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VECTOR BUNDLES WITH TRIVIAL DETERMINANT AND SECOND CHERN CLASS ONE ON SOME NONKÄHLER SURFACES

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Introduction

In this paper we investigate holomorhic rank-2 vector bundles with trivial determinant and second Chern class one on some nonKähler surfaces. The main dificulty one encounters when dealing with holomorphic vector bundles over nonprojective manifolds, is the presence of *nonfiltrable* such bundles (that is, bundles with no filtration by torsion-free coherent subsheaves) or even of *irreducible* ones, that is, bundles with no proper coherent subsheaf (see for instance the exhaustive monograph [Br1]). However, as we shall show, on a certain class of nonKähler surfaces - which will be described below -, such vector bundles are filtrable, hence one may classify them via the classical technique of classification of extensions as it is furthermore shown.

The class of surfaces we are working on is the class of nonKähler *elliptic principal bundles*, (that is surfaces X which admit a bundle map $\pi : X \to B$ over a smooth projective curve B with fiber and structure group a fixed elliptic smooth curve F) satisfying the extra conditions: $NS(X) = 0$ and B is nonhypereliptic.

For nonKä hler elliptic principal bundles, it was proven by Brînzănescu in [Br2] (see also [Br3]) that

$$NS(X)/Tors(NS(X)) = Hom(Alb(B), F).$$

We see that for the "generic" choice of B and F, the Neron-Severi group of X is a torsion group, and hence, up to a finite etalé cover, even vanishing.

Thus we see that the class of surfaces we are working on is "general enough" in the class of nonKähler elliptic surfaces.

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1 Notations and basic facts.

Through this paper we shall denote by F_b the fiber of π over $b \in B$ and by g the genus of the base B. In general, the bundles from $Pic(X)$ will be denoted by $I, L, L', ...$, those from $Pic(B)$ by $\mathcal{I}, \mathcal{L}, \mathcal{L}', ...$, and those from $Pic(F)$ by $\xi, \xi', ...$

We shall next cite several results that will be nedeed.

Theorem 1.1 *Let F_b be some arbitrary fiber of π; then one has the exact sequence:*

$$0 \to Pic(B) \xrightarrow{\pi^*} Pic^0(X) \xrightarrow{r} Pic^0(F_b) \to 0$$

Corollary 1

$$Pic^0(X)/\pi^*(Pic(B)) = F$$

(as complex Lie groups)

The canonical involution $L \mapsto L^\vee$ on $Pic^0(X)$ defines, by passing to the quotient, the canonical involution on F; the quotient of F through this involution is the complex line $\mathbf{P}^1$. We shall denote by $\tilde{L} \in \mathbf{P}^1$ the class of a line bundle $L \in Pic^0(X)$ obtained after these factorisations. In other words, the equality $\tilde{L} = \tilde{L'}$ means: $\exists \mathcal{L} \in Pic(X)$ such that $L = L' \otimes \pi^*(\mathcal{L})$ or $L = (L')^\vee \otimes \pi^*(\mathcal{L})$.

Another fact that shall be needed is:

Theorem 1.2 *(Atiyah, [At])*

The rank-2 holomorphic vector bundles $\mathcal{E}$, with trivial determinant, on a smooth elliptic curve F are of one of the following types:

i) $\mathcal{E} \simeq \xi_0 \oplus \xi_0^\vee$, with $\xi_0 \in Pic^0(F)$;

ii) a nonsplit extension of the form:

$$0 \to \xi_0 \to \mathcal{E} \to \xi_0^\vee \to 0$$

where $\xi_0 \in Pic^0(F)$ is such that $\xi_0^{\otimes 2} = \mathcal{O}_F$.

iii) $\mathcal{E} \simeq \xi \oplus \xi^\vee$, with $\xi \in Pic(F)$, $deg(\xi) \geq 1$.

Finally, let us quote also the following classical fact (for a proof see [Ba-St]).

Theorem 1.3 *Let $\pi : X \to B$ be as above, $\mathcal{F} \in Coh(X)$ flat relatively to B, $b \in B$ fixed, arbitrary. The following assertions are equivalent:*

i) $\mathcal{R}^1\pi_*(\mathcal{F})_{(b)} = 0$,

ii) $H^1(F_b; \mathcal{F}_{|F_B}) = 0$.

2 Filtrability of rank-2 vector bundles with trivial determinant and second Chern class one

Let us now state the main result.

Theorem 2.1 *Let X be a surface as stated above. Then every rank-2 holomorphic vector bundle E on X, with trivial determinant and second Chern class one is filtrable.*

Proof. Assume there exists on X some nonfiltrable E as in the statement (that is $det \simeq \mathcal{O}_X$ and $c_2(E) = 1$).

First notice that one has:

Lemma 2.1 *Forall $L \in Pic(X)$ we have:*
i) $\pi_*(E \otimes L) = 0$, and
ii) $\mathcal{R}^1\pi_*(E \otimes L)$ is a sky-scraper sheaf, of length one.

The proof of this Lemma will be done later. Now observe that we have: $b \in Supp(\mathcal{R}^1\pi_*(E \otimes L))$ iff $H^0(F_b; E \otimes L_{|F_b}) \neq 0$ (this follows from the point i) from the above Lemma, the Theorem 1.3 from above, and from the fact that as $deg(E \otimes L_{|F_b}) = 0$, we get that $H^0(F_b; E \otimes L_{|F_b}) \neq 0$ iff $H^1(F_b; E \otimes L_{|F_b}) \neq 0$, as F_b is elliptic).

Consider now the Poincaré bundle on $\tilde{X} \stackrel{def}{=} X \times Pic^0(X)$; set $\tilde{B} \stackrel{def}{=} B \times Pic^0(X)$ and respectively $\tilde{\pi} : \tilde{X} \to \tilde{B}$ for the application defined by $\tilde{\pi} = (\pi, id_{Pic^0(X)})$.

Let now

$$\tilde{Z} \stackrel{def}{=} supp(\mathcal{R}^1\tilde{\pi}_*(\tilde{E} \otimes U))$$

Acording to Grauert's theorem, $\tilde{Z}$ is some analytic subspace of $\tilde{B}$.

Be

$$\theta : \tilde{B} \to B \times \mathbf{P}^1$$

the map given by $\theta \stackrel{def}{=} (id_B, \tilde{\ })$, where the application $\tilde{\ }$ is the composed of the canonical projections:

$$Pic^0(X) \to Pic^0(X)/\pi^*(Pic(B)) \to \mathbf{P}^1,$$

and put $Z = \theta(\tilde{Z})$. Again, Z is an analytic subspace of $B \times \mathbf{P}^1$, which we shall considered with its reduced structure. We need again a Lemma, which will also be proved later

Lemma 2.2 *In the conditions from above, one has (denoting by $pr_1 : B \times \mathbf{P}^1 \to B$, and respectively $pr_2 : B \times \mathbf{P}^1 \to \mathbf{P}^1$ the canonical projections), that $pr_{1|Z} : Z \to B$ and $pr_{1|Z} : Z \to \mathbf{P}^1$ are bijective.*

This Lemma is the "main ingredient" in order to to obtain a contradiction. To get it, (denoting by $M \overset{def}{=} B \times \mathbf{P}^1$), let us first observe that $H^2(M;\mathbf{Z})$ is the free abelian group generated by $[B]$ and $[\mathbf{P}^1]$ (we denoted by $[B]$, and respectively $[\mathbf{P}^1]$, the Poncaré duals of the fundamental cycles of the fibers of the canonical projections of M onto its factors). These two generators satisfy the following relations, (with respect to the intersection form); $([B])^2 = ([\mathbf{P}^1])^2 = 0$ and $([B].[\mathbf{P}^1]) = 1$.

Also, the canonical bundle $\mathcal{K}_M$ can be expressed as follows: $\mathcal{K}_M = p_1^*(\mathcal{K}_B) \otimes p_2^*(\mathcal{K}_{\mathbf{P}^1})$, hence, numerically one has: $[\mathcal{K}_M] \equiv (2g-2)[\mathbf{P}^1] + (-2)[B]$

Using Lemma 2.2, we get that Z is an irreducible curve. The arithmetic genus of Z can easily be computed as Lemma 2.2 implies immediately that $[Z][\mathbf{P}^1] = [Z][B] = 1$, hence $[Z] \equiv [B] + [\mathbf{P}^1]$. Eventually, we get $g_{aritm}(Z) = g$ (we recall that we denoted by g the genus of the base B).

Let now Z^* be the normalised of Z; then $g(Z^*) \le g(Z)$. As the normalisation morphism $Z^* \to Z$ is surjective, we get by composing with: $pr_{1|Z} : Z \to B$, a surjection $Z^* \to B$. As $g(Z^*) \le g(B)$, we get that $Z^* \simeq B$, and in this way we derived a contradiction, as by Lemma 2.2, there would exist a holomorphic bijection between B and $\mathbf{P}^1$, absurde.

It remains to prove the Lemmas.

Proof of Lemma 2.1.

First i). Assume $\pi_*(E \otimes L) \neq 0$; choosing $\mathcal{H} \in Pic(B)$ ample, we get that there exists $n \gg 0$ such that $\pi_*(E \otimes L) \otimes \mathcal{H}^{\otimes n}$ has nontrivial global sections. But then, keeping into account the projection formula, we would get that $E \otimes L \otimes \pi^*(\mathcal{H})^{\otimes n}$ has nontrivial global sections, hence E would be filtrable.

For ii), first notice that as the dualising sheaf $\omega_{X|B}$ is trivial, we get by relative duality and by the point i), that the rank of $\mathcal{R}^1\pi_*(E \otimes L)$ is zero. To compute its length, use the Leray spectral sequence

$$H^p(B; \mathcal{R}^q\pi_*(E \otimes L)) \Rightarrow H^{p+q}(X; E \otimes L)$$

deducing:

$$0 \to H^1(B; \pi_*(E \otimes L)) \to H^1(X; E \otimes L) \to H^0(B; \mathcal{R}^1\pi_*(E \otimes L)) \to 0$$

and keeping into account the i) point and the Riemann-Roch formula for E, we are done.

Proof of Lemma 2.2.

First pr_2. To show it, it suffices to show that $\forall \tilde{L} \in \mathbf{P}^1$, $\exists! b \in B$ such that $(b, \tilde{L}) \in Z$. But $(b, \tilde{L}) \in Z$ if and only if $(b, L) \in \tilde{Z}$, hence, according to the Theorem 1.3, that is equivalent to

$$H^0(\tilde{E} \otimes U_{|\tilde{\pi}^{-1}(b,L)}) \neq 0.$$

Under canonical identifications, $\tilde{E} \otimes U_{|\tilde{\pi}^{-1}(b,L)}$ becomes $E \otimes L_{|\pi^{-1}(b)}$ and the existence of b is proved.

To show the uniqueness, first observe that if $L' \in \tilde{L}$, then $L' \simeq L \otimes \pi^*(\mathcal{L})$, or $L' \simeq L^\vee \otimes \pi^*(\mathcal{L})$, and as the determinant of E is trivial, we get that $E \otimes L_{|\pi^{-1}(b)} \neq 0$ if and only if $E \otimes L'_{|\pi^{-1}(b)} \neq 0$.

Now pr_1. We first show the surjectivtiy. Assume this map would not be surjective, and choose some $b_0 \in B$ which does not belong to its image. Then for this point we would have

$$\forall L \in Pic(X), H^0(E \otimes L_{|\pi^*(b_0)}) = 0$$

there would exists some $\xi \in Pic^0(F_{b_0})$ such that $H^0(E \otimes L_{|\pi^*(b_0)} \otimes \xi) = 0$, finding a "lift" $I \in Pic(X)$ for it, and using Riemann-Roch, we get that $\exists I \in Pic(X)$ such that $H^0(X; E \otimes I) \neq 0$, hence E is filtrable.

Now the injectivity of the considered map. We show that if $b \in B$ is arbitrary, and $\tilde{L}$, $\tilde{L}'$ are such that $(b, \tilde{L}) \in Z$, $(b, \tilde{L}') \in Z$, then $\tilde{L} = \tilde{L}'$ To do this, first notice that the restriction of some nonfiltrable E to some arbitrary fiber cannot be of type iii) from the theorem of Atiyah. Indeed, if this would hapen, then forall $L \in Pic^0(X)$, the restriction of $E \otimes L$ to that fiber would have nontrivial sections; we would get that the considered map would have only the projection (through π) of that fiber as image, absurde (as we already proved that it is surjective). Now using the anterior observation we get that $L_{|F_{b_0}} \simeq L'_{|F_{b_0}}$ or $L^\vee_{|F_{b_0}} \simeq L'_{|F_{b_0}}$. Finally, keeping into account Theorem 1.1 (more specfically the exactness in the middle of the sequence from Theorem 1.1) we get there exists $\mathcal{L} \in Pic(B)$ such that $L \simeq L' \otimes \pi^*(\mathcal{L})$ or $L^\vee \simeq L' \otimes \pi^*(\mathcal{L})$, QED.

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