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Ordered Model of an Exchange Economy

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Abstract

An importat role in market economies, as ordered economies models, is played by the preference relationship and by the utility function. These notions are studied in the followings on the bases of a build and analyzed demand-offer model, placed within the context of a finite-dimensional exchange economy.

1 The Stocks-Prices ordered space

The number of mathematical models used in economy is impressive. An optimal command problem on national level may be a model of economical growth and consume. Optimum problems for entreprise profit, assurance problems, or even problems concerning the financial chaos, all have as mathematical bases optimum problems, riscs functions and models, or fractals [1, 2, 5]. Despite this, we are concerned on economical equilibrium using linear ordered spaces.

In the following, we consider the real situation in which only a finite number of products are produced, exchanged or consumed. Such a model was first presented in [4]. The stocks set has to be considered as having a triple structure; linear, ordered and topological one. This means:

1. a linear structure, since by summing two stocks or by multiplying a stock with a scalar we still obtain also stocks.
2. an ordered structure: if two stocks x, y are in the relation $x > y$, this means that "x is bigger than y" and this is taken as negative input for production (because it supposes consuming) and as output having positive sign.

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3. a topological structure, because in the finite dimensional case, the demand-offer function continuously influences the prices (this fits with the fact that small price changes involve small demand-offer changes).

The economy models with finite numner of exchanged, produced or consumed assets are introduced by Arrow-Debreu. They consider the n-stocks space as R^n.

Generally, we will denote the stock space by E.

The fact that the product j may be exchanged at the market level with the product i is defined by $\frac{p_i}{p_j}$ where p_i and p_j are non-negative reals and $p_j > 0$. So, $\frac{p_i}{p_j}$ is the quantity of the product j which may be for against the product i, using the prices given by the vector $p = (p_1, p_2, \ldots, p_n)$.

Given a price vector $p = (p_1, \ldots, p_n)$ and an asset vector $x = (x_1, \ldots, x_n)$, the "value" of x at the price p is $p \cdot x = \sum_{i=1}^{n} p_i x_i$. This way, each price vector defines a linear functional on the asset space E and defines the price space as the dual E' of E. In the case $E = R^n$, $E' = R^n$. This good-price duality was first introduced in [3].

Because the asset vector x is greater than the asset vector y, there is an ordered structure on E. This particular structure of the price space E was succefully applied in equilibrium economical analyzes [6].

In the followings we are using a natural order structure on R^n based on the notion of *preference*.

2 Preliminaries in preference and utility functions

In an economic theory, as an axiom, the economic agents are rationals, in the sense that they know their interests and act in order to maximize their capital. To this end, we found interesting to present economical models by computing their price, utility and equilibrium parameters as were introduced in [4, 6].

We say that we are in an equilibrium price if the asset price is the used one on international market, within a group of rich countries.

Definition 2.1 *A reflexive, transitive and total relationship is called* ***preference*** *on a nonempty subset $X \subseteq R^n$.*

Such a preference is denoted by $\succeq$. We say that x is at least as good as y or that x is no worse than y if $x \succeq y$. Consequently, it is denoted by $x \succ y$ the fact that x is preferred to y or x is better than y, if $x \succeq y$ and $y \not\succeq x$.

If $x \succeq y$ and $y \succeq x$, then x is equally prefered as y. This is denoted as $x \sim y$.

If $x \in X$ then $\{y \in X \ : \ y \succ x\}$ is the set of preferred elements of x and is called "the set of the better than" x, and the set $\{y \in X \ : \ x \succ y\}$ is called "the set of the worse than" x.

More, if X is a topological space, the preference relationship is called:

1. superior semicontinuous if for each $x \in X$ the set $\{y \in X \ : \ y \succeq x\}$ is closed;
2. inferior semicontinuous if for each $x \in X$ the set $\{y \in X \ : \ x \succeq y\}$ is closed;
3. continuous if it is both superior semicontinuous and inferior semicontinuous.

In [4] it is presented the *utility function* using the notion of *preference*:

Definition 2.2 *The function $u : X \mapsto R$ is called an **utility function** regarding to the preference $\succeq$ on X if $x \succeq y$ holds iff $u(x) \geq u(y)$.*

The utility function is not uniquely determined. In other words, if u is the utility function, then u^3, u^5, e^u are also utility functions. In fact, given a function $u : X \mapsto R$, by defining the relation $x \succeq y$ if and only if $u(x) \geq u(y)$, we obtain a preference relationship.

When a preference relationship may be represented by the mean of an utility function? The answer is given by the following:

Theorem 2.1 *Each continuous preference relation inside a topological space having a countable open basis can be represented using a continuous utility function.*

Due to the fact that any open set with an open countable basis is separable the theorem conditions assures the separability. Moreover, the continuity of the preference relation became a permanent property of the utility function.

Definition 2.3 *A preference relation defined on a convex set X of a linear space is called:*

1. ***convex***
   if $y \succeq x$ and $z \succeq x$ in X, $0 < \alpha < 1$ then $\alpha y + (1 - \alpha)z \succeq x$;
2. ***strictly convex***
   if $y \succeq x$ and $z \succeq x$ in X, $0 < \alpha < 1$ then $\alpha y + (1 - \alpha)z \succ x$.

We note that the preference relation is convex if and only if the set of "better than" x is convex for each $x \in X$.

A utility function obtained from a convex preference is a quasi-concave function. Similarly, a utility function generated by a strictly convex preference is a strictly quasi-concave function.

Definition 2.4 *The function $u : C \mapsto R$, defined on a convex non-empty subset $C \subseteq X$ within a linear space is called*

1. ***quasi-concave*** *if $\forall x, y \in C, x \neq y,\ 0 < \alpha < 1$, then*
$$u(\alpha x + (1-\alpha)y) \geq \min\{u(x), u(y)\};$$
2. ***strictly quasi-concave*** *if $\forall x, y \in C, x \neq y,\ 0 < \alpha < 1$, then*
$$u(\alpha x + (1-\alpha)y) \geq \min\{u(x), u(y)\};$$
3. ***concave*** *if $\forall x, y \in C, x \neq y,\ 0 < \alpha < 1$, then*
$$u(\alpha x + (1-\alpha)y) \geq \alpha u(x) + (1-\alpha)u(y);$$
4. ***strictly concave*** *if $\forall x, y \in C, x \neq y,\ 0 < \alpha < 1$, then*
$$u(\alpha x + (1-\alpha)y) \geq \alpha u(x) + (1-\alpha)u(y).$$

Definition 2.5 *A preference relation $\succeq$ on a non-empty set X within an ordered linear space is called*

1. ***monotone****, if for $x, y \in X$ and $x > y$, then $x \succeq y$;*
2. ***strictly monotone****, if for $x, y \in X$ and $x > y$, then $x \succ y$.*

Definition 2.6 *Let $\succeq$ be a preference relation on a subset X of a linear space E. The vector $v \in E$ is called the **preferred vector**, if:*

1. *$x + \alpha v \in X$, for any $x \in X$ and $\alpha > 0$;*
2. *$x + \alpha v \succ x$, for any $x \in X$ and $\alpha > 0$.*

Definition 2.7 The budget set *for the price p of the vector $\omega \in R^n_+$ (also called **initial capital**) is the set*

$$B_\omega(p) \ := \ \{x \in R^n_+ \ : \ p \cdot x \leq p \cdot \omega\}.$$

The budget line $B_\omega(p)$ *is the set*

$$\{x \in B_\omega(p) \ : \ p \cdot x = p \cdot \omega\}.$$

The demand vector $x_\omega(p)$ *corresponding to the strictly positive price* $p \in R^n$*, initial capital* $\omega \in R^n$ *and continuous strictly convexe preference* $\succeq$ *(on a subset* $X \subseteq R^n$*) that has the preferred vector, is the maximal element of the relationship* $\succeq$ *inside the budget set* $B_\omega(p)$*. (It can be shown that, in these conditions,* $\succeq$ *have exactly one maximal element on the bugdet line* $B_\omega(p)$*.)*

Demand function $x_\omega \ : \ int\ R^n_+ \mapsto R^n_+$ *corresponding to the preference relationship* $\succeq$ *on* R^n_+ *is the function* $p \longmapsto x_\omega(p)$ *where* $x_\omega(p)$ *is the corresponding demand vector.*

It is clear that, inside the market, the preference relationships as well as the utility functions are not observable. Moreover, the transactional agents behavior depends on the market prices. This is why the demand functions are usefull in the economic agents activity description.

If $x_\omega(p) = (x_1(p), \ldots, x_n(p))$ with $x_i(p) \in R_+$ for $i = 1, \ldots, n$ is the demand vector, then l_1-norm

$$||x_\omega(p)||_1 \ := \ \sum_{i=1}^{n} x_i(p)$$

is called **the total number of good units demanded by the consumers**. When the prices are close enough to the boundary ∂R^n_+, some goods/assets became more expensive so, the corresponding demand will decrease. In other words, when the prices decrease to zero (the general demand, not individual one, for each asset) goes to infinity.

3 A model for demand-offer

In the followings we'll consider the utility function on $X \subset R^3$, $u : X \to R$,

$$u(x, y, z) = \frac{x}{1 + x} + \ln(1 + y) + y + \sqrt{z}$$

with $\omega = (1, 1, 1)$ as initial capital. We will analyze this utility function by verifying its monotony, convexity and continuity attributes and by studying the budget $B_\omega(p)$, and hence, as suggested in [4], we'll obtain a preference relation $\succeq$ on R^3_+.

Let us note that this model simulates the demand-offert process, and its behaviour can be described as follows:

1. u is strictly monotone.

2. u is strictly concave.

3. u is a continuous function.

4. For any $x > 0$, $u(0, x + y, z) > u(x, y, z)$, when $y, z \in R_+$.

5. If the price $p = (p_1, p_2, p_3) >> 0$ has $p_1 = p_2$, then the demand vector $x_\omega(p) = (x(p), y(p), z(p))$ satisfies the condition $x(p) = 0$. Together with the previous properties of u, the budget set has exactly one maximal element. If this element $x_\omega(p)$ should have $x(p) > 0$, then due to 4) with $p_1 = p_2$, then there exists an element $x \in B_\omega(p)$ such that $x \succ x_\omega(p)$, and that gives a nonsense. So, $x(p) = 0$.

6. If $p_n = (\frac{1}{n}, \frac{1}{n}, 1) \to (0, 0, 1)$, then $x_\omega(p_n) = (0, y(p_n), z(p_n))$ is the demand function ($n \in N$). Doing so, the demand function for the first asset remains bounded, due to fact that the prices of the first asset goes to zero.

7. $y(p_n) \to \infty$ when $n \to \infty$. This means that the price goes to zero when the collective demand goes to infinity.

4 Exchange economies in finite dimensions

Definition 4.1 *An* ***exchange economy*** *is a function $\mathcal{E} : A \to R_+^n \times P$, where $i \to (\omega_i, \succeq_i)$. Here $A = \{1, 2, ..., m\}$ is the finite set of agents (consumers) i, $\omega_i \in R_+^n$ is the initial capital of the agent i, $\succeq_i$ is the preference of the agent i, and P is the set of all the preference relationships on R_+^n. If $p \in R_+^n$ is a price, then the positive number $p \cdot \omega_i$ is called the* ***agent income*** *i.*

Definition 4.2 *The vector $\omega = \sum_{i=1}^n \omega_i$ is the* ***total capital*** *of the economy $\mathcal{E}$.*

Definition 4.3 *A preference relation $\succeq$ in R_+^n is called neoclassical if:*

(a) $\succeq$ is strictly monotone (i.e. $x > y$ in R_+^n implies $x \succeq y$ in R_+^n and strictly convex in R_+^n).

(b) $\succeq$ is strictly monotone and strictly convex in $intR_+^n$ and each interior point is preferred by each point from the border.

Definition 4.4 *A* ***neoclassical exchange economy*** *is an economy $\mathcal{E}$ such that, for any $i = 1, \ldots, m$, $\omega_i > 0$ and $\succeq_i$ is a neoclassical preference on R_+^n.*

Definition 4.5 *The* ***excess demand function*** $\xi : intR^n_+ \to R^n$ *is defined by* $\xi(p) = \sum_{i=1}^n x_i(p) - \omega$. *Here,* $x_i(p)$ *is the demand vector corresponding to the preference relationship* $\succeq_i$ *and the strictly positive price* $p \in intR^n_+$ *(denoted by* $p >> 0$*).*

Definition 4.6 *A strictly positive price* p *is called an* ***equilibrium price*** *for the economy* $\mathcal{E}$ *if* $\xi(p) = 0$.

In [4] it was proposed a model for an exchange economy having R^2 as assets space and three economic agents. In fact, this economic model is a neoclassical one, having the equilibrium price in the vecinity of the first bisection $p = (p_1, p_2)$ with $p_1 = p_2$.

The three agents are:

- the agent 1, with $\omega_1 = (1,2)$ and $u_1 = \sqrt{x} + \sqrt{y}$,
- the agent 2, with $\omega_2 = (1,1)$ and $u_2 = xe^y$,
- the agent 3, with $\omega_3 = (1,2)$ and $u_3 = xy$.

The preference relationships $\succeq_i$ $(i = 1,2,3)$ represented by these three utility functions u_i of the agents, are neoclassical. In other words, each $\succeq_i$ is strictly monotone and convex on R^2_+ and each $\succeq_i$ is strictly monotone and strictly concave on *int* R^2_+. Moreover, each interior point is preferred of each point from the border.

The demand function of the first agent is

$$x_1(p) = \left(\frac{p_2(p_1 + 2p_2)}{p_1(p_1 + p_2)}, \frac{p_1(p_1 + 2p_2)}{p_2(p_1 + p_2)} \right)$$

or

$$x_1(t) = \left(\frac{t(1 + 2t)}{1 + t}, \frac{1 + 2t}{t(1 + t)} \right),$$

where $t = \frac{p_2}{p_1}$.

The demand function of the second agent is

$$x_2(p) = \left(\frac{p_2}{p_1}, \frac{p_1}{p_2} \right)$$

or

$$x_2(t) = \left(t, \frac{1}{t} \right),$$

with $t = \frac{p_2}{p_1}$.

With the function u_3 and with the link $\phi_3(x,y) = p_1x + p_2y - (p_1 + 2p_2)$, we obtain

$$x_3(p) = \left(\frac{p_1 + 2p_2}{2p_1}, \frac{p_1 + 2p_2}{2p_2}\right)$$

or

$$x_3(t) = \left(\frac{1}{2} + t, \frac{1}{2t} + 1\right),$$

with $t = \frac{p_2}{p_1}$.

This way, the excess demand function for our proposed economy is $\xi : intR^2_+ \to R^2$ given by $\xi(p) = x_1(p) + x_2(p) + x_3(p) - (\omega_1 + \omega_2 + \omega_3)$ or

$$\xi(t) = \left(\frac{8t^2 + 7t + 1}{2(1+t)} - 3, \frac{2t^2 + 9t + 5}{2t(1+t)} - 5\right),$$

which gives us

$$\xi(p) = \left(\frac{8t^2 + t - 5}{2(1+t)}, \frac{-8t^2 - t + 5}{2t(1+t)}\right)$$

or

$$\xi(p) = \left(\frac{8p_2^2 + p_1p_2 - 5p_1}{2p_1(p_1 + p_2)}, \frac{-8p_2 - p_1p_2 + 5p_1}{2p_2(p_1 + p_2)}\right).$$

The following attributes of ξ can be seen:

1. ξ is zero degree homogenous, i.e. $\xi(\lambda p) = \xi(p)$ for any $p >> 0$ and $\lambda > 0$;

2. ξ is continuous and bounded;

3. ξ satisfies the Walras law, i.e. $p \cdot \xi(p) = 0$ for any $p >> 0$.

5 Conclusions

Despite the finite dimensional approach, we are convinced that its utility can be proved in management, decision and economical equilibrium real problems. To this end, it is neccesary to bring into discussion the model adaptability to crises situation as well as its extension at infinite dimensional, or better to infinite dimensional under a directed order; as order in which each element can be obtained as difference of positive elements, and which is less pretentious than the latticeal one.

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