quad \textit{chosen arbitrarily}, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \quad n \geq 0. \end{cases} \tag{1.2}$$

He pointed out that the conditions $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$ are necessary in the sense that, if the iteration scheme (1.2) converges to a fixed point of T, then these conditions must be satisfied.

Two classical iteration processes are often used to approximate a fixed point of a nonexpansive mapping. The first one is introduced by Mann [11] and is defined as

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad n \geq 0, \tag{1.3}$$

where the initial guess x_0 is taken in C arbitrarily and the sequence $\{\alpha_n\}_{n=0}^{\infty}$ is in the interval $[0, 1]$.

The second iteration process is referred to as Ishikawa's iteration process [8] which is defined recursively by

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Ty_n, \quad n \geq 0, \end{cases} \tag{1.4}$$

where the initial guess x_0 is taken in C arbitrarily, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in the interval $[0, 1]$. But both (1.3) and (1.4) have only weak convergence, in general (see [6,24] for an example). For example, Reich [18], showed that if E is a uniformly convex and has a Fréchet differentiable norm and if the sequence $\{\alpha_n\}$ is such that $\alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by processes (1.3) converges weakly to a point in $F(T)$. (An extension of this result to processes (1.4) can be found in [24].) Therefore, many authors attempt to modify (1.3) and (1.4) to have strong convergence [9,15,16,21,26].

Recently, Noor [12] and Xu and Noor [27] suggested and analyzed three-step iterative methods for solving different classes of variational inequalities. It has been shown that three-step schemes are numerically better than two-step and one-step methods. Therefore, many authors [4,12-14,17,22,27,28] studied the three-step iterative process for nonexpansive mappings and asymptotically nonexpansive mappings. Related to the variational inequalities, it is the problem of finding the fixed points of the nonexpansive mappings, which is the subject of current interest in functional analysis. Motivated by the research going on these fields, we introduce a modified three-step iterative methods for finding a fixed point of nonexpansive mappings. We prove the convergence criteria of this new iterative schemes under some mild conditions and also give two strong convergence theorems of modified Ishikawa (two-step) iterative process and modified Mann (one-step) iterative process as spacial cases.

In this paper, we introduces the following modified iteration scheme

$$\begin{cases} x_0 \in C, \\ w_n = \delta_n x_n + (1-\delta_n)Tx_n, \\ z_n = \gamma_n x_n + (1-\gamma_n)Tw_n, \\ y_n = \beta_n x_n + (1-\beta_n)Tz_n, \\ x_{n+1} = \alpha_n u + \alpha'_n x_n + \alpha''_n y_n, \quad n \geq 0, \end{cases} \tag{1.5}$$

where the initial guess x_0 is taken in C arbitrarily, $u \in C$ is an arbitrary (but fixed) element in C and sequence $\{\alpha_n\}$, $\{\alpha'_n\}$, $\{\alpha''_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are sequences in $(0,1)$. We prove, under certain appropriate assumptions on the control sequences that $\{x_n\}$ defined by (1.5) converges to a fixed point of T.

Next, we consider some special cases of the three-step iterative scheme (1.5)

If $\{\delta_n\} = 1$ for all $n \geq 0$ in (1.5), then (1.5) collapses to

$$\begin{cases} x_0 \in C, \\ z_n = \gamma_n x_n + (1-\gamma_n)Tx_n, \\ y_n = \beta_n x_n + (1-\beta_n)Tz_n, \\ x_{n+1} = \alpha_n u + \alpha'_n x_n + \alpha''_n y_n, \quad n \geq 0. \end{cases} \tag{1.6}$$

If $\{\gamma_n\} = 1$ for all $n \geq 0$ in (1.5), then (1.5) reduces to

$$\begin{cases} x_0 \in C, \\ y_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n u + \alpha'_n x_n + \alpha''_n y_n, \quad n \geq 0. \end{cases} \tag{1.7}$$

Our purpose in this paper is to introduce this general modified three-step iteration scheme for approximating a fixed point of nonexpansive mappings in the framework of uniformly smooth Banach spaces and reflexive Banach spaces which have a weakly continuous duality map, respectively. We establish the strong convergence theorems of the general modified three-step iterative process and also give two strong convergence theorems of modified Ishikawa and modified Mann iterative process under some mild conditions as applications. The results improve and extend results announced by many other authors.

Let E be a real Banach space and let J denote the normalized duality mapping from E into 2^{E^*} given by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \quad x \in E,$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. The norm of E is said to be *Gâteaux differentiable* (and E is said to be *smooth*) if

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1.8}$$

exists for each x, y in its unit sphere $U = \{x \in E : \|x\| = 1\}$. It is said to be *uniformly Fréchet differentiable* (and E is said to be *uniformly smooth*) if the limit in (1.8) is attained uniformly for $(x, y) \in U \times U$.

We need the following definitions and lemmas for the proof of our main results.

Recall that a gauge is a continuous strictly increasing function $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. Associated to a gauge φ is the duality map $J_\varphi : E \to E^*$ defined by

$$J_\varphi(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|\varphi(\|x\|), \|x^*\| = \varphi(\|x\|)\},\ x \in E.$$

Following Browder [3], we say that a Banach space X has a *weakly continuous duality map* if there exists a gauge φ for which the duality map $J_\varphi(x)$ is single-valued and *weak*-to-*weak** sequentially continuous (i.e., if $\{x_n\}$ is a sequence in X weakly convergent to a point x, then the sequence $J_\varphi(x_n)$ converges weak*ly to J_φ). It is known that l^p has a weakly continuous duality map for all $1 < p < \infty$. Set

$$\Phi(t) = \int_0^t \varphi(\tau) d_\tau,\ t \geq 0.$$

Then

$$J_\varphi(x) = \partial\Phi(\|x\|), \quad x \in E,$$

where ∂ denotes the sub-differential in the sense of convex analysis. The first part of the next Lemma is an immediate consequence of the sub-differential inequality and the proof of the second part can be found in [10].

Lemma 1.1. *Assume that E has a weakly continuous duality map J_φ with gauge φ.*
(i) For all $x, y \in E$, there holds the inequality

$$\Phi(\|x+y\|) \le \Phi(\|x\|) + \langle y, J_\varphi(x+y)\rangle.$$

(ii) Assume that a sequence x_n in E is weakly convergent to a point x. The there holds the identity

$$\limsup_{n\to\infty} \Phi(\|x_n - y\|) = \limsup_{n\to\infty} \Phi(\|x_n - x\|) + \Phi(\|y - x\|),\ x, y \in E.$$

Notation: " $\rightharpoonup$ " stands for weak convergence and " $\to$ " for strong convergence.

Lemma 1.2. *A Banach space E is uniformly smooth if and only if the duality map J is single-valued and norm-to-norm uniformly continuous on bounded sets of E.*

Lemma 1.3. *In a Banach space E, there holds the inequality*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad x, y \in E,$$

where $j(x+y) \in J(x+y)$.

Recall that, if C and D are nonempty subsets of a Banach space E such that C is nonempty closed convex and $D \subset C$, then a map $Q : C \to D$ is sunny ([5], [20]) provided $Q(x + t(x - Q(x))) = Q(x)$ for all $x \in C$ and $t \ge 0$ whenever $x + t(x - Q(x)) \in C$. A sunny nonexpansive retraction is a sunny retraction,which is also nonexpansive. Sunny nonexpansive retractions play an important role in our argument. They are characterized as follows [1,5,20]: if E is a smooth Banach space, then $Q : C \to D$ is a sunny nonexpansive retraction if and only if there holds the inequality

$$\langle x - Qx, J(y - Qx)\rangle \le 0 \quad \forall x \in C, \text{and } y \in D.$$

Reich [19] showed that, if E is uniformly smooth and if D is the fixed point set of a nonexpansive mapping from C into itself, then there is a sunny nonexpansive retraction from C onto D and it can be constructed as follows.

Lemma 1.4. *Let E be a uniformly smooth Banach space and let $T : C \to C$ be a nonexpansive mapping with a fixed point $x_t \in C$ of the contraction $C \ni x \mapsto tu + (1-t)Tx$ converging strongly as $t \to 0$ to a fixed point of T. Define $Q : C \to F(T)$ by $Qu = s - \lim_{t\to 0} x_t$. Then Q is the unique sunny nonexpansive retract from C onto $F(T)$; that is, Q satisfies the property*

$$\langle u - Qu, J(z - Qu)\rangle \le 0, u \in C, \quad z \in F(T). \tag{1.9}$$

Lemma 1.5 (Xu [25]). *Let $\{\alpha_n\}$ be a sequence of nonnegative real numbers satisfying the property*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \gamma_n\sigma_n, \quad n \geq 0,$$

where $\{\gamma_n\}_{n=0}^{\infty} \subset (0,1)$ and $\{\sigma_n\}_{n=0}^{\infty}$ such that

(i) $\lim_{n\to\infty} \gamma_n = 0$ *and* $\sum_{n=0}^{\infty} \gamma_n = \infty$,

(ii) *either* $\limsup_{n\to\infty} \sigma_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\gamma_n\sigma_n| < \infty$.

Then $\{\alpha_n\}_{n=0}^{\infty}$ converges to zero.

Lemma 1.6 (Xu [26]). *Let X be a reflexive Banach space and have a weakly continuous duality map $J_\varphi(x)$. Let C be closed convex subset of X and let $T : C \to C$ be a nonexpansive mapping. Fix $u \in C$ and $t \in (0,1)$. Let $x_t \in C$ be the unique solution in C to Eq.(1.1). Then T has a fixed point if and only if x_t remains bounded as $t \to 0^+$, and in this case, $\{x_t\}$ converges as $t \to 0^+$ strongly to a fixed point of T.*

Under the condition of Lemma 1.6, we define a map $Q : C \to F(T)$ by $Q(u) := \lim_{t\to 0} x_t$, $u \in C$. From [26 Theorem 3.2] we know Q is the sunny nonexpansive retraction from C onto $F(T)$.

Lemma 1.7 (Suzuki [23]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let β_n be a sequence in [0,1] with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and*

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

2. Main Results

Theorem 2.1. *Let C be a closed convex subset of a uniformly smooth Banach space E and $T : C \to C$ a nonexpansive mapping. Given a point $u \in C$, the initial guess $x_0 \in C$ is chosen arbitrarily and given sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\alpha_n'\}_{n=0}^{\infty}$, $\{\alpha_n''\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$ and $\{\delta_n\}_{n=0}^{\infty}$ in $[0,1]$, the following conditions are satisfied*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0$;

(ii) $\alpha_n + \alpha_n' + \alpha_n'' = 1$;

(iii) $0 < \liminf_{n\to\infty} \alpha'_n \leq \limsup_{n\to\infty} \alpha'_n < 1$;

(iv) $\beta_n + (1+\beta_n)(1-\gamma_n)(2-\delta_n) \leq a < 1,\ for\ some\ a \in (0,1)$;

(v) $\lim_{n\to\infty} |\beta_{n+1} - \beta_n| = 0, \lim_{n\to\infty} |\gamma_{n+1} - \gamma_n| = 0$ *and* $\lim_{n\to\infty} |\delta_{n+1} - \delta_n| = 0$.

Let $\{x_n\}_{n=1}^{\infty}$ *be the composite process defined by (1.5). Then* $\{x_n\}_{n=1}^{\infty}$ *converges strongly to a fixed point of* T.

Proof. First, we observe that $\{x_n\}_{n=0}^{\infty}$ is bounded. Indeed, taking a fixed point p of T, we have

$$\|w_n - p\| \leq \delta_n \|x_n - p\| + (1-\delta_n)\|Tx_n - p\| \leq \|x_n - p\|.$$

It follows that

$$\begin{aligned} \|z_n - p\| &\leq \gamma_n \|x_n - p\| + (1-\gamma_n)\|Tw_n - p\| \\ &\leq \gamma_n \|x_n - p\| + (1-\gamma_n)\|w_n - p\| \\ &\leq \gamma_n \|x_n - p\| + (1-\gamma_n)\|x_n - p\| \end{aligned}$$

and

$$\begin{aligned} \|y_n - p\| &\leq \beta_n \|x_n - p\| + (1-\beta_n)\|Tz_n - p\| \\ &\leq \beta_n \|x_n - p\| + (1-\beta_n)\|z_n - p\| \\ &\leq \|x_n - p\|. \end{aligned}$$

Hence, we have

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n \|u - p\| + (1-\alpha_n)\|y_n - p\| \\ &\leq \alpha_n \|u - p\| + (1-\alpha_n)\|x_n - p\| \\ &\leq \max\{\|u - p\|, \|x_n - p\|\}. \end{aligned}$$

Now, an induction yields

$$\|x_n - p\| \leq \max\{\|u - p\|, \|x_0 - p\|\}, \quad n \geq 0.$$

This implies that $\{x_n\}$ is bounded, so are $\{y_n\}$, $\{z_n\}$ and $\{w_n\}$.

Next, we claim that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{2.1}$$

In order to prove (2.1), from

$$\begin{cases} w_n = \delta_n x_n + (1-\delta_n)Tx_n, \\ w_{n-1} = \delta_{n-1} x_{n-1} + (1-\delta_{n-1})Tx_{n-1}, \end{cases}$$

we obtain

$$w_n - w_{n-1} = (1-\delta_n)(Tx_n - Tx_{n-1}) + \delta_n(x_n - x_{n-1}) + (\delta_{n-1} - \delta_n)(Tx_{n-1} - x_{n-1}).$$

It follows that

$$\|w_n - w_{n-1}\| \leq \|x_n - x_{n-1}\| + |\delta_{n-1} - \delta_n| M_1, \tag{2.2}$$

where M_1 is an appropriate constant such that $M_1 \geq \sup_{n\geq 1}\{\|Tx_{n-1} - x_{n-1}\|\}$ Observing that

$$\begin{cases} z_n = \gamma_n x_n + (1-\gamma_n)Tw_n, \\ z_{n-1} = \gamma_{n-1}x_{n-1} + (1-\gamma_{n-1})Tw_{n-1}, \end{cases}$$

we have

$$\begin{aligned} z_n - z_{n-1} = \gamma_n(x_n - x_{n-1}) + (1-\gamma_n)(Tw_n - Tw_{n-1}) + \\ + (\gamma_n - \gamma_{n-1})(x_{n-1} - Tw_{n-1}). \end{aligned}$$

It follows from (2.2) that

$$\begin{aligned} &\|z_n - z_{n-1}\| \leq \\ &\leq \gamma_n\|x_n - x_{n-1}\| + (1-\gamma_n)\|w_n - w_{n-1}\| + |\gamma_n - \gamma_{n-1}|\|x_{n-1} - Tw_{n-1}\| \leq \\ &\leq \gamma_n\|x_n - x_{n-1}\| + (1-\gamma_n)(\|x_n - x_{n-1}\| + |\delta_{n-1} - \delta_n|M_1) + \\ &+ |\gamma_n - \gamma_{n-1}|\|x_{n-1} - Tw_{n-1}\| \leq \\ &\leq \|x_n - x_{n-1}\| + M_2(|\delta_{n-1} - \delta_n| + |\gamma_n - \gamma_{n-1}|), \end{aligned}$$

where M_2 is an appropriate constant such that

$$M_2 = \max\{M_1, \sup_{n\geq 1}\{\|x_{n-1} - Tw_{n-1}\|\}\}.$$

Similarly, we can prove that

$$\|y_n - y_{n-1}\| \leq \|x_n - x_{n-1}\| + M_3(|\delta_{n-1} - \delta_n| + |\gamma_n - \gamma_{n-1}| + |\beta_n - \beta_{n-1}|), \tag{2.3}$$

where M_3 is an appropriate constant such that

$$M_3 = \max\{M_1, M_2, \sup_{n\geq 1}\{\|Tz_{n-1} - x_{n-1}\|\}\}.$$

Put $l_n = \frac{x_{n+1} - \alpha'_n x_n}{1-\alpha'_n}$. That is, $x_{n+1} = (1-\alpha'_n)l_n + \alpha'_n x_n$. Now, we compute

$l_{n-1}-l_n$. Observing that

$$
\begin{aligned}
l_{n-1}-l_n &= \frac{\alpha_{n-1}u+\alpha''_{n-1}y_{n-1}}{1-\alpha'_{n-1}} - \frac{\alpha_n u+\alpha''_n y_n}{1-\alpha'_n} = \\
&= \left(\frac{\alpha_{n-1}}{1-\alpha'_{n-1}} - \frac{\alpha_n}{1-\alpha'_n}\right)u + \frac{\alpha''_{n-1}}{1-\alpha'_{n-1}}(y_{n-1}-y_n) + \left(\frac{\alpha''_{n-1}}{1-\alpha'_{n-1}} - \frac{\alpha''_n}{1-\alpha'_n}\right)y_n = \\
&= \left(\frac{\alpha_{n-1}}{1-\alpha'_{n-1}} - \frac{\alpha_n}{1-\alpha'_n}\right)(u-y_n) + \frac{\alpha''_{n-1}}{1-\alpha'_{n-1}}(y_{n-1}-y_n),
\end{aligned} \tag{2.4}
$$

we have

$$
\|l_{n-1}-l_n\| \le \left|\frac{\alpha_{n-1}}{1-\alpha'_{n-1}} - \frac{\alpha_n}{1-\alpha'_n}\right| \|u-y_n\| + \frac{\alpha''_{n-1}}{1-\alpha'_{n-1}}\|y_{n-1}-y_n\|. \tag{2.5}
$$

Substituing (2.3) into (2.5) yields that

$$
\begin{aligned}
\|l_{n-1}-l_n\| \le & \left|\frac{\alpha_{n-1}}{1-\alpha'_{n-1}} - \frac{\alpha_n}{1-\alpha'_n}\right| \|u-y_n\| + \|x_n-x_{n-1}\| \\
& + M_3(|\delta_{n-1}-\delta_n| + |\gamma_n-\gamma_{n-1}| + |\beta_n-\beta_{n-1}|).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|l_{n-1}-l_n\| - \|x_n-x_{n-1}\| \le & \left|\frac{\alpha_{n-1}}{1-\alpha'_{n-1}} - \frac{\alpha_n}{1-\alpha'_n}\right| \|u-y_n\| + \\
& + M_3(|\delta_{n-1}-\delta_n| + |\gamma_n-\gamma_{n-1}| + |\beta_n-\beta_{n-1}|).
\end{aligned}
$$

Observe the conditions (i), (v) and take the limits as $n \to \infty$, which gets

$$
\limsup_{n\to\infty}(\|l_{n+1}-l_n\| - \|x_{n+1}-x_n\|) \le 0.
$$

We can obtain $\lim_{n\to\infty}\|l_n-x_n\| = 0$ easily by Lemma 1.7. It follows that

$$
\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}(1-\alpha'_n)\|l_n-x_n\| = 0. \tag{2.6}
$$

Observing that

$$
\begin{aligned}
\|y_n-x_n\| &\le \|x_{n+1}-x_n\| + \|x_{n+1}-y_n\| \\
&\le \|x_{n+1}-x_n\| + \alpha_n\|\gamma f(x_n) - Ay_n\| + \alpha'_n\|x_n-y_n\|,
\end{aligned}
$$

and the conditions (i) and (iii), we can easily get

$$
\lim_{n\to\infty}\|y_n-x_n\| = 0. \tag{2.7}
$$

It follows that

$$\begin{aligned}\|Tx_n - x_n\| \leq & \|x_n - y_n\| + \|y_n - Tz_n\| + \|Tz_n - Tx_n\| \leq \\ \leq & \|x_n - y_n\| + \beta_n\|x_n - Tz_n\| + \|Tz_n - Tx_n\| \leq \\ \leq & \|x_n - y_n\| + \beta_n\|x_n - Tx_n\| + (1+\beta_n)\|Tz_n - Tx_n\| \leq \\ \leq & \|x_n - y_n\| + \beta_n\|x_n - Tx_n\| + (1+\beta_n)(1-\gamma_n)\|Tw_n - x_n\| \leq \\ \leq & \|x_n - y_n\| + \beta_n\|x_n - Tx_n\| + (1+\beta_n)(1-\gamma_n)\|Tx_n - x_n\| + \\ & + (1+\beta_n)(1-\gamma_n)\|w_n - x_n\| \leq \\ \leq & \|x_n - y_n\| + \beta_n\|x_n - Tx_n\| + (1+\beta_n)(1-\gamma_n)\|Tx_n - x_n\| + \\ & + (1+\beta_n)(1-\gamma_n)(1-\delta_n)\|Tx_n - x_n\|.\end{aligned}$$

From the condition (iv), we have

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0. \tag{2.8}$$

Next, we claim that

$$\limsup_{n\to\infty} \langle u - q, J(x_n - q)\rangle \leq 0,$$

where $q = Qu = s - \lim_{t\to 0} z_t$ with z_t being the fixed point of the contraction

$$z \mapsto tu + (1-t)Tz.$$

From that z_t solves the fixed point equation

$$z_t = tu + (1-t)Tz_t,$$

we have

$$\|z_t - x_n\| = \|(1-t)(Tz_t - x_n) + t(u - x_n)\|.$$

It follows from Lemma 1.3 that

$$\begin{aligned}\|z_t - x_n\|^2 \leq & (1-t)^2\|Tz_t - x_n\|^2 + 2t\langle u - x_n, J(z_t - x_n)\rangle \leq \\ \leq & (1-t)^2(\|Tz_t - Tx_n\| + Tx_n - x_n\|)^2 + 2t\langle u - x_n, J(z_t - x_n)\rangle \leq \\ \leq & (1 - 2t + t^2)\|z_t - x_n\|^2 + f_n(t) + \\ & + 2t\langle u - z_t, J(z_t - x_n)\rangle + 2t\|z_t - x_n\|^2,\end{aligned} \tag{2.9}$$

where

$$\lim_{n\to\infty} f_n(t) = (2\|z_t - x_n\| + \|x_n - Tx_n\|)\|x_n - Tx_n\| = 0. \tag{2.10}$$

It follows that

$$\langle z_t - u, J(z_t - x_n)\rangle \leq \frac{t}{2}\|z_t - x_n\|^2 + \frac{1}{2t} f_n(t). \tag{2.11}$$

Letting $n \to \infty$ in (2.11) and noting (2.10), we obtain

$$\limsup_{n\to\infty}\langle z_t - u, J(z_t - x_n)\rangle \leq \frac{t}{2}M, \tag{2.12}$$

where $M > 0$ is a constant such that $M \geq \|z_t - x_n\|^2$ for all $t \in (0,1)$ and $n \geq 1$. Letting $t \to 0$ from (2.12) we have

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle z_t - u, J(z_t - x_n)\rangle \leq 0.$$

So, for any $\epsilon > 0$, there exists a positive number δ_1 such that, for $t \in (0, \delta_1)$ we get

$$\limsup_{n\to\infty}\langle z_t - u, J(z_t - x_n)\rangle \leq \frac{\epsilon}{2}. \tag{2.13}$$

On the other hand, since $z_t \to q$ as $t \to 0$, from Lemma 1.2, there exists $\delta_2 > 0$ such that, for $t \in (0, \delta_2)$, we have

$$\begin{aligned}
&|\langle u - q, J(x_n - q)\rangle - \langle z_t - u, J(z_t - x_n)\rangle| \leq \\
&\leq |\langle u - q, J(x_n - q)\rangle - \langle u - q, J(x_n - z_t)\rangle| + \\
&+ |\langle u - q, J(x_n - z_t)\rangle - \langle z_t - u, J(z_t - x_n)\rangle| \leq \\
&\leq |\langle u - q, J(x_n - q) - J(x_n - z_t)\rangle| + |\langle z_t - q, J(x_n - z_t)\rangle| \leq \\
&\leq \|u - q\|\|J(x_n - q) - J(x_n - z_t)\| + \|z_t - q\|\|x_n - z_t\| < \frac{\epsilon}{2}.
\end{aligned}$$

Choosing $\delta = \min\{\delta_1, \delta_2\}, \forall t \in (0, \delta)$, we arrive at

$$\langle u - q, J(x_n - q)\rangle \leq \langle z_t - u, J(z_t - x_n)\rangle + \frac{\epsilon}{2}.$$

That is,

$$\limsup_{n\to\infty}\langle u - q, J(x_n - q)\rangle \leq \lim_{n\to\infty}\langle z_t - u, J(z_t - x_n)\rangle + \frac{\epsilon}{2}.$$

It follows from (2.13) that

$$\limsup_{n\to\infty}\langle u - q, J(x_n - q)\rangle \leq \epsilon.$$

Since ϵ is chosen arbitrarily, we obtain

$$\limsup_{n\to\infty}\langle u - q, J(x_n - q)\rangle \leq 0. \tag{2.14}$$

Finally, we show that $x_n \to q$ strongly and this concludes the proof. From Lemma 1.3, we obtain

$$\begin{aligned}\|x_{n+1} - q\|^2 &= \|\alpha''_n(y_n - q) + \alpha'_n(x_n - q) + \alpha_n(u - q)\|^2 \leq \\ &\leq \|\alpha''_n(y_n - q) + \alpha'_n(x_n - q)\|^2 + 2\alpha_n\langle u - q, J(x_{n+1} - q)\rangle \leq \\ &\leq (1 - \alpha_n)\|x_n - q\|^2 + 2\alpha_n\langle u - q, J(x_{n+1} - q)\rangle.\end{aligned}$$

It is easy to see that $\|x_n - q\| \to 0$ as $n \to \infty$ by Lemma 1.5. This completes the proof.

Next, we consider the iterative algorithm 1.5 in a reflexive Banach spaces.

Theorem 2.2. *Let E be a reflexive Banach space which has a weakly continuous duality map J_φ with gauge φ. Let $\{x_n\}$ T, C, $\{\alpha_n\}$, $\{\alpha'_n\}$, $\{\alpha''_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be as Theorem 2.1. Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of T.*

Proof. We only include the differences. Observe that

$$\|Tx_n - x_{n+1}\| \leq \|x_n - x_{n+1}\| + \|x_n - Tx_n\|.$$

It follows from (2.6) and (2.8) that

$$\lim_{n\to\infty} \|Tx_n - x_{n+1}\| = 0. \tag{2.15}$$

Next, we prove that

$$\limsup_{n\to\infty} \langle u - Q(u), J_\varphi(x_n - Q(u)\rangle \leq 0. \tag{2.16}$$

By Lemma 1.6, we have the sunny nonexpansive retraction $Q : C \to F(T)$. Take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle u - Q(u), J_\varphi(x_n - Q(u)\rangle = \lim_{k\to\infty} \langle u - Q(u), J_\varphi(x_{n_k} - Q(u)\rangle. \tag{2.17}$$

Since E is reflexive, we may further assume that $x_{n_k} \rightharpoonup p$, for some $p \in C$. Since J_φ is weakly continuous, we have by Lemma 1.1,

$$\limsup_{n\to\infty} \Phi(\|x_{n_k} - x\|) = \limsup_{n\to\infty} \Phi(\|x_{n_k} - p\|) + \Phi(\|x - p\|),\ x, y \in E.$$

Put

$$f(x) = \limsup_{n\to\infty} \Phi(\|x_{n_k} - x\|), \quad x \in E.$$

It follows that

$$f(x) = f(p) + \Phi(\|x - p\|), \quad x \in E.$$

From Theorem 2.1, we get

$$\|x_{n_k} - Tx_{n_k}\| \to 0.$$

It follows that

$$\begin{aligned} f(Tp) &= \limsup_{n\to\infty} \Phi(\|x_{n_k} - Tp\|) = \limsup_{n\to\infty} \Phi(\|Tx_{n_k} - Tp\|) \\ &\le \limsup_{n\to\infty} \Phi(\|x_{n_k} - p\|) = f(p). \end{aligned} \tag{2.18}$$

On the other hand, we have

$$f(Tp) = f(p) + \Phi(\|Tp - p\|). \tag{2.19}$$

Combine (2.18) and (2.19) yields that

$$\Phi(\|Tp - p\|) \le 0.$$

Hence $Tp = p$ and $p \in F(T)$. That is, $p \in F$. Hence by (2.17) and (1.9) we have

$$\limsup_{n\to\infty} \langle u - Q(u), J_\varphi(x_n - Q(u))\rangle = \langle u - Q(u), J_\varphi(p - Q(u))\rangle \le 0.$$

Hence (2.16) holds. Finally, we prove that $x_n \to p$ as $n \to \infty$. Notice that

$$\begin{aligned} \Phi(\|w_n - p\|) &= \Phi(\|\delta_n(x_n - p) + (1 - \delta_n)(Tx_n - p)\|) \le \\ &\le \Phi(\delta_n\|x_n - p\| + (1 - \delta_n)\|Tx_n - p\|) \le \\ &\le \Phi(\|x_n - p\|). \end{aligned}$$

Similarly, we can prove $\Phi(\|z_n - p\|) \le \Phi(\|x_n - p\|)$ and

$$\Phi(\|y_n - p\|) \le \Phi(\|x_n - p\|). \tag{2.20}$$

Therefore, from (2.20) we obtain

$$\begin{aligned} \Phi(\|x_{n+1} - p\|) &= \Phi(\|\alpha_n(u - p) + \alpha'_n(x_n - p) + \alpha''_n(y_n - p)\|) \le \\ &\le \Phi(\|\alpha'_n(x_n - p) + \alpha''_n(y_n - p)\|) + \alpha_n\langle u - p, J_\varphi(x_{n+1} - p)\rangle \le \\ &\le (1 - \alpha_n)\Phi(\|x_n - p\|) + \alpha_n\langle u - p, J_\varphi(x_{n+1} - p)\rangle. \end{aligned}$$

An application of Lemma 1.5 yields that $\Phi(\|x_n - p\|) \to 0$; that is $\|x_n - p\| \to 0$ as $n \to \infty$. This completes the proof.

3. Applications

In this section, we shall prove two strong convergence theorems of modified Ishikawa (two-step) iterative process and modified Mann (one-step) iterative process under some mild conditions in Banach spaces.

Theorem 3.1. *Let C be a closed convex subset of a uniformly smooth Banach space E and $T : C \to C$ a nonexpansive mapping. Given a point $u \in C$, the initial guess $x_0 \in C$ is chosen arbitrarily and given sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\alpha'_n\}_{n=0}^{\infty}$, $\{\alpha''_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ in $[0,1]$, the following conditions are satisfied*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0$;

(ii) $\alpha_n + \alpha'_n + \alpha''_n = 1$;

(iii) $0 < \liminf_{n\to\infty} \alpha'_n \leq \limsup_{n\to\infty} \alpha'_n < 1$;

(iv) $\beta_n + (1 + \beta_n)(1 - \gamma_n) \leq a < 1,\ for\ some\ a \in (0,1)$;

(v) $\lim_{n\to\infty} |\beta_{n+1} - \beta_n| = 0, \lim_{n\to\infty} |\gamma_{n+1} - \gamma_n| = 0$.

Let $\{x_n\}_{n=1}^{\infty}$ be the composite process defined by (1.6). Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of T.

Proof. By taking $\{\delta_n\} = 1$ in Theorem 2.1, it is easy to get the desired conclusion.

Theorem 3.2. *Let C be a closed convex subset of a uniformly smooth Banach space E and let $T : C \to C$ be a nonexpansive mapping. Given a point $u \in C$, the initial guess $x_0 \in C$ is chosen arbitrarily and given sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\alpha'_n\}_{n=0}^{\infty}$, $\{\alpha''_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in [0,1], the following conditions are satisfied*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0$;

(ii) $\alpha_n + \alpha'_n + \alpha''_n = 1$;

(iii) $0 < \liminf_{n\to\infty} \alpha'_n \leq \limsup_{n\to\infty} \alpha'_n < 1$;

(iv) $\beta_n \le a < 1,\ for\ some\ a \in (0,1)$;

(v) $\lim_{n\to\infty} |\beta_{n+1} - \beta_n| = 0$.

Let $\{x_n\}_{n=1}^{\infty}$ be the composite process defined by (1.7). Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of T.

Proof. By taking $\{\gamma_n\} = 1$ in Theorem 2.1, we can obtain the desired conclusion immediately.

Remark 3.3. Theorem 3.2 relaxes the assumptions imposed on the control sequences by Kim and Xu [9]. To be more precise, we remove $\beta_n \to 0$ and also relaxes the restricts on $\{\alpha_n\}$, respectively.

Remark 3.4. From the proof of Theorem 2.2, we see that Theorem 3.1 and Theorem 3.2 still hold in the framework of reflexive Banach spaces.

Remark 3.5. If $f : C \to C$ is a contraction map and we replace u by $f(x_n)$ in the recursion formula (1.5), we obtain what some authors now call viscosity iteration method. We note that our theorems in this paper carry over trivially to the so-called viscosity process. One simply replaces u by $f(x_n)$, and using the fact that f is a contraction map, one can repeat the argument of this paper.

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