An. Şt. Univ. Ovidius Constanţa Vol. 16(1), 2008, 19–30

# Compactness and Radon-Nikodym properties on the Banach space of convergent series

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**Abstract**

We characterize the bounded linear operators on the Banach space $\gamma$ of convergent complex sequences. The class of infinite matrices that determine such operators is determined, as well as those that induce conservative, regular or compact operators. It is seeing that $\gamma$ does not have the Radon-Nikodym property and hence it is deduced its non reflexivity and its non uniform convexity.

## 1 Preliminaries

In this article we investigate the structure of bounded operators on the sequence space $\gamma$ of convergent series, as well as some of their geometric properties. In 1949 R. G. Cooke characterized the class of bounded operators between $\gamma$ and the Banach space $\mathrm{c}$ of the convergent sequences (cf. [5]). For a further study and a more complete list of references and reader can see [9]. The following facts are well-known:

**Theorem 1** *Let* $\mathrm{c}$ *be the space of convergent complex sequences endowed with the supremum norm, i.e.* $\|z\|_{\mathrm{c}} = \sup_{n\geq 1} |z_n|$ *for* $z \in \mathrm{c}$.

*(i) With the usual coordinate operations* $\mathrm{c}$ *becomes a Banach space, as it is a closed subspace of* $\mathrm{l}^{\infty}$.

*(ii) If* $\mathrm{e} = (1, 1, ...)$ *then* $\mathrm{c} = \mathrm{c}_0 \oplus \mathbb{C}\cdot \mathrm{e}$, *where* $\mathrm{c}_0$ *is the Banach subspace of* $\mathrm{c}$ *of sequences that converge to zero.*

Key Words: Conservative and regular operators; Radon-Nikodym type properties; Hausdorff measure of non compactness; Uniform convexity of Banach spaces.
Mathematics Subject Classification: 40C05; 40E05
Received: 8 November, 2007
Accepted: January, 2008

*(iii)* $c_0^* \approx l^1$, *where* $\approx$ *denotes an isometric isomorphism of Banach spaces and* $l^1$ *is the usual Banach space of absolutely convergent series.*

*(iv) Every* $\varphi \in c^*$ *can be written in a unique way as*

$$\varphi(z) = \widetilde{a}_0 \cdot \lambda(z) + \sum_{n=1}^{\infty} a_n \cdot (z_n - \lambda(z)) \ \text{ if } z \in c, \tag{1}$$

*where* $\lambda \in c^*$ *is defined as* $\lambda(z) = \lim_{n\to\infty} z_n$. *Further,* $a_n = \varphi(e_n)$ *if* $n \geq 1$, $\{a_n\}_{n=1}^{\infty} \in l^1$ *and* $\widetilde{a}_0 = \varphi(e)$. *By (1) we can write*

$$\varphi(z) = \left(\widetilde{a}_0 - \sum_{n=1}^{\infty} a_n\right) \lambda(z) + \sum_{n=1}^{\infty} a_n \cdot z_n.$$

*(v) Moreover, if* $a_0 \triangleq \widetilde{a}_0 - \sum_{n=1}^{\infty} a_n$ *then* $\|\varphi\| = \sum_{n=0}^{\infty} |a_n|$.

In order to be more clear and self-contained we prove the following:

**Corollary 2** *(cf. [12]) A linear operator* $A : c \to c$ *is bounded if and only if there is a unique bi-index sequence* $\{a_{n,m}\}_{n,m=0}^{\infty}$ *so that*

$$A(z) = \left\{ a_{n,0} \cdot \lambda(z) + \sum_{m=1}^{\infty} a_{n,m} \cdot z_m \right\}_{n=1}^{\infty} \ \text{ if } z \in c \tag{2}$$

*and*

$$\|A\| = \sup_{n\in\mathbb{N}} \sum_{m=0}^{\infty} |a_{n,m}|. \tag{3}$$

*In particular, the following limits exist:*

$$a_{0,0} \triangleq \lim_{n\to\infty} \sum_{m=0}^{\infty} a_{n,m}, \tag{4}$$

$$a_{0,m} \triangleq \lim_{n\to\infty} a_{n,m}, \ m \in \mathbb{N} \tag{5}$$

*and* $\{a_{0,m}\}_{m=1}^{\infty} \in l^1$.

**Proof.** If $A \in \mathcal{B}(c)$, $n \in \mathbb{N}$ and $\chi_n \in c^*$ is the projection onto the $n$-th coordinate, by *(iv)* there is a unique sequence $\{a_{n,m}\}_{m=0}^{\infty} \in l^1$ so that

$$(\chi_n \circ A)(z) = a_{n,0} \cdot \lambda(z) + \sum_{m=1}^{\infty} a_{n,m} \cdot z_m \text{ if } z \in c. \tag{6}$$

Indeed, $a_{n,0} = (\chi_n \circ A)(\mathrm{e}) - \sum_{m=1}^{\infty} a_{n,m}$ and $\|\chi_n \circ A\| = \sum_{m=0}^{\infty} |a_{n,m}|$. Since $\{(\chi_n \circ A)(z)\}_{n=1}^{\infty} \in \mathrm{c}$ if $z \in \mathrm{c}$ an application of the uniform boundedness principle gives

$$\sigma \triangleq \sup_{n \in \mathbb{N}} \sum_{m=0}^{\infty} |a_{n,m}| < \infty. \tag{7}$$

By (6) we see that $\|A\| \leq \sigma$ and we can assume that $\sigma > 0$. If $0 < \varepsilon < \sigma$ let $n \in \mathbb{N}$ so that $\sum_{m=0}^{\infty} |a_{n,m}| > \sigma - \varepsilon/2$. Then choose $m_0 \in \mathbb{N}$ so that

$$\sum_{m=0}^{m_0} |a_{n,m}| > \sigma - \varepsilon/2 \quad \text{and} \quad \sum_{m>m_0} |a_{n,m}| < \varepsilon/2.$$

As in Theorem 1*(v)*, let

$$z = \sum_{m=1}^{m_0} \overline{u(a_{n,m})} \cdot e_m + \overline{u(a_{n,0})} \cdot \left(\mathrm{e} - \sum_{m=1}^{m_0} e_m\right).$$

Then $\|z\|_{\mathrm{c}} = 1$ and

$$\begin{aligned}
\|A\| \geq |(\chi_n \circ A)(z)| &= \\
&= \left| \sum_{m=0}^{m_0} |a_{n,m}| + \overline{u(a_{n,0})} \sum_{m>m_0} a_{n,m} \right| \geq \\
&\geq \sum_{m=0}^{m_0} |a_{n,m}| - \left| \sum_{m>m_0} a_{n,m} \right| \geq \\
&\geq \sigma - \varepsilon/2 - \sum_{m>m_0} |a_{n,m}| \geq \\
&\geq \sigma - \varepsilon.
\end{aligned}$$

Since $\varepsilon$ is arbitrary, (3) follows. As $(\chi_n \circ A)(e) = \sum_{m=0}^{\infty} a_{n,m}$, the limit in (4) exists. The existence of the limits in (5) is immediate as $a_{0,m} = \lambda(A(e_m))$ for $m \in \mathbb{N}$ and the conditions are necessary.

On the other hand, let $\{a_{n,m}\}_{n,m=0}^{\infty}$ be a given sequence of complex scalars so that (7), (4) and (5) hold. We shall show that (2) defines an operator $A \in \mathcal{B}(\mathrm{c})$. For, by (7) giving $z \in \mathrm{c}$ and $n \in \mathbb{N}$ the series $\sum_{m=1}^{\infty} a_{n,m} \cdot z_m$ converges. Further,

$$a_{n,0} \cdot \lambda(z) + \sum_{m=1}^{\infty} a_{n,m} \cdot z_m = \lambda(z) \cdot \sum_{m=0}^{\infty} a_{n,m} + \sum_{m=1}^{\infty} a_{n,m} \cdot (z_m - \lambda(z)). \tag{8}$$

By (5) and (7) we see that if $M \in \mathbb{N}$. Then

$$\sum_{m=1}^{M} |a_{0,m}| = \lim_{n\to\infty} \sum_{m=1}^{M} |a_{n,m}| \leq \sigma,$$

i.e. $\{a_{0,m}\}_{m=1}^{\infty} \in \mathrm{l}^1$. Therefore, by (8), (4) and (7), we conclude that

$$\lim_{n\to\infty} \left\{ a_{n,0} \cdot \lambda(z) + \sum_{m=1}^{\infty} a_{n,m} \cdot z_m \right\} = a_{0,0} \cdot \lambda(z) + \sum_{m=1}^{\infty} a_{0,m} \cdot (z_m - \lambda(z)), \quad (9)$$

i.e. $A$ is well defined. That $A$ is bounded is immediate by (7).

**Corollary 3** *Let $A \in \mathcal{B}(\mathrm{c})$. The unique bi-index sequence $\{a_{n,m}\}_{n,m=0}^{\infty}$ that determinates $A$ is defined as follows:*

$$a_{n,m} = \begin{cases} (\chi_n \circ A)(e_m) & if \quad n, m \in \mathbb{N}, \\ (\chi_n \circ A)(\mathrm{e}) - \sum_{m=1}^{\infty} (\chi_n \circ A)(e_m) & if \quad n \in \mathbb{N},\ m = 0, \\ \lambda(A(e_m)) & if \quad n = 0, m \in \mathbb{N}, \\ \lambda(A(\mathrm{e})) & if \quad n = m = 0. \end{cases}$$

**Remark 4** *A bi-index sequence $\{a_{n,m}\}_{n,m=0}^{\infty}$ is called* c-***conservative*** *if it satisfies the conditions of Corollary 2. Further, $\{a_{n,m}\}_{n,m=0}^{\infty}$ is said* c-***regular*** *if it is conservative and its induced bounded operator $A$ on* c *preserves limits, i.e. $\lambda(A(z)) = \lambda(z)$ for all $z \in \mathrm{c}$. From (9) it is easily seeing that $\{a_{n,m}\}_{n,m=0}^{\infty}$ is* c*-regular if and only if $a_{0,0} = 1$ and $a_{0,m} = 0$ for all $m \in \mathbb{N}$.*

The study of the properties of $\gamma$ listed in this paper are motivated on recent works about the structure and behauviour of derivations on certain Banach algebras (cf [1], [2]). In particular, intrinsic connections between derivations on non-amenable nuclear Banach algebras whose underlying space has a shrinking basis and the corresponding multiplier Banach sequence space were recently established (cf. [3]). There is a huge literature on the structure of operators on classic Banach sequence spaces, but we believe that a careful look of $\gamma$ will allow a more deeper understand of its Banach algebra of bounded operators. With this aim we shall try to write this article in order that it be self-contained as well as possible.

In Section 2 we introduce the Banach space of complex convergent series and we characterize in Theorem 7 its bounded operators. The so called $\gamma$-regular and $\gamma$-conservative operators are determined. In Section 3 we analyze the Radon-Nikodym property on $\gamma$ and then we deduce that it is not reflexive and not uniformly convex Banach space. Finally, in Section 4 it is characterized the class of compact operators on $\gamma$.

## 2 Concerning to the space $\gamma$ of convergent series

Let $\gamma$ be the set of complex convergent series endowed with its natural vector space structure. If $z \in \gamma$ set

$$\|z\|_{\gamma} = \sup_{m \geq 1} \left| \sum_{n=1}^{m} z_n \right|.$$

We shall write $S_m(z) = \sum_{n=1}^{m} z_n$ and $S(z) = \sum_{n=1}^{\infty} z_n$ for $m \in \mathbb{N}$ and $z \in \gamma$. Clearly $\left(\gamma, \|\circ\|_{\gamma}\right)$ is a complex normed space and $\{S\} \cup \{S_n\}_{n=1}^{\infty} \subseteq \overline{\mathrm{B}}_{\gamma^*}(0,1)$.

**Proposition 5** $\left(\gamma, \|\circ\|_{\gamma}\right)$ *is a complex Banach space.*

**Proof.** Let $\left\{z^k\right\}_{k=1}^{\infty}$ be a Cauchy sequence in $\gamma$, with $z^k = \left(z_n^k\right)_{n=1}^{\infty}$ if $k \in \mathbb{N}$. Since $\left|z_1^k - z_1^{k+h}\right| \leq \left\|z^k - z^{k+h}\right\|_{\gamma}$ if $k, h \in \mathbb{N}$ then $\left(z_1^k\right)_{k=1}^{\infty}$ becomes a Cauchy sequence in $\mathbb{C}$. Thus it has a limit, say $\lim_{k\to\infty} z_1^k \triangleq z_1$. Further, let assume that the limits $\lim_{k\to\infty} z_j^k \triangleq z_j$ exist if $1 \leq j < J$. Since

$$\left| \sum_{n=1}^{J} \left(z_n^k - z_n^{k+h}\right) \right| \geq \left|z_J^k - z_J^{k+h}\right| - \left| \sum_{n=1}^{J-1} \left(z_n^k - z_n^{k+h}\right) \right|$$

we see that $\left|z_J^k - z_J^{k+h}\right| \leq 2\left\|z^k - z^{k+h}\right\|_{\gamma}$ if $k, h \in \mathbb{N}$, i.e. $\left(z_J^k\right)_{k=1}^{\infty}$ is a Cauchy sequence. Hence we can write $\lim_{k\to\infty} z_J^k \triangleq z_J$ and inductively we constructed a sequence $z \triangleq (z_n)_{n=1}^{\infty}$. If $\varepsilon > 0$ let $k(\varepsilon) \in \mathbb{N}$ be so that $\left\|z^k - z^{k+h}\right\|_{\gamma} \leq \varepsilon/4$ if $k \geq k(\varepsilon)$ and $h \in \mathbb{N}$. Whence, if $m \in \mathbb{N}$ then

$$\left| \sum_{n=1}^{m} \left(z_n^k - z_n^{k+h}\right) \right| \leq \left\|z^k - z^{k+h}\right\|_{\gamma} \leq \varepsilon/4$$

and letting $h \to \infty$ we deduce that $\left|\sum_{n=1}^{m} \left(z_n^k - z_n\right)\right| \leq \varepsilon$. Since $z^{k(\varepsilon)} \in \gamma$ there exists $k_0 \in \mathbb{N}$ so that $\left|\sum_{n=k}^{k+h} z_n^{k(\varepsilon)}\right| \leq \varepsilon/2$ if $k \geq k_0$ and $h \in \mathbb{N}$. Finally, if $k \geq k_0$ and $h \in \mathbb{N}$ then

$$\left| \sum_{n=k}^{k+h} z_n \right| = \left| \sum_{n=1}^{k+h} (z_n - z_n^{k(\varepsilon)}) - \sum_{n=1}^{k-1} (z_n - z_n^{k(\varepsilon)}) + \sum_{n=k}^{k+h} z_n^{k(\varepsilon)} \right| \leq \varepsilon$$

and so $z \in \gamma$. By the previous reasoning we get $\left\|z - z^k\right\|_{\gamma} \leq \varepsilon$ if $k \geq k(\varepsilon)$.

**Theorem 6** *A linear form $t$ on $\gamma$ is bounded if and only if there is a unique sequence $a=(a_n)_{n=1}^{\infty}$ so that*

$$t(z)=\lambda(a)\cdot S(z)+\sum_{n=1}^{\infty}(a_n-a_{n+1})\cdot S_n(z) \text{ if } z\in\gamma \tag{10}$$

*and $(a_n-a_{n+1})_{n=1}^{\infty}\in \mathrm{l}^1$. Indeed,*

$$\|t\|=|\lambda(a)|+\sum_{n=1}^{\infty}|a_n-a_{n+1}|. \tag{11}$$

**Proof.** The map $S_0:\gamma\to \mathrm{c}$ so that $S_0(z)=\{S_n(z)\}_{n=1}^{\infty}$ if $z\in\gamma$ is a well defined linear isomorphism of $\gamma$ onto c whose inverse for $w\in \mathrm{c}$ is given by $S_0^{-1}(w)=(w_1,w_2-w_1,w_3-w_2,...)$. Thus, if $t\in\gamma$ and $\widetilde{t}\triangleq t\circ S_0^{-1}$ then $\widetilde{t}\in \mathrm{c}^*$. By Th.1(iv) we know that $\left\{\widetilde{t}(e_n)\right\}_{n=1}^{\infty}\in \mathrm{l}^1$. Hence $\{t(e_n)\}_{n=1}^{\infty}\in \mathrm{c}$ and if $z=S_0^{-1}(w)$ for $w\in \mathrm{c}$ and $z\in\gamma$ then

$$\begin{aligned} t(z)=\widetilde{t}(w)&=\lim_{n\to\infty}t(e_n)\cdot\lambda(w)+\sum_{n=1}^{\infty}t(e_n-e_{n+1})\cdot w_n \\ &=\lim_{n\to\infty}t(e_n)\cdot S(z)+\sum_{n=1}^{\infty}t(e_n-e_{n+1})\cdot S_n(z), \end{aligned}$$

i.e. $a\triangleq\{t(e_n-e_{n+1})\}_{n=1}^{\infty}$. The uniqueness of this sequence follows from Theorem 1. On the other hand, clearly (10) defines a linear form $t\in\gamma^*$ if the sequence $a=(a_n)_{n=1}^{\infty}$ is choosen so that $(a_n-a_{n+1})_{n=1}^{\infty}\in \mathrm{l}^1$. Further, since $S_0$ is a linear isometric isomorphism of $\gamma$ onto c then (11) holds.

**Theorem 7** *There is an 1-1 correspondence between $\mathcal{B}(\gamma)$ and the set of infinite complex matrices $\{a_{m,p}\}_{m,p=1}^{\infty}$ so that*

*(i)* $\sup_{m\in\mathbb{N}}\left\{|\lim_{p\to\infty}a_{m,p}|+\sum_{p=1}^{\infty}|a_{m,p}-a_{m,p+1}|\right\}<\infty.$

*(ii) Letting $a_m\triangleq\{a_{m,n}\}_{n=1}^{\infty}$ then*

$$\sup_{n\in\mathbb{N}}\left\{\sum_{p=1}^{\infty}\left|\sum_{m=1}^{n}(a_{m,p}-a_{m,p+1})\right|+\left|\sum_{m=1}^{n}\lambda(a_m)\right|\right\}<\infty. \tag{12}$$

*(iii)* $\{a_{m,1}\}_{m=1}^{\infty}\in\gamma.$

*(iv)* $\left\{\{a_{m,p}-a_{m,p+1}\}_{m=1}^{\infty}:p\in\mathbb{N}\right\}\subseteq\gamma.$

**Proof.** Take $\pi_n : \gamma \to \mathbb{C}$ so that $\pi_n(z) = z_n$ if $z \in \gamma$. Then $\sup_{n\in\mathbb{N}} \|\pi_n\|_{\gamma^*} \leq 2$. So, given $B \in \mathcal{B}(\gamma)$ and $m \in \mathbb{N}$ by Theorem 6 there is a uniquely determined sequence $a_m \triangleq \{a_{m,n}\}_{n=1}^{\infty}$ so that $\{a_{m,n} - a_{m,n+1}\}_{n=1}^{\infty} \in \mathrm{l}^1$,

$$(\pi_m \circ B)(z) = \lambda(a_m) \cdot S(z) + \sum_{n=1}^{\infty} (a_{m,n} - a_{m,n+1}) \cdot S_n(z) \text{ if } z \in \gamma$$

and

$$\|\pi_m \circ B\|_{\gamma^*} = |\lambda(a_m)| + \sum_{p=1}^{\infty} |a_{m,p} - a_{m,p+1}| .$$

Now *(i)* follows by the uniform boundedness principle. If $A \triangleq S_0 \circ B \circ S_0^{-1}$ then $A \in \mathcal{B}(c)$ and

$$(\chi_n \circ A)(w) = \lambda(w) \cdot \sum_{m=1}^{n} \lambda(a_m) + \sum_{p=1}^{\infty} w_p \sum_{m=1}^{n} (a_{m,p} - a_{m,p+1})$$

if $w \in \mathrm{c}$ and $n \in \mathbb{N}$. We shall write

$$\theta_{n,p} = \begin{cases} \sum_{m=1}^{n} \lambda(a_m) & if \quad p = 0, \\ \sum_{m=1}^{n} (a_{m,p} - a_{m,p+1}) & if \quad p \in \mathbb{N}. \end{cases} \tag{13}$$

If $n \in \mathbb{N}$, by Theorem 1*(v)*, we know that $\|\chi_n \circ A\|_{c^*} = \sum_{p=0}^{\infty} |\theta_{n,p}|$ and, by the uniform boundedness principle, we get (12). Moreover, by Corollary 2 the following limits exist

$$\theta_{0,0} \triangleq \lim_{n\to\infty} \sum_{p=0}^{\infty} \theta_{n,p} \tag{14}$$

$$= \lim_{n\to\infty} \sum_{m=1}^{n} \left[ \lambda(a_m) + \sum_{p=1}^{\infty} (a_{m,p} - a_{m,p+1}) \right] = \sum_{m=1}^{\infty} a_{m,1},$$

$$\theta_{0,p} \triangleq \lim_{n\to\infty} \theta_{n,p} = \lim_{n\to\infty} \sum_{m=1}^{n} (a_{m,p} - a_{m,p+1}), \text{ with } p \in \mathbb{N}, \tag{15}$$

i.e. *(iii)* and *(iv)* hold.

Let $\{a_{m,p}\}_{m,p=1}^{\infty}$ be a given sequence so that *(i)*, *(ii)*, *(iii)* and *(iv)* hold. If $m \in \mathbb{N}$ set

$$B_m(z) = \lambda(a_m) \cdot S(z) + \sum_{p=1}^{\infty} (a_{m,p} - a_{m,p+1}) \cdot S_p(z) \text{ for } z \in \gamma. \tag{16}$$

By *(i)* and Theorem 6 we know that $B_m \in \gamma^*$. Using the above notation, if $z \in \gamma$ and $n \in \mathbb{N}$ by (16) we have

$$\begin{aligned}\sum_{m=1}^{n} B_m(z) &= S(z)\sum_{m=1}^{n}\lambda(a_m) + \sum_{p=1}^{\infty}\sum_{m=1}^{n}(a_{m,p} - a_{m,p+1})\cdot S_p(z) \\ &= \theta_{n,0}\cdot\lambda(S_0(z)) + \sum_{p=1}^{\infty}\theta_{n,p}\cdot\chi_p(S_0(z)) \\ &= \lambda(S_0(z))\cdot\sum_{p=0}^{\infty}\theta_{n,p} + \sum_{p=1}^{\infty}\theta_{n,p}\cdot(\chi_p(S_0(z)) - \lambda(S_0(z))).\end{aligned}$$

By *(i)*, *(iii)* and *(iv)* the limits in (14) and (15) exist. Further, by *(ii)* we see that $\sup_{n\in\mathbb{N}}\sum_{p=0}^{\infty}|\theta_{n,p}| < \infty$. Hence $\{\theta_{0,p}\}_{p=1}^{\infty} \in l^1$ and then

$$\sum_{m=1}^{\infty} B_m(z) = \lambda(S_0(z))\cdot\sum_{m=1}^{\infty} a_{m,1} + \sum_{p=1}^{\infty}\theta_{0,p}(\chi_p(S_0(z)) - \lambda(S_0(z))).$$

Whence $B(z) \triangleq \{B_m(z)\}_{m=1}^{\infty}$ if $z \in \gamma$ defines a linear mapping on $\gamma$ that is now clearly bounded.

**Remark 8** *$\{a_{m,p}\}_{m,p=1}^{\infty}$ is $\gamma$-**conservative** if it verifies the conditions of Theorem 7. Further, it called $\gamma$-**regular** if it is $\gamma$-conservative and its induced bounded operator $B$ on $\gamma$ preserves sums, i.e. $\lambda(S_0(B(z))) = \lambda(S_0(z))$ for all $z \in \gamma$. It is now readily seeing that $\{a_{m,p}\}_{m,p=1}^{\infty}$ is $\gamma$-regular if and only if $\sum_{m=1}^{\infty} a_{m,p} = 1$ for all $p \in \mathbb{N}$.*

**Example 9** *If $\{a_m\}_{m=1}^{\infty} \in \gamma$ set $a_{m,p} = a_m$, $m, p \in \mathbb{N}$. Then*

$$B(z) \triangleq \{a_m \cdot S(z)\}_{m=1}^{\infty},\ z \in \gamma,$$

*defines a bounded linear functional on $\gamma$.*

**Example 10** *Let $a_{m,p} = m^{-s}\cdot p^{-t}$, where $m, p \in \mathbb{N}$, $s > 1, t > 0$. Then we get*

$$B(z) = \left\{m^{-s}\left(S(z) + \sum_{p=1}^{\infty}\left((p+1)^{-t} - p^{-t}\right)S_p(z)\right)\right\}_{m=1}^{\infty},\ z \in \gamma.$$

**Example 11** *Let $a_{m,p} = p/(1 + pm^s)$, where $m, p \in \mathbb{N}$, $s > 1$. In this case*

$$B(z) = \left\{\frac{S(z)}{m^s} - \sum_{p=1}^{\infty}\frac{S_p(z)}{(1 + pm^s)(1 + (p+1)m^s)}\right\}_{m=1}^{\infty},\ z \in \gamma.$$

## 3 Radon-Nikodym type properties of $\gamma$

**Proposition 12** *The Banach space $\gamma$ does not have the Radon-Nikodym property.*

**Proof.** We shall use a modified crude argument of J. Diestel & J. J. Uhl (cf. [6], p. 60). Let $\Lambda$ be the $\sigma$-field of Lebesgue measurable subsets of $[0,1]$. If $n \in \mathbb{N}$ and $E \in \Lambda$ we set

$$H_n(E) = \begin{cases} \int_E \sin(2\pi t)\, dm(t) & if \quad n = 1, \\ -2\int_E \sin\left(2^{n-2}\pi t\right)\cos\left(3 \cdot 2^{n-2}\pi t\right) dm(t) & if \quad n \geq 2, \end{cases}$$

where $m$ denotes the Lebesgue measure on $[0,1]$. Thus $H$ is a well $\gamma$-valued function of $\Lambda$. For, if $E \in \Lambda$ and $m \in \mathbb{N}$ we have

$$\sum_{n=1}^{m} H_n(E) = \int_E \sin(2^m \pi t)\, dm(t). \tag{17}$$

According to the Riemann-Lebesgue lemma, we conclude that $\sum_{n=1}^{\infty} H_n(E) = 0$. Indeed, by (17) we have that $\|H(E)\|_\gamma \leq m(E)$ for all $E \in \Lambda$. Now, it is clear that $H$ is countable additive, $m$-continuous and of bounded variation. Suppose there exist a Bochner integrable $h : [0,1] \to \gamma$ so that $H(E) = \int_E h(t) dm(t)$ for all $E \in \Lambda$. Then, if $h_n = \pi_n \circ h$ for $n \in \mathbb{N}$ and $E \in \Lambda$ we obtain

$$\pi_n(H(E)) = \int_E \pi_n(h(t))\, dm(t) = \int_E h_n(t) dm(t).$$

Hence it can be easily deduced that for almost all $t \in [0,1]$ we have

$$h_n(t) = \begin{cases} \sin(2\pi t) & if \quad n = 1, \\ -2\sin\left(2^{n-2}\pi t\right)\cos\left(3 \cdot 2^{n-2}\pi t\right) & if \quad n \geq 2. \end{cases}$$

Then if $m \in \mathbb{N}$ is

$$\sum_{n=1}^{m} h_n(t) = \sin(2^m \pi t) \text{ a.e..} \tag{18}$$

We already know that $\operatorname{Im}(H) \subseteq \ker(S)$. Since $S \in \gamma^*$ and $h$ is Bochner integrable then $S \circ h \in \mathrm{L}^1[0,1]$ and $\int_E S(h(t)) dm(t) = 0$ for all $E \in \Lambda$ (cf. [8]). So by (18) we have $S(h(t)) = \lim_{m\to\infty} \sin(2^m \pi t) = 0$ almost everywhere on $[0,1]$. However, given $n \in \mathbb{N}$ we set

$$E_n \triangleq \left\{ t \in [0,1] : |\sin(2^n \pi t)| \geq 1/\sqrt{2} \right\}.$$

If $t \in E_n$ there is an integer $1 \leq k \leq 2^{n-1}$ so that

$$\frac{2(k-1)+1/4}{2^n} \leq t \leq \frac{2(k-1)+3/4}{2^n} \text{ or } \frac{2(k-1)+5/4}{2^n} \leq t \leq \frac{2(k-1)+7/4}{2^n}.$$

Consequently $m(E_n) = 1/2$ for all $n \in \mathbb{N}$ and

$$m\left(\overline{\lim}_{n\to\infty} E_n\right) = \lim_{n\to\infty} m\left(\cup_{l=n}^{\infty} E_l\right) \geq 1/2.$$

But certainly $S(h(t)) \neq 0$ on $\overline{\lim}_{n\to\infty} E_n$. Therefore $h$ is not almost everywhere $\gamma$-valued and thus $G$ has no Radon-Nikodym derivative with respect to $m$.

**Corollary 13** *$\gamma$ is not reflexive nor uniformly convex.*

**Proof.** Since the Radon-Nikodym property does not hold on $\gamma$ this claim follows from R. S. Phillips theorem (cf. [11]). On the other hand, it is well known that any uniformly convex Banach space is reflexive (cf. [4]).

## 4 Compact operators on $\gamma$

The notion of *Hausdorff measure of non compactness* provides a way to characterize compact operators acting on certain Banach spaces. Precisely, given a bounded subset $Q$ of a normed space $X$ set

$$q(Q) = \inf\{\varepsilon > 0 : Q \text{ has a finite } \varepsilon\text{-net in } X\}.$$

The function $q$ is called the Hausdorff measure of non compactness (cf. [7]). If $X, Y$ are Banach spaces and $T \in \mathcal{B}(X,Y)$ we write $\|T\|_q \triangleq q(TB_X[0,1])$, where $B_X[0,1]$ is the closed unit ball of $X$ centered at zero. Consequently, $T$ becomes compact if and only if $\|T\|_q = 0$. With the notation of Corollary 2 the following result of B. de Malafosse, E. Malkowsky & V. Rakočević holds:

**Theorem 14** *(cf. [10]) If $A \in \mathcal{B}(\mathrm{c})$ then*

$$\frac{1}{2}\overline{\lim}_{n\to\infty}\left(\left|a_{n,0} - a_{0,0} + \sum_{m=1}^{\infty} a_{0,m}\right| + \sum_{m=1}^{\infty} |a_{n,m} - a_{0,m}|\right)$$
$$\leq \|A\|_q \leq \overline{\lim}_{n\to\infty}\left(\left|a_{n,0} - a_{0,0} + \sum_{m=1}^{\infty} a_{0,m}\right| + \sum_{m=1}^{\infty} |a_{n,m} - a_{0,m}|\right).$$

**Theorem 15** *Let $B \in \mathcal{B}(\gamma)$ be the unique bounded operator induced by a bi-index sequence $\{a_{m,p}\}_{m,p=1}^{\infty}$ that verifies the conditions of Theorem 7. Then $B$ is compact if and only if*

$$\overline{\lim}_{n\to\infty}\left\{\left|\lim_{p\to\infty}\sum_{m=n}^{\infty} a_{m,p}\right| + \sum_{p=1}^{\infty}\left|\sum_{m=n}^{\infty}(a_{m,p} - a_{m,p+1})\right|\right\} = 0.$$

**Proof.**We use the notation of Theorem 7. By Theorem 14 we see that $B \in C(\gamma)$ if and only if

$$\overline{\lim}_{n\to\infty}\left(\left|\theta_{n,0}-\theta_{0,0}+\sum_{m=1}^{\infty}\theta_{0,m}\right|+\sum_{m=1}^{\infty}|\theta_{n,m}-\theta_{0,m}|\right)=0.$$

By (13), (14) and (15) if $n \in \mathbb{N}$ we obtain

$$\theta_{n,0}-\theta_{0,0}+\sum_{m=1}^{\infty}\theta_{0,m}=\sum_{m=1}^{n}\lambda(a_m)-\sum_{m=1}^{\infty}a_{m,1}+\sum_{p=1}^{\infty}\sum_{m=1}^{\infty}(a_{m,p}-a_{m,p+1}). \quad (19)$$

Using *(iii)* and *(iv)* of Theorem 7, it is seeing recursively that $\sum_{m=1}^{\infty} a_{m,p}$ converge for all $p \in \mathbb{N}$. Therefore in (19) we have

$$\begin{aligned}\theta_{n,0}-\theta_{0,0}+\sum_{m=1}^{\infty}\theta_{0,m} &= \sum_{m=1}^{n}\lambda(a_m)-\lim_{p\to\infty}\sum_{m=1}^{\infty}a_{m,p} \quad (20)\\ &= -\lim_{p\to\infty}\sum_{m=n+1}^{\infty}a_{m,p}.\end{aligned}$$

Analogously,

$$\sum_{p=1}^{\infty}|\theta_{n,p}-\theta_{0,p}|=\sum_{p=1}^{\infty}\left|\sum_{m=n+1}^{\infty}(a_{m,p}-a_{m,p+1})\right| \quad (21)$$

and the claim follows from (20) and (21).

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