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Extension of the Kirk-Saliga fixed point theorem

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Abstract

A technical extension is given for the fixed point statement in Kirk and Saliga [Nonlinear Analysis, 47 (2001), 2765-2778].

1. Introduction

Let (M, d) be a complete metric space; and $x \vdash \varphi(x)$, some function from M to $R_+ := [0, \infty[$ with

$$\begin{aligned}&\varphi \text{ is lsc from above on } M\text{:}\\ &x_n \to x \text{ and } (\varphi(x_n)) \text{ descending imply } \lim_n \varphi(x_n) \geq \varphi(x).\end{aligned} \tag{1.1}$$

Further, let $x \vdash Tx$ be a selfmap of M. The following 1975 statement in Caristi and Kirk [6] (referred to as the Caristi-Kirk fixed point theorem; in short: CK-fpt) is our starting point.

Theorem 1. *Assume that (in addition)*

$$d(x, Tx) \leq \varphi(x) - \varphi(Tx), \quad \textit{for each } x \in M. \tag{1.2}$$

Then, T has at least one fixed point in M.

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[As a matter of fact, the quoted result is with (1.1) substituted by

$$\varphi \text{ is lsc on } M \ (\liminf_n \varphi(x_n) \geq \varphi(x), \text{ whenever } x_n \to x). \tag{1.3}$$

But the authors' argument also works in this relaxed setting].

The original proof of Theorem 1 is by transfinite induction; see also Wong [17]. Note that, in terms of the associated (to φ) order on M

$$(x, y \in M) \quad x \leq y \quad \text{iff } d(x, y) \leq \varphi(x) - \varphi(y)$$

the contractivity condition (1.2) becomes

$$x \leq Tx, \text{ for each } x \in M \quad (\text{i.e.: } T \text{ is } \textit{progressive} \text{ on } M). \tag{1.4}$$

So, by the Bourbaki "duality" principle [3], Theorem 1 is logically equivalent with Zorn's maximality result [19] subsumed to this order; i.e., with Ekeland's variational principle [7]. Hence, the sequential type argument used by the quoted author to get his statement is also working in our precised setting; see also Pasicki [12]. A proof of Theorem 1 involving the chains of the structure $(M, \leq)$ may be found in Turinici [16]. Further aspects (involving the general case) may be found in Brunner [5]; see also Taskovic [14].

Now, CK-fpt found (especially via Ekeland's approach) some basic applications to control and optimization, generalized differential calculus, critical point theory and normal solvability; see the above references for details. As a consequence, many extensions of Theorem 1 were proposed. Here, we shall concentrate on the 2001 statement obtained in Kirk and Saliga [10] (referred to as the Kirk-Saliga fixed point theorem; in short: KS-fpt).

Theorem 2. *Assume that (in addition to (1.1))*

$$\varphi(x) \geq \varphi(Tx), \quad \text{for each } x \in M \tag{1.5}$$

$$d(x, Tx) \leq \varphi(x) - \varphi(T^p x), \text{ for each } x \in M \text{ and some } p \geq 1. \tag{1.6}$$

Then T has at least one fixed point in M.

The argument used by the authors is (again) the transfinite induction. A direct analysis of its mechanism reveals certain possibilities of extending Theorem 2; this will be done in Section 3. The preliminaries for our approach are given in Section 2. Finally, Section 4 is devoted to some particular versions of our main result.

2. Preliminaries

(A) Let W stand for the class of ordinal numbers, introduced in a "factorial" way; cf. Kuratowski and Mostowski [11, Ch 7, Sect 2]. Precisely, call the partially ordered structure $(P, \leq)$, *well ordered* if each part of P admits a first element. Given a couple $(P, \leq)$, $(Q, \leq)$ of such objects, put

$$(P, \leq) \equiv (Q, \leq) \text{ if there exists a strictly increasing bijection:} P \to Q.$$

This is an equivalence relation; the order type of $(P, \leq)$ (denoted $\mathrm{ord}(P, \leq)$) is just its equivalence class; also referred to as an *ordinal*.

Note that W is not a set, as results from the Burali-Forti paradox; cf. Sierpinski [13, Ch 14, Sect 2]. However, when one restricts to a *Grothendieck universe* $\mathcal{G}$ (taken as in Hasse and Michler [8, Ch 1, Sect 2]) this contradictory character is removed for the class $W(\mathcal{G})$ of all *admissible* (modulo $\mathcal{G}$) ordinals (generated by (non-contradictory) well ordered parts of $\mathcal{G}$). In the following, we drop any reference to this universe, for simplicity. So, by an *ordinal* in W one actually means a $\mathcal{G}$-admissible ordinal with respect to a "sufficiently large" Grothendieck universe $\mathcal{G}$. Clearly,

$$\xi\text{=admissible ordinal and } \eta \leq \xi \text{ imply } \eta\text{=admissible ordinal.}$$

Hence, in the formulae

$$W(\alpha) = \{\xi \in W; \xi < \alpha\}, \quad W[\alpha] = \{\xi \in W; \xi \leq \alpha\},$$

the symbol W in the brackets is the "absolute" class of all ordinals.

Now, an enumeration of W is realized via the immediate successor map

$$\mathrm{suc}(M) = \min\{\xi \in W; M < \xi\}, M \subseteq W \text{ (hence } \mathrm{suc}(\alpha) = \alpha + 1, \forall \alpha \in W).$$

(Here, $M < \xi$ means: $\lambda < \xi, \forall \lambda \in M$). It begins with the natural numbers $N = \{0, 1, ...\}$. Their immediate successor is $\omega = \mathrm{suc}(N)$ (the first transfinite ordinal); the next in this enumeration is $\omega + 1$, and so on.

In parallel to this, we may (construct and) enumerate the class of all admissible cardinals. Let P and Q be nonempty sets; we put

$$P \preceq Q (P \sim Q) \text{ iff there exists an injection (bijection):} P \to Q.$$

The former is a quasi-order; while the latter is an equivalence. Denote also

$$P \prec Q \text{ if and only if } P \preceq Q \text{ and } \neg(P \sim Q).$$

This relation is *irreflexive* ($\neg(P \prec P)$, for each P) and *transitive*; hence a *strict order*. Let $\alpha > 0$ be an (admissible) ordinal; we say that it is an (admissible) *cardinal* if $W(\xi) \prec W(\alpha)$, for each $\xi < \alpha$. The class of all these will be denoted by Z. Now, the enumeration we are looking for is realized via the immediate successor (in Z) map

$$\mathrm{SUC}(M) = \min\{\eta \in Z; M < \eta\}, \quad M \subseteq Z.$$

Precisely, this begins with the natural numbers $0, 1, ...$ The immediate successor (in Z) of all these is (again) $\omega = \mathrm{SUC}(N)$ (the first transfinite cardinal). To describe the remaining ones, we may introduce via transfinite recursion the function $\lambda \vdash \aleph_\lambda$ from W to Z as

$$\begin{aligned}
&\aleph_0 = \omega; \quad \text{and, for each } \lambda > 0,\\
&\aleph_\lambda = \mathrm{SUC}(\aleph_{\lambda-1}), \quad \text{if } \lambda - 1 \text{ exists}\\
&\aleph_\lambda = \mathrm{SUC}\{\aleph_\xi; \xi < \lambda\}, \quad \text{if } \lambda - 1 \text{ does not exist.}
\end{aligned}$$

Note that, in such a case, the order structure of $Z(\omega, \leq) = \{\xi \in Z; \omega \leq \xi\}$ is completely reducible to the one of W; further details may be found in Sierpinski [op. cit., Ch 15, Sect 7].

Any nonempty part P with $P \prec W(\omega)$ ($P \sim W(\omega)$) is termed *finite* (*effectively countable*); the union of these ($P \preceq W(\omega)$) is referred to as P is *countable*. When $P = W(\xi)$, all such properties will be transferred to ξ.

Now, the immediate successor in Z of $\omega = \aleph_0$ is $\Omega = \aleph_1$ (the first uncountable ordinal). The motivation of our convention comes from

$$\xi \text{ is countable, for each } \xi < \Omega; \quad \text{but } \Omega \text{ is not countable.} \tag{2.1}$$

A basic consequence of this is precised in the statement below (to be found, e.g., in Alexandrov [1, Ch 3, Sect 4]):

Proposition 1. *The following are valid:*

i) *The ordinal* Ω *cannot be attained via sequential limits of countable ordinals. That is: if* (α_n) *is an ascending sequence of countable ordinals then*

$$\alpha = \sup_n(\alpha_n)(= \lim_n(\alpha_n)) \tag{2.2}$$

is countable too.

ii) *Each second kind countable ordinal is attainable via such sequences. In other words: if* $\alpha < \Omega$ *is of second kind then, there exists a strictly ascending sequence* (α_n) *of countable ordinals with the property (2.2).*

(B) Let M be a nonempty set; and $(\leq)$, some *order* (=antisymmetric quasi-order) on it. By a $(\leq)$-*chain* of M we shall mean any (nonempty) part A of M with $(A, \leq)$ being well ordered (see above). Note that any such object may be written as $A = \{a_\xi; \xi < \lambda\}$, where the net $\xi \vdash a_\xi$ is strictly ascending ($\xi < \eta \Longrightarrow a_\xi < a_\eta$); the uniquely determined ordinal λ is just $\mathrm{ord}(A, \leq)$. Now, by the remark above, A is countable $\Longleftrightarrow \mathrm{ord}(A, \leq) < \Omega$. If, moreover, $\mathrm{ord}(A, \leq) \leq \omega$, we say that A is *normally countable*; equivalently, this reads

$$A = \{b_n; n < \omega\}, \text{ where } n \vdash b_n \text{ is ascending } (p < q \Longrightarrow b_p \leq b_q). \tag{2.3}$$

Let P, Q be nonempty parts with $P \supseteq Q$. We say that P is *majorized* by Q (and write $P \propto Q$) provided Q is cofinal in P ($\forall x \in P, \exists y \in Q$ with $x \leq y$). The $(\leq)$-chain $S \subseteq M$ is called *upper countable* in case

$$S \propto T, \text{ for some normally countable } (\leq)\text{-chain } T \subseteq S. \tag{2.4}$$

Clearly, this happens if S is normally countable. The following completion of it is available (via Proposition 1):

Proposition 2. *The generic relation holds*

$$(\forall (\leq)\text{-}chain) \textit{ countable} \Longrightarrow \textit{upper countable}. \tag{2.5}$$

Hence, the $(\leq)$*-chain* $S \subseteq M$ *is upper countable if and only if*

$$S \propto T, \textit{ for some countable } (\leq)\textit{-chain } T \subseteq S. \tag{2.6}$$

Remark. The converse of (2.5) is not in general true; just take any $(\leq)$-chain S of M with $\Omega \leq \mathrm{ord}(S, \leq)=$ first kind ordinal.

(C) Let us now return to our initial setting. We say that the order structure $(M, \leq)$ is *separable* if (cf. Zhu, Fan and Zhang [16])

$$\text{any } (\leq)\text{-chain of } M \text{ is upper countable.} \tag{2.7}$$

For example, this holds (under (2.5)) whenever

$$(M, \leq) \text{ is } \textit{strongly separable}\text{: each } (\leq)\text{-chain of } M \text{ is countable.} \tag{2.8}$$

In fact, the reciprocal holds too; so that, we may formulate

Proposition 3. *Under these conventions,*

$$(\forall (M, \leq)\textit{=ordered structure}) \textit{ separable} \Longleftrightarrow \textit{strongly separable}. \tag{2.9}$$

[The proof is essentially based on Proposition 1; we do not give details].

A basic example of such structures may be given along the following lines. By a *topology* over M we mean, as usually, any family $\mathcal{T} \supseteq \{\emptyset, M\}$ of parts in M, invariant to arbitrary unions and finite intersections. Assume that we fixed such an object; and let "cl" stand for the associated *closure* operator. Any subfamily $\mathcal{B} \subseteq \mathcal{T}$ with the property that each $D \in \mathcal{T}$ is a union of members in $\mathcal{B}$, will be referred to as a *basis* for $\mathcal{T}$. If, in addition, $\mathcal{B}$ is countable, then $\mathcal{T}$ will be called *second countable*. Finally, term $(\leq)$, *closed from the left* provided $M(x, \geq) := \{y \in M; x \geq y\}$ is closed, for each $x \in M$.

Proposition 4. *Assume that* $\mathcal{T}$ *is second countable and* $(\leq)$ *is closed from the left. Then,* $(M, \leq)$ *is (strongly) separable.*

Proof. Let $\mathcal{B} = \{B_n; n < \omega\}$ stand for a countable basis of $\mathcal{T}$. Further, take some choice function "Ch" of the nonempty parts in M [$\mathrm{Ch}(X) \in X$, for each $X \subseteq M, X \neq \emptyset$]. Given the arbitrary fixed $(\leq)$-chain S of M, denote $T = \{\mathrm{Ch}(B \cap S); B \in \mathcal{B}\}$ (hence $T \subseteq S$). For the moment, T is countable (because $T \preceq \mathcal{B}$). In addition, we claim that $\mathrm{cl}(T) \supseteq S$ [wherefrom, T is dense in S]. In fact, let s be some point of S; and U stand for an open neighborhood of it. By definition, U=union of members in $\mathcal{B}$; so

$$U \supseteq B \ni s \text{ (hence } U \ni \mathrm{Ch}(B \cap S)), \quad \text{for some } B \in \mathcal{B};$$

and our claim follows. If T is cofinal in S, we are done (cf. Proposition 2). Otherwise, there must be some $s \in S$ with $T \subseteq M(s, \geq)$; wherefrom

$$S \subseteq \mathrm{cl}(T) \subseteq \mathrm{cl}(M(s, \geq)) = M(s, \geq);$$

i.e., $\{s\}$ is cofinal in S. The proof is thereby complete. ■

It remains now to establish under which conditions is $\mathcal{T}$, second countable. An appropriate answer is to be given in a *metrizable* context:

$$\text{there exists a metric } d : M \times M \to R_+ \text{ which generates } \mathcal{T}.$$

Then, e.g., the condition below yields the desired property for $\mathcal{T}$:

$$M \text{ has a countable dense subset } P \text{ (in the sense: } \mathrm{cl}(P) = M). \tag{2.10}$$

The proof is to be found in Bourbaki [4, Ch 9, Sect 2.8]; see also Alexandrov [op. cit., Ch 4, Sect 4].

In particular, let R stand for the real axis. Denote by $(\leq, d)$ the usual order and metric. Take any bounded from above part M of R ($M \leq v$, for some $v \in R$). The structure $(M, \leq)$ fulfills (via (2.10)) conditions of Proposition 4; wherefrom, $(M, \leq)$ is (strongly) separable. A similar conclusion is valid for the dual order $(\geq)$. Precisely, for each bounded from below part M of R ($M \geq u$, for some $u \in R$), one has that $(M, \geq)$ is (strongly) separable. This will be useful for our future developments.

3. Main result

With these informations at hand, we may now return to the questions of the introductory section. Let M be some nonempty set. Take a metric $d : M \times M \to R_+$ and a function $\varphi : M \to R_+ \cup \{\infty\}$ with

$$\begin{array}{l} (d, \varphi) \text{ is descending complete:} \\ \text{for each Cauchy sequence } (x_n) \text{ in } M \text{ with } (\varphi(x_n)) \text{ descending} \\ \text{there exists } x \in M \text{ with } x_n \to x \text{ and } \lim_n \varphi(x_n) \geq \varphi(x). \end{array} \tag{3.1}$$

Further, let $x \vdash \psi(x)$ stand for another function from M to $R_+ \cup \{\infty\}$ with

$$\psi \text{ is proper} \quad (\mathrm{Dom}(\psi) := \{x \in M; \psi(x) < \infty\} \text{ is nonempty}). \tag{3.2}$$

Finally, take a selfmap $T : M \to M$. Our main result is

Theorem 3. *Let the above data be such that (1.5) is true, as well as*

$$[\varphi(T^n x) \geq \varphi(T^n y),\ n = 0, 1, ...] \Longrightarrow \psi(Tx) \geq \psi(Ty) \tag{3.3}$$

$$d(x, Tx) \leq \psi(x) - \psi(Tx), \quad \textit{for each } x \in \mathrm{Dom}(\psi). \tag{3.4}$$

Then, for each $u \in \mathrm{Dom}(\psi)$ there exists $v \in \mathrm{Dom}(\psi)$ with

$$v = Tv \quad \textit{and} \quad d(u, v) \leq \psi(u) - \psi(v). \tag{3.5}$$

Proof. Denote for simplicity

$$M_u = \{x \in M; \varphi(u) \geq \varphi(x),\ d(u, x) \leq \psi(u) - \psi(x)\}$$

(where u is the above precised one). Clearly, $\emptyset \neq M_u \subseteq \mathrm{Dom}(\psi)$; and

$$M_u \text{ is invariant under } T \text{ (in view of (1.5)+(3.4))}. \tag{3.6}$$

Assume by contradiction that there is no fixed point of T in M_u:

$$d(x, Tx) > 0, \text{ (hence } \psi(x) > \psi(Tx), \text{ via (3.4)), for each } x \in M_u. \tag{3.7}$$

We shall prove by transfinite induction that this cannot be in agreement with some statements in Section 2. Put $a(0) = u$, $a(1) = T(a(0))$; note that $a(1) \in M_u$ (by (3.6)); and (cf. (1.5)+(3.4)+(3.7))

$$\begin{aligned}&\varphi(a(0)) \geq \varphi(a(1));\ 0 < d(a(0), a(1)) \leq \psi(a(0)) - \psi(a(1))\\ &\text{(hence } \psi(a(0)) > \psi(a(1))).\end{aligned}$$

Generally, assume that, for the ordinal $\mu < \Omega$, we constructed a net $(a(\xi); \xi < \mu)$ in M_u so that: for each $\lambda < \mu$

$$\xi < \xi + 1 \leq \lambda \Longrightarrow a(\xi + 1) = T(a(\xi)) \tag{$A(\lambda)$}$$

$$\xi \leq \lambda \Longrightarrow \varphi(a(\xi)) \geq \varphi(a(\lambda)) \tag{$B(\lambda)$}$$

$$\xi \leq \lambda \Longrightarrow d(a(\xi), a(\lambda)) \leq \psi(a(\xi)) - \psi(a(\lambda)) \tag{$C(\lambda)$}$$

$$\xi < \lambda \Longrightarrow \psi(a(\xi)) > \psi(a(\lambda)). \tag{$D(\lambda)$}$$

Two cases are open before us.

i) μ is a first kind ordinal: $\lambda = \mu - 1$ exists. Put $a(\mu) = T(a(\lambda))$; hence $a(\mu) \in M_u$ (by (3.6)). In addition (from (1.5)+(3.4))

$$\varphi(a(\lambda)) \geq \varphi(a(\mu));\ d(a(\lambda), a(\mu)) \leq \psi(a(\lambda)) - \psi(a(\mu));$$
$$\text{hence } \psi(a(\lambda)) > \psi(a(\mu)) \text{ (by (3.7), with } x = a(\lambda));$$

and, from this, $(A(\mu)) - (D(\mu))$ follow.

ii) μ is a second kind ordinal: $\mu - 1$ does not exist. For the moment, it is clear that $(A(\mu))$ holds; because $\xi < \xi + 1 \leq \mu \Longrightarrow \xi < \xi + 1 < \mu$; and then, by $(A(\xi + 1))$, we are done. The remaining conclusions necessitate a special construction. Let (λ_n) be a strictly ascending sequence of ordinals in $W(\mu)$ with $\sup_n(\lambda_n)(= \lim_n(\lambda_n)) = \mu$ (see Proposition 1); and put for simplicity $b_n = a(\lambda_n)$, $n = 0, 1, \ldots$. By $(C(\lambda) : \lambda < \mu)$ we have an evaluation like

$$d(b_n, b_m) \leq \psi(b_n) - \psi(b_m), \quad \text{whenever } n \leq m. \tag{3.8}$$

The sequence $(\psi(b_n))$ is descending in R_+; hence a Cauchy one. As a consequence, (b_n) is a Cauchy sequence in M; and this, along with $(B(\lambda); \lambda < \mu)$ tells us (via (3.1)) that there must be an element $a(\mu) \in M$ with

$$b_n \to a(\mu) \text{ as } n \to \infty;\ \text{ and } \lim_n \varphi(b_n) \geq \varphi(a(\mu)).$$

The second part of this relation yields, again via $(B(\lambda); \lambda < \mu)$,

$$\varphi(a(\xi)) \geq \varphi(a(\mu)), \forall \xi < \mu;\ \text{ wherefrom } (B(\mu)) \text{ holds.} \tag{3.9}$$

Moreover, a repeated application of $(A(\lambda); \lambda < \mu)$ and (1.5) gives

$$\varphi(T^p(a(\xi))) = \varphi(a(\xi + p)) \geq \varphi(a(\mu)) \geq \varphi(T^p(a(\mu))), \quad \forall p < \omega, \forall \xi < \mu;$$

so, by simply combining with (3.3),

$$\psi(a(\xi)) \geq \psi(a(\mu)),\ \forall \xi < \mu; \quad \text{hence } \psi(b_n) \geq \psi(a(\mu)), \forall n. \tag{3.10}$$

Taking (3.8) into account gives $d(b_n, b_m) \leq \psi(b_n) - \psi(a(\mu))$, for all (n, m) with $n \leq m$; wherefrom (passing to limit as $m \to \infty$) $d(b_n, a(\mu)) \leq \psi(b_n) - \psi(a(\mu))$, for all n. But, from this and the choice of (b_n), the conclusion $(C(\mu))$ is clear; hence (combining with (3.9) above) $a(\mu) \in M_u$. Finally, $(D(\mu))$ results from $(D(\lambda); \lambda < \mu)$ and (3.10). Summing up, the recursive construction of $(a(\xi))$ is possible over $\xi \in W(\Omega)$. But, in this case, $(D(\lambda); \lambda < \Omega)$ tells us that $(\psi(a(\xi)); \xi < \Omega)$ is a $(\geq)$-chain in R_+, of order type Ω; in contradiction with Proposition 4 above. Hence, the working assumption (3.7) about our data cannot be accepted; and the conclusion follows. ■

4. Particular aspects

The key regularity condition of Theorem 3 is evidently (3.3). So, it would be natural to have it expressed in convenient ways, useful for applications. A basic construction of this type is to be performed under the lines below. Let again M be a nonempty set. Take a metric $d : M \times M \to R_+$ and a function $\varphi : M \to R_+ \cup \{\infty\}$ fulfilling (3.1). Further, let $(R_+ \cup \{\infty\})^N$ stand for the class of all sequences $(s_0, s_1, ...)$ with positive terms $(s_n \geq 0, n = 0, 1, ...)$. Take a map $(s_0, s_1, ...) \vdash F(s_0, s_1, ...)$ from $(R_+ \cup \{\infty\})^N$ to $R_+ \cup \{\infty\}$ with the global increasing property

$$[s_n \leq t_n, n = 0, 1, ...] \Longrightarrow F(s_0, s_1, ...) \leq F(t_0, t_1, ...). \tag{4.1}$$

Finally, take a self-map $T : M \to M$. The composed function

$$(\psi : M \to R_+ \cup \{\infty\})\ \psi(x) = F(\varphi(x), \varphi(Tx), ...),\ x \in M \tag{4.2}$$

fulfills (3.3). Moreover, the properness condition (3.2) reads

$$\Delta(\varphi, F; T) := \{x \in M; F(\varphi(x), \varphi(Tx), ...) < \infty\} \text{ is nonempty.} \tag{4.3}$$

By Theorem 3 we then have

Theorem 4. *Assume (1.5)+(4.3) are valid, as well as* $(\forall x \in \Delta(\varphi, F; T))$

$$d(x, Tx) \leq F(\varphi(x), \varphi(Tx), ...) - F(\varphi(Tx), \varphi(T^2x), ...). \tag{4.4}$$

Then, for each $u \in \Delta(\varphi, F; T)$ *there exists* $v \in \Delta(\varphi, F; T)$ *in such a way that (3.5) holds, where* $\psi : M \to R_+ \cup \{\infty\}$ *is that of (4.2).*

Now, by an appropriate choice of the function $(s_0, s_1, ...) \vdash F(s_0, s_1, ...)$ appearing in (4.1) one gets some useful particular cases.

I) For example, let the function $F_p : (R_+ \cup \{\infty\})^N \to R_+ \cup \{\infty\}$ be taken as (for some $p \geq 1$)

$$F_p(s_0, s_1, ...) = s_0 + ... + s_{p-1}, \quad (s_0, s_1, ...) \in (R_+ \cup \{\infty\})^N. \tag{4.5}$$

Clearly, (4.1) holds in such a case; and (4.3) reads

$$\Delta(\varphi, F_p; T) := \{x \in M; \{x, ..., T^{p-1}x\} \subseteq \mathrm{Dom}(\varphi)\} \text{ is nonempty.} \tag{4.6}$$

The associated by (4.2) function ψ has the form $\psi(x) = \varphi(x) + ... + \varphi(T^{p-1}x)$, $x \in M$. By Theorem 4 we get:

Theorem 5. *Assume that (1.5)+(4.6) are valid, as well as*

$$d(x, Tx) \leq \varphi(x) - \varphi(T^p x), \quad \textit{for each } x \in \Delta(\varphi, F_p; T). \tag{4.7}$$

Then, for each $u \in \Delta(\varphi, F_p; T)$ *there exists* $v \in \Delta(\varphi, F_p; T)$ *with*

$$v = Tv \text{ and } d(u,v) \le (\varphi(u) - \varphi(v)) + ... + (\varphi(T^{p-1}u) - \varphi(v)). \tag{4.8}$$

The obtained fact may be viewed as a completion of the Kirk-Saliga fixed point result [10] (subsumed to Theorem 2). Moreover, it also extends a related statement in Bîrsan [2]; see also Taskovic [15].

II) Another interesting choice for this function is the *limit* of the preceding one as $p \to \infty$; namely

$$F_\infty(s_0, s_1, ...) = s_0 + s_1 + ..., \quad (s_0, s_1, ...) \in (R_+ \cup \{\infty\})^N. \tag{4.9}$$

As before, (4.1) holds in such a case; and (4.3) reads

$$\Delta(\varphi, F_\infty; T) := \{x \in M; \sum_n \varphi(T^n x) < \infty\} \neq \emptyset \tag{4.10}$$

The associated by (4.2) function ψ has the form $\psi(x) = \varphi(x) + \varphi(Tx) + ...$, $x \in M$. Since the series in the right member converges on $\Delta(\varphi, F_\infty; T)$, we must have $\lim_n(\varphi(T^n x)) = 0$, for all $x \in \Delta(\varphi, F_\infty; T)$; wherefrom

$$\begin{aligned}&\psi(x) - \psi(Tx) = \\ &\lim_n[(\varphi(x) + ... + \varphi(T^n x)) - (\varphi(Tx) + ... + \varphi(T^{n+1}x))] = \\ &\lim_n[\varphi(x) - \varphi(T^{n+1}x)] = \varphi(x), \quad \forall x \in \Delta(\varphi, F_\infty; T).\end{aligned}$$

By Theorem 4 we derive a "limit" counterpart of Theorem 5 above:

Theorem 6. *Assume that (1.5)+(4.10) are valid, as well as*

$$d(x, Tx) \le \varphi(x), \quad \text{for each } x \in \Delta(\varphi, F_\infty; T). \tag{4.11}$$

Then, for each $u \in \Delta(\varphi, F_\infty; T)$ *there exists* $v \in \Delta(\varphi, F_\infty; T)$ *with*

$$v = Tv \text{ (hence } \varphi(v) = 0\text{) and } d(u,v) \le \sum_n \varphi(T^n u). \tag{4.12}$$

Finally, we note that the construction (4.1) is not the only possible one so as to satisfy conditions of Theorem 3; this fact will be developed elsewhere. Some related aspects may be found in the 2003 survey paper by Kirk [9].

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