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A BOUND OF THE DEGREE OF SOME RATIONAL SURFACES IN $\mathbf{P}^4$ *

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Abstract

In this paper we find a bound of the degree of rational surfaces embedded in $\mathbf{P^4}$ with a linear system of type

$$\mid \mathcal{L} - px_0 - x_1 - \ldots - x_r \mid$$

We determine all the possible (families of) rational surfaces embedded in $\mathbf{P^4}$ with a linear system as above, for the particular case $p = 2$.

1 Introduction

A few years ago, Ellingsrud and Peskine ([EP]) proved that there are only finitely many families of surfaces embedded in $\mathbf{P^4}$, not of general type. Then, in particular, it is theoretically possible to find all the rational surfaces embedded in $\mathbf{P^4}$.

A natural point of view of the classification is to determine all the surfaces with a given degree. For the moment, there have been classified the (families of) rational surfaces in $\mathbf{P^4}$ with degree ≤ 10; all these surfaces can be represented as blow-ups of the plane $\mathbf{P^2}$.

The starting point of this paper is to give another point of view in the classification: our idea is to determine all the surfaces with a given type for the linear system of hyperplane sections. In [V] we give a complete classification of rational surfaces $S \subset \mathbf{P^4}$ embedded with a linear system of type

$$\mid f^*\mathcal{L} - E_1 - \ldots - E_r \mid,$$

where $\mathcal{L}$ is an ample linear system on a minimal rational surface, f is a birational morphism and $E_1,\ E_2, \ldots,\ E_r$ are the exceptional curves of f. In

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the present paper we continue this classification: we prove that there are at most five (families of) rational surfaces in $\mathbf{P^4}$ embedded with a linear system of type

$$| f^*\mathcal{L} - 2E_0 - E_1 - \ldots - E_r |,$$

where $f : S \to S_{min}$ is a birational morphism and $\mathcal{L}$ is an ample linear system on S_{min}.

A second important problem in the classification of rational surfaces in $\mathbf{P^4}$ is to determine an explicit bound of their degree. An actual bound is 66, but all the known (families of) rational surfaces in $\mathbf{P^4}$ have degree ≤ 12. In this direction, we prove that every rational surface embedded in $\mathbf{P^4}$ with a linear system of type

$$| f^*(\mathcal{L}) - pE_0 - E_1 - \ldots - E_r |,$$

where f is a birational morphism on a minimal rational surface, have degree ≤ 14.

2 Preliminaries and notations

In this paper, **surface** means a projective, smooth, irreducible, non-degenerate rational surface $S \subset \mathbf{P^4}$.

We will use standard notations, as, for instance, those in ([H]). For a surface S we denote by $d = d(S)$ and $\pi = \pi(S)$ the degree and the sectional genus, respectively. Recall that

$$d = (H^2) \text{ and } 2\pi - 2 = d + (H.K),$$

where H and K are a hyperplane section and the canonical divisor of S, respectively.

Every rational surface is a blow-up of $\mathbf{P^2}$ or of one of the Hirzebruch surfaces $\mathbf{F_n}$, for $n \neq 1$ (see, e.g., [H]).

The numerical invariants of a surface $S \subset \mathbf{P^4}$ verify double point formula:

$$d^2 - 10.d + 12\chi(O_S) = 5(H.K) + 2(K^2)$$

(see, e.g., [H]) For a rational surface, double point formula becomes:

$$d^2 - 10d + 12 = 5(H.K) + 2(K^2).$$

We will use the following result:

PROPOSITION 1 *(Voica, [V]). Let S be a rational surface in $\mathbf{P^4}$. If $d \geq 13$, then $\pi \leq \frac{d^2}{8}$; if $d \geq 9$, then $\pi \leq \frac{d(d-3)}{6}$; if $d \geq 7$, then $\pi \leq 1 + \frac{d(d-3)}{6}$; if $d = 6$, then $\pi \leq 3$; in any case, $\pi \leq 1 + \frac{d(d-1)}{4}$.*

3 The main result

Let S be a rational surface in $\mathbf{P^4}$.

Suppose that S is a blow-up of $\mathbf{P^2}$ such that the linear system of the hyperplane sections of S is of type

$$\mid H \mid = \mid f^*(mL) - pE_0 - E_1 - \ldots - E_r \mid,$$

where $m > 0$ and $E_0,\ E_1, \ldots,\ E_r$ are the exceptional curves of the blow-up f. In this case, the numerical invariants of S are:

$$d = (H^2) = m^2 - p^2 - r,\ (H.K) = -3m + p + r,\ (K^2) = 8 - r$$

and double point formula becomes:

$$d^2 - 10.d - 4 = -15m + 5p + 2r$$

We denote:

$$\alpha = d + r = m^2 - p^2 \text{ and } \beta = 3m - p.$$

Using this notation we obtain that

$$d^2 - 7d - 4 = 3\alpha - 5\beta \tag{1}$$

and

$$2\pi - 2 = 3\alpha - 5\beta.$$

For $d \geq 13$ we have

$$m^2 - p^2 = d + r \geq 14$$

and

$$m > p\ ,\ m \geq 4 \text{ and } \beta \geq 2m + 1 \geq 9.$$

Observe that

$$\alpha = -8m^2 + 6m\beta - \beta^2.$$

From double point formula (1) we obtain that

$$24m^2 - 18m.\beta + d^2 - 7d - 4 + 3\beta^2 + 5\beta = 0.$$

We consider this equality as a quadratic equation in m; then

$$\Delta = 9\beta^2 - 120\beta - 24d^2 + 168d + 96 \geq 0 \tag{2}$$

We write the above inequality (2) as

$$(3\beta - 20)^2 \geq 6(2d - 7)^2 + 10$$

and we obtain that
$$\beta > \frac{2\sqrt{6}}{3}d.$$
Because
$$8\pi \leq d^2 \text{ for } d \geq 13,$$
we deduce
$$d^2 - 7d - 4 = 3(\alpha - \beta) - 2\beta \leq 6(\frac{d^2}{8} - 1) - 2\beta$$
and then
$$d^2 - 4d(7 - \frac{4\sqrt{6}}{3}) + 2 < 0. \tag{3}$$
Using (3) we see that
$$d \leq 14.$$

Suppose now that S is a blow-up of a minimal Hirzebruch rational surface $\mathbf{F_n}, \mathbf{n} \neq \mathbf{1}$. We denote by C anf F the unique section with negative self-intersection on $\mathbf{F_n}$ and a fiber, respectively. Let
$$\mid H \mid = \mid f^*(aC + bF) - pE_0 - E_1 - \ldots - E_r \mid$$
be the linear system of hyperplane sections on S. Recall that
$$(C^2) = -n,\ (F.C) = 1,\ (F^2) = 0,\ a > 0 \text{ and } b > an$$
(see, e.g., [H]). Then the numerical invariants of S are:
$$d = -na^2 + 2ab - p^2 - r,\ (H.K) = an - 2a - 2b + p + r,$$
$$(K^2) = 7 - r,\ \pi = -na^2 + 2ab - 2a - 2b - p^2 + p.$$
Denote
$$\alpha = 2b - na,\ \beta = 2a;$$
using this notation, we obtain that
$$d = \frac{\alpha.\beta}{2} - p^2 - r,\ (H.K) = -\alpha - \beta + p + r,\ 4\pi = (\alpha - 2)(\beta - 2) + 2p - 2p^2.$$
From double point formula we have:
$$2d^2 - 14d - 4 = 3\alpha\beta - 10(\alpha + \beta) + 10p - 6p^2.$$
First of all, observe that $\alpha \geq 2$ and $\beta \geq 2$: in the contrary case, the sectional genus can not be a positive integer. For $d \geq 13$ we have
$$9(\alpha - 2)(\beta - 2) - 12(\alpha + \beta - 4) \leq 36\pi - 48\sqrt{\pi} + 9p^2 - 9p$$

and we can use Proposition (1) to obtain that

$$6d^2 - 42d + 80 + 18p^2 - 30p \leq \frac{9d^2}{2} + 9p^2 - 9p - 24\sqrt{\frac{d^2}{2} + p^2 - p}.$$

From the above inequality we have

$$3d^2 - d(84 - 24\sqrt{2}) + 160 + 36p^2 - 60p \leq 0. \tag{4}$$

If $p = 1$, in [V] we prove that $d \leq 5$; for $p \geq 2$ we have

$$18p^2 - 30p + 80 \geq 92$$

and we use (4) to obtain that $d \leq 14$.

All these considerations prove

PROPOSITION 2 . *Let $S \subset \mathbf{P^4}$ be a rational surface such that the hyperplane section of S is of type*

$$\mid f^*(\mathcal{L}) - pE_0 - E_1 - \ldots - E_r \mid,$$

where $f : S \to S_{min}$ is a blow-up morphism and $E_0, E_1, \ldots, E_r$ are the exceptional curves of f.

Then $deg(S) \leq 14$.

4 A class of rational surfaces in $\mathbf{P^4}$

Let S be a rational surface in $\mathbf{P^4}$, let $f : S \to S_{min}$ be a birational morphism and let $E_0,\ E_1, \ldots, E_r$ be the exceptional curves of f. The aim of this section is to classify all the surfaces as above if the hyperplane section of S is of type

$$\mid f^*(\mathcal{L}) - 2E_0 - E_1 - \ldots - E_r \mid .$$

From Proposition 2 we know that $deg(S) \leq 14$.

First of all, suppose that $S_{min} = \mathbf{P^2}$ and let $\mathcal{L} = mL$. The numerical invariants of S are:

$$d = m^2 - 4 - r,\ (H.K) = -3m + r + 2,\ \pi = \frac{m(m-3)}{2}.$$

Using this relation, double point formula becomes:

$$r^2 - r(2m^2 - 15) + (m^4 - 18m^2 + 15m + 42) = 0.$$

Consider this equality as a equation; we obtain that there exists an integer k such that

$$k^2 = 12m^2 - 60m + 57$$

or, equivalently, that

$$k^2 = 3(2m-5)^2 - 18.$$

Observe that 3 must divide k and denote $k = 3y$ and $2m - 5 = x$. Then

$$3x^2 - y^2 = 2 \tag{5}$$

and

$$m = \frac{3x+5}{2},\ r = \frac{2m^2 - 15 \pm 3y}{2},\ d = \frac{7 \mp 3y}{2}. \tag{6}$$

Since $\mathbf{Z}[\sqrt{3}]$ is an Euclidean ring, we know (see, e.g., [AI]) that the solutions (x, y) of equation (5) verify:

$$\mid y \mid + \mid x \mid \sqrt{3} = (1+\sqrt{3})(2+\sqrt{3})^n, \tag{7}$$

for a natural number n. Since

$$m > \sqrt{d+5} \text{ and } 3 \leq d \leq 14$$

we obtain that

$$1 \leq x \text{ and } 1 \leq y \leq 7.$$

Then the acceptable solutions of (5) are:

$$x = 1,\ y = 1 \text{ or } x = 3,\ y = 5.$$

The numerical invariants of the corresponding surfaces are

$$d = 5,\ \pi = 2,\ m = 4,\ r = 7$$

and, respectively,

$$d = 11,\ \pi = 14,\ m = 7,\ r = 34.$$

Suppose now that $S_{min} = \mathbf{F_n},\ n \neq 1$ and let $\mathcal{L} = aC + bF$. With the notations

$$\alpha = 2b - na,\ \beta = 2a$$

used also in the above section, the numerical invariants of S are:

$$d = \frac{\alpha\beta}{2} - 4 - r,\ (H.K) = -\alpha - \beta + 2 + r$$

and

$$4\pi = (\alpha - 2)(\beta - 2) - 4.$$

From double point formula we obtain

$$2d^2 - 14d = 3\alpha\beta - 10(\alpha + \beta)$$

or, equivalently,

$$6d^2 - 42d + 100 = (3\alpha - 10)(3\beta - 10).$$

It will be sufficient to assign to d all the values from 3 to 14, to compute the possible values of $\alpha,\ \beta$ and π and to use Proposition 1 in order to decide if it is possible that the corresponding surface exists. Note that $\alpha \geq 3,\ \beta \geq 4$ and that $3\alpha - 10$ and $3\beta - 10$ are $\equiv 2(mod\ 3)$. In addition, we have $2\alpha > n\beta$ and $n \neq 1$. The numerical invariants of the corresponding surfaces are:

$$d = 6,\ \pi = 3,\ r = 8;$$

$$d = 10,\ \pi = 11,\ r = 26$$

and

$$d = 11,\ \pi = 14,\ r = 33.$$

The above considerations prove:

PROPOSITION 3 . *There are at most five (families of) rational surfaces $S \subset \mathbf{P^4}$ embedded with a linear system of type*

$$\mid f^*(\mathcal{L}) - 2E_0 - E_1 - \ldots - E_r \mid .$$

Remark. We do not know if some one of the found (families of) surfaces exists; we can not decide the inexistence using only the inequalities between the sectional genus and the degree of a rational surface proved in [V].

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