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Optimaly conditons and duality for a minimax nondifferentiable programming problem, involving (η, ρ, θ)-invex functions

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Abstract

We establish necessary and sufficient optimality condition, involving (η, ρ, θ)-invex functions, for a class of minimax programming problems with square-root terms in the objective function. Subsequently, we apply the optimality condition to formulate a parametric dual problem and we prove weak duality, direct duality, and strict converse duality theorems.

1 Introduction

Let us consider the following continuous differentiable mappings:

$$f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, \quad g = (g_1, \ldots, g_p) : \mathbb{R}^n \to \mathbb{R}^p.$$

We denote

$$\mathcal{P} = \left\{ x \in \mathbb{R}^n \mid g_i(x) \le 0,\ i = \overline{1,p} \right\}. \tag{1}$$

and consider $Y \subseteq \mathbb{R}^m$ to be a compact subset of $\mathbb{R}^m$. For $r = 1, 2, \ldots, q$, let B_r be $n \times n$ positive semidefinite matrices such that for each $(x, y) \in \mathcal{P} \times Y$, we have:

$$f(x, y) + \sum_{r=1}^{q} \sqrt{x^\top B_r x} \ge 0.$$

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In this paper we consider the following nondifferentiable minimax programming problem:

$$\inf_{x\in\mathcal{P}}\sup_{y\in Y}\left\{f(x,y)+\sum_{r=1}^{q}\sqrt{x^{\top}B_r x}\right\} \tag{P}$$

Many authors investigated the optimality conditions and duality theorems for minimax programming problems. For details, one can consult [1, 2, 10]. Problems which contain square root terms were also considered by Preda [9], and Preda and Köller [11].

In an earlier work, under conditions of convexity, Schmittendorf [12] established necessary and sufficient optimality conditions for the problem:

$$\inf_{x\in\mathcal{P}}\sup_{y\in Y}\psi(x,y), \tag{P1}$$

where $\psi:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}$ is a continuous differentiable mapping. Later, Yadev and Mukherjee [14] employed the optimality conditions of Schmittendorf [12] to construct two dual problems for which they derived some duality theorems for fractional minimax programming problems, involving convex differentiable functions.

In this paper, we derive sufficient optimality conditions for (P) and we apply the optimality conditions to construct a parametric dual problem for which we formulate a weak duality, a direct duality and a strictly converse duality theorem. Some definitions and notations are given in Section 2. In Section 3, we derive sufficient optimality conditions under the assumption of a particular form of generalized convexity. Using the optimality conditions we define in Section 4 a parametric dual problem for which we prove the above mentioned duality results.

2 Notation and Preliminary Results

Throughout this paper, we denote by $\mathbb{R}^n$ the n-dimensional Euclidean space and by $\mathbb{R}^n_+$ its nonnegative orthant. Let us consider the set $\mathcal{P}$ defined by (1), and for each $x\in\mathcal{P}$, we define

$$\begin{aligned} J(x) &= \{j\in\{1,2,\ldots,p\}\mid g_j(x)=0\},\\ Y(x) &= \left\{y\in Y \;\middle|\; f(x,y)+\sum_{r=1}^{q}\sqrt{x^{\top}B_r x}=\sup_{z\in Y}\left(f(x,z)+\sum_{r=1}^{q}\sqrt{x^{\top}B_r x}\right)\right\}. \end{aligned}$$

Let A be an $m\times n$ matrix and let M, M_i, $i=1,\cdots,k$, be $n\times n$ symmetric positive semidefinite matrices.

Lemma 1 [13] *We have*

$$Ax \geq 0 \;\Rightarrow\; c^{\top}x + \sum_{i=1}^{k} \sqrt{x^{\top}M_i x} \geq 0,$$

if and only if there exist $y \in \mathbb{R}_{+}^{m}$ *and* $v_i \in \mathbb{R}^n$, $i = \overline{1,k}$, *such that*

$$Av_i \geq 0, \quad v_i^{\top}M_i v_i \leq 1,\; i = \overline{1,k}, \quad A^{\top}y = c + \sum_{i=1}^{k} M_i v_i.$$

If all $M_i = 0$, Lemma 1 becomes the well-known Farkas lemma.
We shall use the generalized Schwarz inequality:

$$x^{\top}Mv \leq \sqrt{x^{\top}Mx}\,\sqrt{v^{\top}Mv}. \tag{2}$$

We note that equality holds in (2) if $Mx = \tau Mv$ for some $\tau \geq 0$.
Obviously, from (2), we have

$$v^{\top}Mv \leq 1 \;\Rightarrow\; x^{\top}Mv \leq \sqrt{x^{\top}Mx}. \tag{3}$$

The following lemma is given by Schmittendorf [12] for the problem (P1):

Lemma 2 [12] *Let* x_0 *be a solution of the minimax problem (P1) and the vectors* $\nabla g_j(x_0)$, $j \in J(x_0)$ *are linearly independent. Then there exist a positive integer* $s \in \{1, \ldots, n+1\}$, *real numbers* $t_i \geq 0$, $i = \overline{1,s}$, $\mu_j \geq 0$, $j = \overline{1,p}$, *and vectors* $\bar{y}_i \in Y(x_0)$, $i = \overline{1,s}$, *such that*

$$\begin{gathered}\sum_{i=1}^{s} t_i \nabla_x \psi(x_0, \bar{y}_i) + \sum_{j=1}^{p} \mu_j \nabla g_j(x_0) = 0,\\ \mu_j g_j(x_0) = 0, \quad j = \overline{1,p},\\ \sum_{i=1}^{s} t_i \neq 0.\end{gathered}$$

Now we give the definitions of (η, ρ, θ)-quasi-invexity and (η, ρ, θ)-pseudo-invexity as extensions of the invexity notion. The invexity notion of a function was introduced into optimization theory by Hanson [5] and the name of invex function was given by Craven [3]. Some extensions of invexity as pseudo-invexity, quasi-invexity and ρ-invexity, ρ-pseudo-invexity, ρ-quasi-invexity are presented in Craven and Glover [4], Kaul and Kaur [6], Preda [8], Mititelu and Stancu-Minasian [7].

Definition 3 *A differentiable function $\varphi : C \subseteq \mathbb{R}^n \to \mathbb{R}$ is (η, ρ, θ)-**invex** at $x_0 \in C$ if there exist functions $\eta : C \times C \to \mathbb{R}^n$, $\theta : C \times C \to \mathbb{R}_+$ with $\theta(x,x) = 0$, and $\rho \in \mathbb{R}$ such that*

$$\varphi(x) - \varphi(x_0) \geq \eta(x, x_0)^\top \nabla\varphi(x_0) + \rho\theta(x, x_0).$$

*If $-\varphi$ is (η, ρ, θ)-invex at $x_0 \in C$, then φ is called (η, ρ, θ)-**incave** at $x_0 \in C$. If the inequality holds strictly, then φ is called to be **strictly** (η, ρ, θ)-**invex**.*

Definition 4 *A differentiable function $\varphi : C \subseteq \mathbb{R}^n \to \mathbb{R}$ is (η, ρ, θ)-**pseudo-invex** at $x_0 \in C$ if there exist functions $\eta : C \times C \to \mathbb{R}^n$, $\theta : C \times C \to \mathbb{R}_+$ with $\theta(x,x) = 0$, and $\rho \in \mathbb{R}$ such that the following holds:*

$$\eta(x, x_0)^\top \nabla\varphi(x_0) \geq -\rho\theta(x, x_0) \implies \varphi(x) \geq \varphi(x_0), \quad \forall x \in C,$$

or equivalently,

$$\varphi(x) < \varphi(x_0) \implies \eta(x, x_0)^\top \nabla\varphi(x_0) < -\rho\theta(x, x_0), \quad \forall x \in C.$$

If $-\varphi$ is (η, ρ, θ)-pseudo-invex at $x_0 \in C$, then φ is called (η, ρ, θ)-pseudo-incave at $x_0 \in C$.

Definition 5 *A differentiable function $\varphi : C \subseteq \mathbb{R}^n \to \mathbb{R}$ is **strictly** (η, ρ, θ)-**pseudo-invex** at $x_0 \in C$ if there exist functions $\eta : C \times C \to \mathbb{R}^n$, $\theta : C \times C \to \mathbb{R}_+$ with $\theta(x,x) = 0$, and $\rho \in \mathbb{R}$ such that the following hold:*

$$\eta(x, x_0)^\top \nabla\varphi(x_0) \geq -\rho\theta(x, x_0) \implies \varphi(x) > \varphi(x_0), \quad \forall x \in C,\ x \neq x_0.$$

Definition 6 *A differentiable function $\varphi : C \subseteq \mathbb{R}^n \to \mathbb{R}$ is (η, ρ, θ)-**quasi-invex** at $x_0 \in C$ if there exist functions $\eta : C \times C \to \mathbb{R}^n$, $\theta : C \times C \to \mathbb{R}_+$ with $\theta(x,x) = 0$, and $\rho \in \mathbb{R}$ such that the following hold:*

$$\varphi(x) \leq \varphi(x_0) \implies \eta(x, x_0)^\top \nabla\varphi(x_0) \leq -\rho\theta(x, x_0), \quad \forall x \in C.$$

If $-\varphi$ is (η, ρ, θ)-quasi-invex at $x_0 \in C$, then φ is called (η, ρ, θ)-quasi-incave at $x_0 \in C$.

If in the above definitions the corresponding property of a differentiable function $\varphi : C \subseteq \mathbb{R}^n \to \mathbb{R}$ is satisfied for any $x_0 \in C$, then φ has that (η, ρ, θ)-property on C.

3 Necessary and Sufficient Optimality Conditions

For any $x \in \mathcal{P}$, let us denote the following index sets:

$$\begin{aligned}\mathcal{B}(x) &= \left\{r \in \{1,2,\ldots,q\} \mid x^\top B_r x > 0\right\},\\ \overline{\mathcal{B}}(x) &= \{1,2,\ldots,q\} \setminus \mathcal{B}(x) = \left\{r \mid x^\top B_r x = 0\right\},\end{aligned}$$

Using Lemma 2, we may prove the following necessary optimality conditions for problem (P).

Theorem 7 (necessary conditions) *If x_0 is an optimal solution of the problem (P) for which $\overline{\mathcal{B}}(x_0) = \emptyset$ and $\nabla g_j(x_0)$, $j \in J(x_0)$, are linearly independent, then there exist an integer number $s \in \{1,\ldots,n+1\}$ and the vectors $\bar{y}_i \in Y(x_0)$, $i = \overline{1,s}$, $t \in \mathbb{R}^s_+$, $w_r \in \mathbb{R}^n$, $r = \overline{1,q}$, and $\mu \in \mathbb{R}^p_+$ such that*

$$\sum_{i=1}^{s} t_i \left(\nabla f(x_0,\bar{y}_i) + \sum_{r=1}^{q} B_r w_r\right) + \sum_{j=1}^{p} \mu_j \nabla g_j(x_0) = 0, \tag{4}$$

$$\sum_{j=1}^{p} \mu_j g_j(x_0) = 0, \tag{5}$$

$$\sum_{i=1}^{s} t_i > 0, \tag{6}$$

$$w_r^\top B_r w_r \leq 1, \quad x_0^\top B_r w_r = \sqrt{x_0^\top B_r x_0}, \quad r = \overline{1,q}. \tag{7}$$

Proof. Since for all $r = \overline{1,q}$ B_r are positive definite and f is a differentiable function, it follows that the function

$$f(x,y) + \sum_{r=1}^{q} \sqrt{x^\top B_r x}$$

is differentiable with respect to x for any given $y \in \mathbb{R}^m$. In Lemma 2, the differentiable function ψ in (P1) is replaced by the objective function of (P), and using the Kuhn-Tucker type formula, it follows that there exist a positive integer $s \in \{1,\ldots,n+1\}$, and vectors $t \in \mathbb{R}^s_+$, $\mu \in \mathbb{R}^p_+$, $\bar{y}_i \in Y(x_0)$, $i = \overline{1,s}$, such that

$$\sum_{i=1}^{s} t_i \left(\nabla f(x_0,\bar{y}_i) + \sum_{r=1}^{q} \frac{B_r x_0}{\sqrt{x_0^\top B_r x_0}}\right) + \sum_{j=1}^{p} \mu_j \nabla g_j(x_0) = 0, \tag{8}$$

$$\sum_{j=1}^{p} \mu_j g_j(x_0) = 0, \tag{9}$$

$$\sum_{i=1}^{s} t_i > 0. \tag{10}$$

If we denote

$$w_r = \frac{x_0}{\sqrt{x_0^\top B_r x_0}},$$

the equation (8) becomes

$$\sum_{i=1}^{s} t_i \left(\nabla f(x_0, \bar{y}_i) + \sum_{r=1}^{q} B_r w_r \right) + \sum_{j=1}^{p} \mu_j \nabla g_j(x_0) = 0.$$

This proves (4) - (6). Furthermore, it verifies easily that for any $r = \overline{1,q}$, we have

$$w_r^\top B_r w_r = 1, \text{ and } x_0^\top B_r w_r = \sqrt{x_0^\top B_r x_0}.$$

So relation (7) also holds, and the theorem is proved.

We notice that, in the above theorem, all matrices B_r are supposed to be positive definite. If $\overline{\mathcal{B}}(x_0)$ is not empty, then the objective function of problem (P) is not differentiable. In this case, the necessary optimality conditions still hold under some additional assumptions. For this purpose, for an integer number $s \in \{1, \ldots, n+1\}$, for which $\bar{y}_i \in Y(x_0)$, $i = \overline{1,s}$, the real numbers $t_i \geq 0$, with $\sum\limits_{i=1}^{s} t_i > 0$, and $x_0 \in \mathcal{P}$, we define the following vector:

$$\alpha = \sum_{i=1}^{s} t_i \left(\nabla f(x_0, \bar{y}_i) + \sum_{r \in \mathcal{B}(x_0)} \frac{B_r x_0}{\sqrt{x_0^\top B_r x_0}} \right)$$

Now we define a set Z as follows:

$$Z_{\bar{y}}(x_0) = \left\{ z \in \mathbb{R}^n \;\middle|\; \begin{array}{l} z^\top \nabla g_j(x_0) \leq 0, \ j \in J(x_0), \\ z^\top \alpha + \sum\limits_{i=1}^{s} t_i \sum\limits_{r \in \overline{\mathcal{B}}(x_0)} \sqrt{z^\top B_r z} < 0. \end{array} \right\}$$

Using Lemma 1, we establish the following result:

Theorem 8 (necessary conditions) *Let x_0 be an optimal solution of the problem (P). We consider that for an integer $s \in \{1, \ldots, n+1\}$, $\bar{y}_i \in Y(x_0)$,*

$i=\overline{1,s}$, and there are real numbers $t_i \geq 0$, with $\sum_{i=1}^{s} t_i > 0$, for which the set $Z_{\bar{y}}(x_0) = \emptyset$. Then there exist vectors $w_r \in \mathbb{R}^n$, $r = \overline{1,q}$, and $\mu \in \mathbb{R}_+^p$ satisfying relations (4) - (7).

Proof. Relation (6) follows directly from the assumptions.
Since $Z_{\bar{y}}(x_0) = \emptyset$, for any $z \in \mathbb{R}^n$ with $-z^\top \nabla g_j(x_0) \geq 0,\ j \in J(x_0)$, we have

$$z^\top \alpha + \sum_{i=1}^{s} t_i \sum_{r \in \overline{\mathcal{B}}(x_0)} \sqrt{z^\top B_r z} \geq 0.$$

Let us denote

$$\lambda = \sum_{i=1}^{s} t_i.$$

Now we apply Lemma 1, by considering

- the rows of matrix A are the vectors $[-\nabla g_j(x_0)]$, $j \in J(x_0)$;
- $c = \alpha$;
- $M_r = \lambda^2 B_r$ for $r \in \overline{\mathcal{B}}(x_0)$.

It follows that there exist the scalars $\mu_j \geq 0$, $j \in J(x_0)$, and the vectors $v_r \in \mathbb{R}^n$, $r \in \overline{\mathcal{B}}(x_0)$, such that

$$-\sum_{j \in J(x_0)} \mu_j \nabla g_j(x_0) = c + \sum_{r \in \overline{\mathcal{B}}(x_0)} M_r v_r \tag{11}$$

and

$$v_r^\top M_r v_r \leq 1, \qquad r \in \overline{\mathcal{B}}(x_0). \tag{12}$$

Since $g_j(x_0) = 0$ for $j \in J(x_0)$, we have: $\mu_j g_j(x_0) = 0$ for $j \in J(x_0)$. If $j \notin J(x_0)$, we put $\mu_j = 0$. It follows:

$$\sum_{j=1}^{p} \mu_j g_j(x_0) = 0$$

which shows that relation (5) holds.

Now we define

$$w_r = \begin{cases} \dfrac{x_0}{\sqrt{x_0^\top B_r x_0}}, & \text{if } r \in \mathcal{B}(x_0) \\ \lambda v_r, & \text{if } r \in \overline{\mathcal{B}}(x_0). \end{cases}$$

With this notations, equality (11) yields relation (4).

From (12) we get $w_r^\top B_r w_r \le 1$ for any $r = \overline{1,q}$. Further, if $r \in \overline{\mathcal{B}}(x_0)$, we have $x_0^\top B_r x_0 = 0$, which implies $B_r x_0 = 0$, and then $\sqrt{x_0^\top B_r x_0} = 0 =$
$= x_0^\top B_r w_r$.
If $r \in \mathcal{B}(x_0)$, we obviously have $x_0^\top B_r w_r = \sqrt{x_0^\top B_r x_0}$, so relation (7) holds. Therefore the theorem is proved.

For convenience, if $x_0 \in \mathcal{P}$ so that the vectors $\nabla g_j(x_0)$, $j \in J(x_0)$, are linear independent and $Z_{\bar{y}}(x_0) = \emptyset$, then such $x_0 \in \mathcal{P}$ is said to satisfy a "*constraint qualification*".

The results of Theorems 7 and 8 are the necessary conditions for the optimal solution of the problem (P). Actually, the conditions (4) - (7) are also the sufficient optimality conditions for (P) if some generalized invexity conditions are fulfilled.

Theorem 9 (sufficient conditions) *Let $x_0 \in \mathcal{P}$ be a feasible solution of (P) such that there exist a positive integer $s \in \{1, \ldots, n+1\}$ and the vectors $\bar{y}_i \in Y(x_0)$, $i = \overline{1,s}$, $t \in \mathbb{R}_+^s$, $w_r \in \mathbb{R}^n$, $r = \overline{1,q}$, and $\mu \in \mathbb{R}_+^p$ for which the relations (4) - (7) are satisfied. We define*

$$\overline{\Phi}(\,\cdot\,) = \sum_{i=1}^{s} t_i \left(f(\,\cdot\,, \bar{y}_i) + \sum_{r=1}^{q} (\,\cdot\,)^\top B_r w_r \right).$$

If any one of the following four conditions holds:

(a) $f(\,\cdot\,, \bar{y}_i) + \sum_{r=1}^{q} (\,\cdot\,)^\top B_r w_r$ is (η, ρ_i, θ)-invex, for $i = \overline{1,s}$, $\sum_{j=1}^{p} \mu_j g_j(\,\cdot\,)$ is (η, ρ_0, θ)-invex, and $\rho_0 + \sum_{i=1}^{s} t_i \rho_i \ge 0$;

(b) $\overline{\Phi}(\,\cdot\,)$ is (η, ρ, θ)-invex and $\sum_{j=1}^{p} \mu_j g_j(\,\cdot\,)$ is (η, ρ_0, θ)-invex, and $\rho + \rho_0 \ge 0$;

(c) $\overline{\Phi}(\,\cdot\,)$ is (η, ρ, θ)-pseudo-invex and $\sum_{j=1}^{p} \mu_j g_j(\,\cdot\,)$ is (η, ρ_0, θ)-quasi-invex, and $\rho + \rho_0 \ge 0$;

(d) $\overline{\Phi}(\,\cdot\,)$ is (η, ρ, θ)-quasi-invex and $\sum_{j=1}^{p} \mu_j g_j(\,\cdot\,)$ is strictly (η, ρ_0, θ)-pseudo-invex, and $\rho + \rho_0 \ge 0$;

then x_0 is an optimal solution of (P).

Proof. On contrary, let us suppose that x_0 is not an optimal solution of (P). Then there exists an $x_1 \in \mathcal{P}$ such that

$$\sup_{y \in Y}\left(f(x_1,y)+\sum_{r=1}^{q}\sqrt{x_1^\top B_r x_1}\right) < \sup_{y \in Y}\left(f(x_0,y)+\sum_{r=1}^{q}\sqrt{x_0^\top B_r x_0}\right)$$

We note that, for $\bar{y}_i \in Y(x_0)$, $i=\overline{1,s}$, we have

$$\sup_{y \in Y}\left(f(x_0,y)+\sum_{r=1}^{q}\sqrt{x_0^\top B_r x_0}\right) = f(x_0,\bar{y}_i)+\sum_{r=1}^{q}\sqrt{x_0^\top B_r x_0},$$

and

$$f(x_1,\bar{y}_i)+\sum_{r=1}^{q}\sqrt{x_1^\top B_r x_1} \le \sup_{y \in Y}\left(f(x_1,y)+\sum_{r=1}^{q}\sqrt{x_1^\top B_r x_1}\right).$$

Thus, we have

$$f(x_1,\bar{y}_i)+\sum_{r=1}^{q}\sqrt{x_1^\top B_r x_1} < f(x_0,\bar{y}_i)+\sum_{r=1}^{q}\sqrt{x_0^\top B_r x_0}, \quad \text{for } i=\overline{1,s}. \tag{13}$$

Using the relations (3), (7), (13), and (6), we obtain

$$\begin{aligned}\overline{\Phi}(x_1) &= \sum_{i=1}^{s} t_i\left(f(x_1,\bar{y}_i)+\sum_{r=1}^{q}x_1^\top B_r w_r\right) \le \sum_{i=1}^{s} t_i\left(f(x_1,\bar{y}_i)+\sum_{r=1}^{q}\sqrt{x_1^\top B_r x_1}\right)\\ &< \sum_{i=1}^{s} t_i\left(f(x_0,\bar{y}_i)+\sum_{r=1}^{q}\sqrt{x_0^\top B_r x_0}\right) = \sum_{i=1}^{s} t_i\left(f(x_0,\bar{y}_i)+\sum_{r=1}^{q}x_0^\top B_r w_r\right)\\ &= \overline{\Phi}(x_0).\end{aligned}$$

It follows that

$$\overline{\Phi}_r(x_1) < \overline{\Phi}_r(x_0). \tag{14}$$

1. If the hypothesis *(a)* holds, then for $i=\overline{1,s}$, we have

$$\begin{aligned}&f(x_1,\bar{y}_i)+\sum_{r=1}^{q}x_1^\top B_r w_r - f(x_0,\bar{y}_i)-\sum_{r=1}^{q}x_0^\top B_r w_r \ge\\ &\ge \eta(x_1,x_0)^\top\left(\nabla f(x_0,\bar{y}_i)+\sum_{r=1}^{q}B_r w_r\right)+\rho_i\theta(x_1,x_0),\end{aligned} \tag{15}$$

Now, multiplying (15) by t_i, and then suming up these inequalities, we obtain

$$\begin{aligned}&\overline{\Phi}(x_1)-\overline{\Phi}(x_0) \ge\\ &\ge \eta(x_1,x_0)^\top \sum_{i=1}^{s} t_i\left(\nabla f(x_0,\bar{y}_i)+\sum_{r=1}^{q}B_r w_r\right)+\\ &+\sum_{i=1}^{s} t_i\rho_i\theta(x_1,x_0).\end{aligned}$$

Further, by (4) and (η,ρ_0,θ)-invexity of $\sum_{j=1}^{p}\mu_j g_j(\,\cdot\,)$, we get

$$\begin{aligned}\overline{\Phi}(x_1)-\overline{\Phi}(x_0) &\geq -\eta(x_1,x_0)^{\top}\sum_{j=1}^{p}\mu_j\nabla g_j(x_0)+\sum_{i=1}^{s}t_i\rho_i\theta(x_1,x_0)\\ &\geq -\sum_{j=1}^{p}\mu_j g_j(x_1)+\sum_{j=1}^{p}\mu_j g_j(x_0)+\left(\rho_0+\sum_{i=1}^{s}t_i\rho_i\right)\theta(x_1,x_0).\end{aligned}$$

Since $x_1\in\mathcal{P}$, we have $g_i(x_1)\leq 0$, $i=\overline{1,s}$, and using (5) it follows

$$\overline{\Phi}(x_1)-\overline{\Phi}(x_0)\geq\left(\rho_0+\sum_{i=1}^{s}t_i\rho_i\right)\theta(x_1,x_0)\geq 0,$$

which contradicts the inequality (14).

2. If the hypothesis *(b)* holds, we have

$$\begin{aligned}&\overline{\Phi}(x_1)-\overline{\Phi}(x_0)\geq\\ &\quad\geq\eta(x_1,x_0)^{\top}\sum_{i=1}^{s}t_i\left(\nabla f(x_0,\bar{y}_i)+\sum_{j=1}^{p}B_r w_r\right)+\rho\theta(x_1,x_0).\end{aligned}$$

Using relation (4) and the (η,ρ_0,θ)-invexity of $\sum_{j=1}^{p}\mu_j g_j(\,\cdot\,)$, we obtain

$$\begin{aligned}\overline{\Phi}(x_1)-\overline{\Phi}(x_0) &\geq -\eta(x_1,x_0)^{\top}\sum_{j=1}^{p}\mu_j\nabla g_j(x_0)+\rho\theta(x_1,x_0)\geq\\ &\geq -\sum_{j=1}^{p}\mu_j g_j(x_1)+\sum_{j=1}^{p}\mu_j g_j(x_0)+(\rho+\rho_0)\theta(x_1,x_0)\geq\\ &\geq (\rho+\rho_0)\theta(x_1,x_0)\geq 0,\end{aligned}$$

which contradicts the inequality (14).

3. If the hypothesis *(c)* holds, using the (η,ρ,θ)-pseudo-invexity of $\overline{\Phi}$, it follows from (14) that

$$\overline{\Phi}(x_1)<\overline{\Phi}(x_0)\implies\eta(x_1,x_0)^{\top}\nabla\overline{\Phi}(x_0)<-\rho\theta(x_1,x_0). \tag{16}$$

Using again relation (4), from (16) and $\rho+\rho_0\geq 0$, we get

$$\eta(x_1,x_0)^{\top}\sum_{j=1}^{p}\mu_j\nabla g_j(x_0)>\rho\theta(x_1,x_0)\geq-\rho_0\theta(x_1,x_0). \tag{17}$$

Since $x_1 \in \mathcal{P}$ imply $g_i(x_1) \leq 0$, $i = \overline{1,s}$, and $\mu \in \mathbb{R}_+^p$, using (5) we have

$$\sum_{j=1}^{p} \mu_j g_j(x_1) \leq 0 = \sum_{j=1}^{p} \mu_j g_j(x_0). \tag{18}$$

Using the (η, ρ_0, θ)-quasi-invexity of $\sum\limits_{j=1}^{p} \mu_j g_j(\cdot)$, we get from the last relation

$$\eta(x_1, x_0)^{\top} \sum_{j=1}^{p} \mu_j \nabla g_j(x_0) \leq \rho_0 \theta(x_1, x_0)$$

which contradicts the inequality (17).

4. If the hypothesis *(d)* holds, the (η, ρ, θ)-quasi-invexity of $\overline{\Phi}$ imply

$$\overline{\Phi}(x_1) \leq \overline{\Phi}(x_0) \implies \eta(x_1, x_0)^{\top} \nabla\overline{\Phi}(x_0) \leq -\rho\theta(x_1, x_0).$$

From here, together with (4) and $\rho + \rho_0 \geq 0$, we have

$$\eta(x_1, x_0)^{\top} \sum_{j=1}^{p} \mu_j \nabla g_j(x_0) \geq \rho\theta(x_1, x_0) \geq -\rho_0\theta(x_1, x_0). \tag{19}$$

Since (18) is true, the strictly (η, ρ, θ)-pseudo-invexity of $\sum\limits_{j=1}^{p} \mu_j g_j(\cdot)$ imply

$$\eta(x_1, x_0)^{\top} \sum_{j=1}^{p} \mu_j \nabla g_j(x_0) < \rho_0 \theta(x_1, x_0)$$

which contradicts the inequality (19).

Therefore the proof of the theorem is complete.

4 Duality

Let us denote

$$K(x) = \left\{ (s,t,y) \;\middle|\; \begin{array}{l} s \in \{1, \ldots, n+1\}, \\ t \in \mathbb{R}_+^s, \text{ and } \sum\limits_{i=1}^{s} t_i = 1, \\ y = (y_1, \cdots, y_s), \text{ with } y_i \in Y(x), \; i = \overline{1,s} \end{array} \right\}.$$

We consider further the set $H(s,t,y)$ consisting of all $(z, \mu, \lambda, w) \in \mathbb{R}^n \times \mathbb{R}_+^p \times \mathbb{R}_+ \times \mathbb{R}^n$ which satisfy the following conditions:

$$\sum_{i=1}^{s} t_i \left(\nabla f(z, y_i) + \sum_{r=1}^{q} B_r w \right) + \sum_{j=1}^{p} \mu_j \nabla g_j(z) \geq 0, \tag{20}$$

$$\sum_{i=1}^{s} t_i \left(f(z, y_i) + \sum_{r=1}^{q} z^\top B_r w \right) \geq \lambda, \tag{21}$$

$$\sum_{j=1}^{p} \mu_j g_j(z) \geq 0, \tag{22}$$

$$(s, t, y) \in K(z) \tag{23}$$

$$w^\top B_r w \leq 1. \tag{24}$$

The optimality conditions, stated in the preceding section for the minimax problem (P), suggest us to define the following dual problem:

$$\max_{(s,t,y)\in K(z)} \sup \{\lambda \mid (z, \mu, \lambda, w) \in H(s, t, y)\} \tag{DP}$$

If, for a triple $(s, t, y) \in K(z)$, the set $H(s, t, y) = \emptyset$, then we define the supremum over $H(s, t, y)$ to be $-\infty$. Further, we denote

$$\Phi(\cdot) = \sum_{i=1}^{s} t_i \left(f(\cdot, y_i) + \sum_{r=1}^{q} (\cdot)^\top B_r w \right)$$

Now, we can state the following weak duality theorem for (P) and (DP).

Theorem 10 (weak duality) *Let $x \in \mathcal{P}$ be a feasible solution of (P) and $(x, \mu, \lambda, w, s, t, y)$ be a feasible solution of (DP). If any of the following four conditions holds:*

(a) $f(\cdot, y_i) + \sum_{r=1}^{q} (\cdot)^\top B_r w$ *is (η, ρ_i, θ)-invex for $i = \overline{1, s}$,* $\sum_{j=1}^{p} \mu_j g_j(\cdot)$ *is (η, ρ_0, θ)-invex, and* $\rho_0 + \sum_{i=1}^{s} t_i \rho_i \geq 0$,

(b) $\Phi(\cdot)$ *is (η, ρ, θ)-invex and* $\sum_{j=1}^{p} \mu_j g_j(\cdot)$ *is (η, ρ_0, θ)-invex, and* $\rho + \rho_0 \geq 0$,

(c) $\Phi(\cdot)$ *is (η, ρ, θ)-pseudo-invex and* $\sum_{j=1}^{p} \mu_j g_j(\cdot)$ *is (η, ρ_0, θ)-quasi-invex, and*
$\rho + \rho_0 \geq 0$,

(d) $\Phi(\cdot)$ *is (η, ρ, θ)-quasi-invex and* $\sum_{j=1}^{p} \mu_j g_j(\cdot)$ *is strictly (η, ρ_0, θ)-pseudo-invex, and* $\rho + \rho_0 \geq 0$,

then

$$\sup_{y\in Y}\left(f(x,y)+\sum_{r=1}^{q}\sqrt{x^{\top}B_rx}\right)\geq\lambda \tag{25}$$

Proof. If we suppose, on contrary, that

$$\sup_{y\in Y}\left(f(x,y)+\sum_{r=1}^{q}\sqrt{x^{\top}B_rx}\right)<\lambda$$

then we have, for all $y\in Y$,

$$f(x,y)+\sum_{r=1}^{q}\sqrt{x^{\top}B_rx}<\lambda.$$

It follows that, for $t_i\geq 0$, $i=\overline{1,s}$, with $\sum\limits_{i=1}^{s}t_i=1$,

$$t_i\left[f(x,y)+\sum_{r=1}^{q}\sqrt{x^{\top}B_rx}-\lambda\right]\leq 0,\quad i=\overline{1,s}, \tag{26}$$

with at least one strict inequality, because $t=(t_1,\cdots,t_s)\neq 0$.

Taking into account the relations (3), (24), (26) and (21), we have

$$\begin{aligned}\Phi(x)-\lambda &= \sum_{i=1}^{s}t_i\left[f(x,y_i)+\sum_{r=1}^{q}x^{\top}B_rw-\lambda\right]\\ &\leq \sum_{i=1}^{s}t_i\left[f(x,y_i)+\sum_{r=1}^{q}\sqrt{x^{\top}B_rx}-\lambda\right]\\ &< 0\leq\sum_{i=1}^{s}t_i\left[f(z,y_i)+\sum_{r=1}^{q}z^{\top}B_rw-\lambda\right]=\Phi(z)-\lambda,\end{aligned}$$

that is

$$\Phi(x)<\Phi(z). \tag{27}$$

1. If hypothesis *(a)* holds, then for $i=\overline{1,s}$, we have

$$\begin{aligned}&f(x,y_i)+\sum_{r=1}^{q}x^{\top}B_rw-f(z,y_i)-\sum_{r=1}^{q}z^{\top}B_rw\geq\\ &\quad\geq\eta(x,z)^{\top}\left(\nabla f(z,y_i)+\sum_{r=1}^{q}B_rw\right)+\rho_i\theta(x,z),\end{aligned} \tag{28}$$

Now, multiplying (28) by t_i, and then sum up these inequalities, we obtain

$$\Phi(x) - \Phi(z) \geq \eta(x,z)^\top \sum_{i=1}^{s} t_i \left(\nabla f(z, y_i) + \sum_{r=1}^{q} B_r w \right) + \sum_{i=1}^{s} t_i \rho_i \theta(x,z).$$

Further, by (20) and (η, ρ_0, θ)-invexity of $\sum_{j=1}^{p} \mu_j g_j(\cdot)$, we get

$$\begin{aligned} \Phi(x) - \Phi(z) &\geq -\eta(x,z)^\top \sum_{j=1}^{p} \mu_j \nabla g_j(z) + \sum_{i=1}^{s} t_i \rho_i \theta(x,z) \\ &\geq -\sum_{j=1}^{p} \mu_j g_j(x) + \sum_{j=1}^{p} \mu_j g_j(z) + \left(\rho_0 + \sum_{i=1}^{s} t_i \rho_i \right) \theta(x,z). \end{aligned}$$

Since $x \in \mathcal{P}$, we have $g_i(x) \leq 0$, $i = \overline{1, s}$, and using (22) it follows

$$\Phi(x) - \Phi(z) \geq \left(\rho_0 + \sum_{i=1}^{s} t_i \rho_i \right) \theta(x,z) \geq 0,$$

which contradicts the inequality (27). Hence, the inequality (25) is true.

2. The case of hypothesis *(b)* follows with the same argument as before by using (a).

3. If hypothesis *(c)* holds, using the (η, ρ, θ)-pseudo-invexity of Φ, we get from (27) that

$$\eta(x,z)^\top \nabla \Phi(z) < -\rho \theta(x,z) \tag{29}$$

Consequently, relations (20), (29) and $\rho + \rho_0 \geq 0$ yield

$$\eta(x,z)^\top \sum_{j=1}^{p} \mu_j \nabla g_j(z) > \rho \theta(x,z) \geq -\rho_0 \theta(x,z). \tag{30}$$

Because $x \in \mathcal{P}$, $\mu \in \mathbb{R}^p_+$, and (22), we have

$$\sum_{j=1}^{p} \mu_j g_j(x) \leq 0 = \sum_{j=1}^{p} \mu_j g_j(z).$$

Using the (η, ρ_0, θ)-quasi-invexity of $\sum_{j=1}^{p} \mu_j g_j(\cdot)$, we get from the last relation

$$\eta(x,z)^\top \sum_{j=1}^{p} \mu_j \nabla g_j(z) \leq \rho_0 \theta(x,z),$$

which contradicts the inequality (30).

4. The result under the hypothesis (d) follows similarly like before in step 3.

Therefore the proof of the theorem is complete.

Theorem 11 (direct duality) *Let $\bar{x}$ be an optimal solution of the problem (P). Assume that $\bar{x}$ satisfies a constraint qualification for (P). Then there exist $(\bar{s}, \bar{t}, \bar{y}) \in K(\bar{x})$ and $\left(\bar{x}, \bar{\mu}, \bar{\lambda}, \bar{w}\right) \in H(\bar{s}, \bar{t}, \bar{y})$ such that $(\bar{x}, \bar{\mu}, \bar{\lambda}, \bar{w}, \bar{s}, \bar{t}, \bar{y})$ is a feasible solution of (DP). If the hypotheses of Theorem 10 are also satisfied, then $\left(\bar{x}, \bar{\mu}, \bar{\lambda}, \bar{w}, \bar{s}, \bar{t}, \bar{y}\right)$ is an optimal solution for (DP), and both problems (P) and (DP) have the same optimal value.*

Proof. By Theorems 7 and 8, there exist $(\bar{s}, \bar{t}, \bar{y}) \in K(\bar{x})$ and $(\bar{x}, \bar{\mu}, \bar{\lambda}, \bar{w}) \in H(\bar{s}, \bar{t}, \bar{y})$ such that $\left(\bar{x}, \bar{\mu}, \bar{\lambda}, \bar{w}, \bar{s}, \bar{t}, \bar{y}\right)$ is a feasible solution of (DP), and

$$\bar{\lambda} = f(\bar{x}, \bar{y}) + \sum_{r=1}^{q} \sqrt{(\bar{x})^{\top} B_r \bar{x}}.$$

The optimality of this feasible solution for (DP) follows from Theorem 10.

Theorem 12 (strict converse duality) *Let $\hat{x}$ and $\left(\bar{z}, \bar{\mu}, \bar{\lambda}, \bar{w}, \bar{s}, \bar{t}, \bar{y}\right)$ be the optimal solutions of (P) and (DP), respectively, and that the hypotheses of Theorem 11 are fulfilled. If any one of the following three conditions holds:*

(a) one of $f(\,\cdot\,, \bar{y}_i) + \sum\limits_{r=1}^{q} (\,\cdot\,)^{\top} B_r \bar{w}$ is strictly (η, ρ_i, θ)-invex for $i = \overline{1, s}$, or $\sum\limits_{j=1}^{p} \bar{\mu}_j g_j(\,\cdot\,)$ is strictly (η, ρ_0, θ)-invex, and $\rho_0 + \sum\limits_{i=1}^{s} \bar{t}_i \rho_i \geq 0$;

(b) either $\sum\limits_{i=1}^{s} \bar{t}_i \left(f(\,\cdot\,, \bar{y}_i) + \sum\limits_{r=1}^{q} (\,\cdot\,)^{\top} B_r \bar{w} \right)$ is strictly (η, ρ, θ)-invex or $\sum\limits_{j=1}^{p} \bar{\mu}_j g_j(\,\cdot\,)$ is strictly (η, ρ_0, θ)-invex, and $\rho + \rho_0 \geq 0$;

(c) the function $\sum\limits_{i=1}^{s} \bar{t}_i \left(f(\,\cdot\,, \bar{y}_i) + \sum\limits_{r=1}^{q} (\,\cdot\,)^{\top} B_r \bar{w} \right)$ is strictly (η, ρ, θ)-pseudo-invex and $\sum\limits_{j=1}^{p} \bar{\mu}_j g_j(\,\cdot\,)$ is (η, ρ_0, θ)-quasi-invex, and $\rho + \rho_0 \geq 0$;

then $\hat{x} = \bar{z}$, that is, $\bar{z}$ is an optimal solution for problem (P) and

$$\sup_{y \in Y} \left(f(\bar{z}, y) + \sum_{r=1}^{q} \sqrt{\bar{z}^{\top} B_r \bar{z}} \right) = \bar{\lambda}.$$

Proof. Suppose on the contrary that $\hat{x} \neq \bar{z}$. From Theorem 11 we know that there exist $\left(\hat{s}, \hat{t}, \hat{y}\right) \in K\left(\hat{x}\right)$ and $\left(\hat{x}, \hat{\mu}, \hat{\lambda}, \hat{w}\right) \in H\left(\hat{s}, \hat{t}, \hat{y}\right)$ such that $\left(\hat{x}, \hat{\mu}, \hat{\lambda}, \hat{w}, \hat{s}, \hat{t}, \hat{y}\right)$ is a feasible solution for (DP) with the optimal value

$$\hat{\lambda} = \sup_{y \in Y} \left(f\left(\hat{x}, y\right) + \sum_{r=1}^{q} \sqrt{\hat{x}^\top B_r \hat{x}} \right).$$

Now, we proceed similarly as in the proof of Theorem 10, replacing x by $\hat{x}$ and $(z, \mu, \lambda, w, s, t, y)$ by $\left(\bar{z}, \bar{\mu}, \bar{\lambda}, \bar{w}, \bar{s}, \bar{t}, \bar{y}\right)$, so that we arrive at the strict inequality

$$\sup_{y \in Y} \left(f\left(\hat{x}, y\right) + \sum_{r=1}^{q} \sqrt{\hat{x}^\top B_r \hat{x}} \right) > \bar{\lambda}.$$

But this contradicts the fact

$$\sup_{y \in Y} \left(f\left(\hat{x}, y\right) + \sum_{r=1}^{q} \sqrt{\hat{x}^\top B_r \hat{x}} \right) = \hat{\lambda} = \bar{\lambda},$$

and we conclude that $\hat{x} = \bar{z}$. Hence, the proof of the theorem is complete.

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