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A note on conformal mappings onto mutually non-overlapping domains

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Abstract

In the paper, an approach is proposed that allowed to establish new upper estimates for products of inner radii of mutually non-overlapping domains.

1 Preliminaries

Let $\mathbb{N}$, $\mathbb{R}$ be the sets of natural and real numbers, respectively, $\mathbb{C}$ be the complex plane, $\overline{\mathbb{C}} = \mathbb{C} \bigcup \{\infty\}$ be its one point compactification, U be the open unit disk in $\mathbb{C}$, $\mathbb{R}^+ = (0, \infty)$.

Let $B \subset \overline{\mathbb{C}}$ be a simply connected domain and a point $a \in B$. According to the Riemann mapping theorem, there is a univalent and conformal mapping f of the domain B onto the unit disk W with center at the origin for which $f(a) = 0$, $f'(a) = 1$. The radius of the circle W is called a conformal radius of the domain B at a point a and is denoted by $R(B, a)$. The concept of conformal radius can be introduced also in the following equivalent way: let the mapping φ perform a conformal and univalent mapping of the unit disk U onto the domain B such that $\varphi(0) = a$, then the concept of the conformal radius of a simply connected domain $B \subset \overline{\mathbb{C}}$ with respect to a point $a \in B$ is defined as

$$R(B, a) = |\varphi'(0)|.$$

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For the case of multiply connected domains, the concept of the conformal radius is analogous to the concept of an inner radius.

Green's function $g_B(z,a)$ of the domain B with a pole at the point $a \in B$ is a real function that is harmonic in z in $B \backslash \{a\}$ and tends to zero when z approaches the boundary B. This function has a correct asymptotic expansion

$$g_B(z,a) = -\ln|z-a| + \delta + o(1), \quad o(1) \to 0, \quad z \to a,$$

in a certain vicinity of the point a. If $a = \infty$ the correct expansion looks like

$$g_B(z,\infty) = \ln|z| + \delta + o(1), \quad o(1) \to 0, \quad z \to \infty.$$

The inner radius $r(B,a)$ of the domain B with respect to the point a is the quantity e^{δ} (see, e.g., [1, 2, 3, 4, 5, 6]).

Note that at the point $a = \infty$, this definition of the inner radius coincides with the definitions given in works [5, 6], in contrast, for example, to works [1, 17].

For a simply connected domain, its inner radius coincides with its conformal radius.

Let B be a domain of the extended complex plane $\overline{\mathbb{C}}_z$. By a quadratic differential in B we mean the expression

$$Q(z)dz^2, \tag{1}$$

where $Q(z)$ is a meromorphic function in B (see, e.g., [3, 5, 6]).

A finite point $z_0 \in B$ is called a zero or a pole of order n of the differential (1) if it is a zero or a pole, respectively, of the function $Q(z)$.

A circle domain for quadratic differential $Q(z)dz^2$ is called simply connected domain $B \subset \mathbb{C}_z$, containing a unique double pole of the quadratic differential $Q(z)dz^2$ at the point $w = a \in B$, such that at univalent conformal mapping $w = f(z)$ $(f(a) = 0)$ the domain B onto the unit disk of the plane $\mathbb{C}_w$, the identity holds

$$Q(z)dz^2 \equiv -k\frac{dw^2}{w^2}, \quad k \in \mathbb{R}^+.$$

Let E be a bounded infinite closed set on $\mathbb{C}$. Let

$$V(z_1, z_2, ..., z_n) = \prod_{\substack{k,l=1 \\ k<l}}^{n} (z_k - z_l),$$

where $n \geqslant 2$ and $z_1, z_2, ..., z_n \in E$. Let $V_n = V_n(E)$ be the maximum value of the module $|V(z_1, z_2, ..., z_n)|$, where $z_1, z_2, ..., z_n$ run through various systems of n points belonging to the set E. Let us denote

$$d_n = V_n^{\frac{2}{n(n-1)}}.$$

The quantity $d(E) = \lim\limits_{n\to\infty} d_n$ is called by the transfinite diameter of the set E. For example, the transfinite diameter of any circle is equal to its radius, and the transfinite diameter of any straight segment is equal to a quarter of its length [1].

For a compact set $E \subset \mathbb{C}$, its logarithmic capacity is determined by the equality

$$\operatorname{cap} E := \frac{1}{r(\overline{\mathbb{C}}\backslash E, \infty)},$$

if the value of $r(\overline{\mathbb{C}}\backslash E, \infty)$ is finite; otherwise, $\operatorname{cap} E := 0$.

It is known [1] that the logarithmic capacity $\operatorname{cap} E$ coincides with the transfinite diameter $d(E)$ of the set E.

In this work, when proving auxiliary statements, the area-minimization theorem is used.

Theorem 1. [1] In the family of all functions $F(z)$, $F(0) = 0$, $F'(0) = 1$, which are regular in a given simply connected domain B that contains the point $z = 0$ and has more than one limit point, the minimum of the quantity

$$H(F) = \iint\limits_{B} |F'(z)|^2 d\sigma,$$

where $d\sigma$ is an element of the area, is achieved at a unique function that univalently maps the domain B onto the full disk $|z| < R$, and this minimum is equal to the conformal radius R of the domain B at the point $z = 0$.

In monograph [1], the problem of minimizing the area of multiply connected domains is also considered.

In the work [7] dated 1934, M.A. Lavrentiev, in particular, solved the problem on the maximum of the product of the conformal radii of two non-overlapping simply connected domains. Namely, the following result is valid.

Theorem 2. [7] *Let a_1 and a_2 be some fixed points of the complex plane $\mathbb{C}$ and B_k, $a_k \in B_k$, $k \in \{1, 2\}$, be an arbitrary non-overlapping simply connected domains in $\mathbb{C}$. Then the following inequality holds:*

$$R(B_1, a_1)R(B_2, a_2) \leqslant |a_1 - a_2|^2, \tag{2}$$

where the equality is achieved for the half-planes B_k and the points a_k that are symmetric with respect to their common boundary.

Later (see, e.g., [8]), the result obtained by M.A. Lavrentiev was generalized to the case of meromorphic functions. Then, for any non-overlapping domains $B_1 \subset \overline{\mathbb{C}}$ and $B_2 \subset \overline{\mathbb{C}}$, inequality (2) survives, and the equality sign is reached if the domains B_1 and B_2 look like

$$B_1 = \left\{ w \in \overline{\mathbb{C}} : \left| \frac{w - a_1}{w - a_2} \right| < \rho \right\}, \quad B_2 = \left\{ w \in \overline{\mathbb{C}} : \left| \frac{w - a_1}{w - a_2} \right| > \rho \right\},$$

where $\rho \in \mathbb{R}^+$. An example of such a domain configuration is the case where one of the domains is bounded by a circle of the above type and the other domain is unbounded, i.e., it is a supplement to the first domain. It should be noted that in this case the family of extremals has the continual capacity.

Further generalizations of the M.A. Lavrentiev problem were made via the increase in the domain number, the refusal to fix the poles of quadratic differentials, the change to an extended complex plane, and the expansion of the analyzed object: the non-overlapping domains are replaced by some class of open sets or partially overlapping domains (see, e.g., [5, 6, 9, 10, 11, 12, 13]). In particular, in the work [14] (see also [15]) the following result was obtained.

Theorem 3. [14, 15] *Let $n \in \mathbb{N}$, $n \geqslant 2$, $a_k \in \mathbb{C}$, $B_k \subset \overline{\mathbb{C}}$, $k = \overline{1,n}$, be, respectively, some set of fixed points and domains of the complex plane such that $a_k \in B_k$, $k = \overline{1,n}$, $B_i \cap B_j = \emptyset$, $i \neq j$. Then the following inequality holds:*

$$\prod_{k=1}^{n} r(B_k, a_k) \leqslant (n-1)^{-\frac{n}{4}} \left(\prod_{1 \leqslant p < k \leqslant n} |a_p - a_k| \right)^{\frac{2}{n-1}}. \tag{3}$$

This paper is devoted to obtaining effective upper estimates for the functional of the following type

$$J_n(\gamma) = \left[r\left(B_0, 0\right) r\left(B_{n+1}, \infty\right) \right]^{\gamma} \prod_{k=1}^{n} r\left(B_k, a_k\right), \tag{4}$$

where $n \in \mathbb{N}$, $\gamma \in \mathbb{R}^+$, $A_n = \{a_k\}_{k=1}^n$ is an arbitrary fixed system of points of the complex plane $\mathbb{C}\backslash\{0\}$; B_0, B_{n+1}, $\{B_k\}_{k=1}^n$ is an arbitrary system of mutually non-overlapping domains such that $a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, $\infty \in B_{n+1} \subset \overline{\mathbb{C}}$, $a_k \in B_k \subset \overline{\mathbb{C}}$ at $k = \overline{1,n}$.

In 1988 for $\gamma = \frac{1}{2}$ and $n \geqslant 2$ estimate of the functional $J_n(\gamma)$ for the system of non-overlapping domains by the method of symmetrization when the points lie on a unit circle was found by V.N. Dubinin [16]. Using the extremal metric method G.V. Kuz'mina [17] for simply connected domains strengthened this result of V.N. Dubinin [16] and showed that the estimate is correct for $\gamma \in \left(0, \frac{n^2}{8}\right]$, $n \geqslant 2$. However, for the case $n = 2$ G.V. Kuz'mina's result completely coincides with the result of the paper [16]. G.V. Kuz'mina [17] also notes that the upper bounds for the parameter γ can be improved. Therefore, the final question about the estimate for γ for the functional (4) remains open. The functional $J_n(\gamma)$ was considered, for example, in the papers [5, 6, 17, 18], in which for $J_n(\gamma)$ in particular cases for some values of γ, the following inequality was established

$$J_n(\gamma) \leqslant \left(\frac{4}{n}\right)^n \frac{\left(\frac{4\gamma}{n^2}\right)^{\frac{2\gamma}{n}}}{\left|1-\frac{4\gamma}{n^2}\right|^{\frac{2\gamma}{n}+\frac{n}{2}}} \left|\frac{n-2\sqrt{\gamma}}{n+2\sqrt{\gamma}}\right|^{2\sqrt{\gamma}}.$$

Equality in this inequality is achieved when the points 0, ∞, a_k and the domains B_0, B_∞, B_k, $k=\overline{1,n}$, are, respectively, poles and circular domains of the quadratic differential

$$Q(w)dw^2 = -\frac{\gamma w^{2n} + (n^2-2\gamma)w^n + \gamma}{w^2(w^n-1)^2}\, dw^2.$$

The method proposed in this paper originates from the paper [19], where the problem of finding the maximum for the product of the inner radii of three mutually non-overlapping domains was considered under the additional condition of symmetry of two of them with respect to the unit circle and the power exponent $\gamma = 1$ for the inner radius of the domain with respect to the coordinate origin. The ideas proposed in [19] were substantially generalized in the works [14, 15, 18, 20, 21, 22, 23, 24, 25, 26, 27].

2 Upper estimate of the functional $J_n(\gamma)$

The following proposition is true.

Theorem 4. *Let $n \in \mathbb{N}$, $\gamma \in \mathbb{R}^+$. Then for any fixed system of different points $A_n = \{a_k\}_{k=1}^n \in \mathbb{C}\backslash\{0\}$ and any set of mutually non-overlapping domains B_0, B_{n+1}, $\{B_k\}_{k=1}^n$, $a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, $a_{n+1} = \infty \in B_{n+1} \subset \overline{\mathbb{C}}$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1,n}$, and besides there exists $\varepsilon > 0$ such that $r\left(B_k, a_k\right) \geqslant \varepsilon$ for an arbitrary $k = \overline{1,n}$, the following inequality holds:*

$$J_n(\gamma) \leqslant \varepsilon^{-\frac{2n\gamma}{n+2}} (n+1)^{-\frac{\gamma(n+1)}{n+2}} \left(\prod_{k=1}^n |a_k|\right)^{\frac{2\gamma}{n+2}} \prod_{k=1}^n r\left(B_k, a_k\right). \tag{5}$$

Proof. First, we prove two auxiliary lemmas. Let us find an estimate for the expression $r\left(B_0, 0\right)$. The following lemma is valid.

Lemma 1. *Let $n \in \mathbb{N}$, $\{a_k\}$, $\{B_k\}$, $k = \overline{0, n+1}$, are the same as in the Theorem 4. Then the following inequality holds:*

$$r\left(B_0, 0\right) \leqslant \varepsilon^{-\frac{n}{n+1}} \frac{\left(\prod\limits_{k=1}^n |a_k|\right)^{\frac{2}{n+1}}}{(n+1)^{\frac{1}{2}} \left(r\left(B_{n+1}, \infty\right)\right)^{\frac{1}{n+1}}}. \tag{6}$$

Proof. Consider the mapping $w = \frac{1}{z}$ and let $B^{+} = \left\{z : \frac{1}{z} \in B\right\}$. Since the inner radius of the domain containing an infinitely distant point is reciprocal to the transfinite diameter of the complement to this domain (see [1, 6]), then

$$r\left(B_0, 0\right) = r\left(B_0^{+}, \infty\right) = \frac{1}{d\left(\overline{\mathbb{C}} \setminus B_0^{+}\right)}. \tag{7}$$

According to the Polya theorem [4], the following inequality is valid:

$$\mu E \leqslant \pi d^2(E),$$

where μE is the Lebesgue measure of the compact set E. Whence we obtain

$$d(E) \geqslant \left(\frac{1}{\pi}\mu E\right)^{\frac{1}{2}}.$$

Thus,

$$\frac{1}{d\left(\overline{\mathbb{C}} \setminus B_0^{+}\right)} \leqslant \frac{1}{\sqrt{\frac{1}{\pi}\mu\left(\overline{\mathbb{C}} \setminus B_0^{+}\right)}}.$$

Taking the monotonicity and additivity of the Lebesgue measure into account, we have

$$\frac{1}{\sqrt{\frac{1}{\pi}\mu\left(\overline{\mathbb{C}} \setminus B_0^{+}\right)}} \leqslant \frac{1}{\sqrt{\frac{1}{\pi}\mu\left(\bigcup\limits_{k=1}^{n+1} \overline{B}_k^{+}\right)}} = \frac{1}{\sqrt{\frac{1}{\pi}\sum\limits_{k=1}^{n+1} \mu\overline{B}_k^{+}}}.$$

Then, from (7), we obtain

$$r\left(B_0, 0\right) \leqslant \left(\frac{1}{\pi}\sum_{k=1}^{n+1} \mu\overline{B}_k^{+}\right)^{-\frac{1}{2}}. \tag{8}$$

From the area-minimization theorem [1] (see also Theorem 1), it follows that

$$\mu(B) \geqslant \pi r^2\left(B, a\right).$$

From inequality (8), it follows immediately that

$$r\left(B_0, 0\right) \leqslant \left[\frac{1}{\pi}\sum_{k=1}^{n+1} \mu\overline{B}_k^{+}\right]^{-\frac{1}{2}} \leqslant \left[\frac{1}{\pi}\sum_{k=1}^{n+1} \mu B_k^{+}\right]^{-\frac{1}{2}} \leqslant \left[\sum_{k=1}^{n+1} r^2\left(B_k^{+}, a_k^{+}\right)\right]^{-\frac{1}{2}}.$$

It directly follows from this

$$r\left(B_0,0\right)\leqslant\frac{1}{\left[\sum\limits_{k=1}^{n+1}r^2\left(B_k^+,a_k^+\right)\right]^{\frac{1}{2}}}. \tag{9}$$

Let us find the values $r\left(B_k^+,a_k^+\right)$ for $k=\overline{1,n}$. Note that

$$g_{B_k}(z,a_k)=-\ln|z-a_k|+\ln r(B_k,a_k)+o(1),\ o(1)\to 0,\ z\to a.$$

Taking advantage of the Green function invariance at conformal and univalent mapping, we have

$$g_{B_k}(z,a_k)=g_{B_k^+}(w^+,a_k^+),\quad w^+=\frac{1}{z}.$$

Then

$$g_{B_k^+}(w^+,a_k^+)=g_{B_k^+}\left(\frac{1}{z},\frac{1}{a_k}\right)=\ln\frac{1}{|\frac{1}{z}-a_k^+|}+\ln r(B_k^+,a_k^+)+o(1).$$

Using simple transformations, we get

$$g_{B_k^+}(w^+,a_k^+)=\ln\frac{1}{|z-a_k|}+\ln|a_k|^2 r(B_k^+,a_k^+)+o(1).$$

Hence,

$$r\left(B_k^+,a_k^+\right)=\frac{r\left(B_k,a_k\right)}{|a_k|^2}. \tag{10}$$

Let us find now $r\left(B_{n+1}^+,a_{n+1}^+\right)$. Since $a_{n+1}=\infty$, then from the invariance of the Green function at conformal and univalent mapping, we obtain

$$g_{B_{n+1}}(z,\infty)=g_{B_{n+1}^+}(w,0)=\ln\frac{1}{|w|}+\ln r(B_{n+1}^+,0)+o(1)=$$

$$=\ln|z|+\ln r(B_{n+1}^+,0)+o(1),$$

and we arrive at the following inequality:

$$r(B_{n+1}^+,0)=r(B_{n+1},\infty). \tag{11}$$

Substituting the equalities (10) and (11) into the inequality (9), we obtain:

$$r\left(B_0,0\right)\leqslant\left[r^2\left(B_{n+1},\infty\right)+\sum_{k=1}^{n}\frac{r^2\left(B_k,a_k\right)}{|a_k|^4}\right]^{-\frac{1}{2}}.$$

From the Cauchy inequality of arithmetic and geometric means, we deduce

$$r^2\left(B_{n+1},\infty\right)+\sum_{k=1}^{n}\frac{r^2\left(B_k,a_k\right)}{|a_k|^4}\geqslant$$
$$\geqslant(n+1)\left(r\left(B_{n+1},\infty\right)\prod_{k=1}^{n}\frac{r\left(B_k,a_k\right)}{|a_k|^2}\right)^{\frac{2}{n+1}},$$

whence it is easy to obtain that

$$r\left(B_0,0\right)\leqslant\left(\prod_{k=1}^{n}r\left(B_k,a_k\right)\right)^{-\frac{1}{n+1}}\frac{\left(\prod\limits_{k=1}^{n}|a_k|\right)^{\frac{2}{n+1}}}{(n+1)^{\frac{1}{2}}\left(r\left(B_{n+1},\infty\right)\right)^{\frac{1}{n+1}}}.$$

Taking into account that according to the condition of the Theorem 4 for $k=\overline{1,n}$ the inequality $r\left(B_k,a_k\right)\geqslant\varepsilon$ holds, we obtain

$$\left(\prod_{k=1}^{n}r\left(B_k,a_k\right)\right)^{-\frac{1}{n+1}}\leqslant\varepsilon^{-\frac{n}{n+1}}.$$

Substituting this expression into the previous inequality, we obtain the inequality (6). □

Let us now find an estimate for the expression $r\left(B_{n+1},\infty\right)$. The following lemma is true.

Lemma 2. *Let $n\in\mathbb{N}$, $\{a_k\}$, $\{B_k\}$, $k=\overline{0,n+1}$, are the same as in the Theorem 4. Then the following inequality holds:*

$$r\left(B_{n+1},\infty\right)\leqslant\frac{\varepsilon^{-\frac{n}{n+1}}}{(n+1)^{\frac{1}{2}}\left(r\left(B_0,0\right)\right)^{\frac{1}{n+1}}}. \tag{12}$$

Proof. By performing transformations similar to those performed in Lemma 1 for the quantity $r\left(B_{n+1},\infty\right)$, we obtain the inequality

$$r\left(B_{n+1},\infty\right)\leqslant\frac{1}{\left[\sum\limits_{k=0}^{n}r^2\left(B_k,a_k\right)\right]^{\frac{1}{2}}}.$$

Further, by the Cauchy inequality of arithmetic and geometric means, we have

$$\sum_{k=0}^{n}r^2\left(B_k,a_k\right)\geqslant(n+1)\left[\prod_{k=0}^{n}r\left(B_k,a_k\right)\right]^{\frac{2}{n+1}},$$

and therefore

$$r\left(B_{n+1},\infty\right)\leqslant\frac{\left(\prod\limits_{k=1}^{n}r\left(B_k,a_k\right)\right)^{-\frac{1}{n+1}}}{(n+1)^{\frac{1}{2}}\left(r\left(B_0,0\right)\right)^{\frac{1}{n+1}}}.$$

Considering that, according to the condition of the Theorem 4 for $k=\overline{1,n}$ the inequality $r\left(B_k,a_k\right)\geqslant\varepsilon$ is valid, thus

$$\left(\prod_{k=1}^{n}r\left(B_k,a_k\right)\right)^{-\frac{1}{n+1}}\leqslant\varepsilon^{-\frac{n}{n+1}}.$$

By substituting this expression into the previous inequality, we get the inequality (12). □

Let us proceed to prove the main result of the Theorem 4. Multiplying the inequalities (6) and (12), we obtain the following chain of inequalities:

$$r\left(B_0,0\right)r\left(B_{n+1},\infty\right)\leqslant\varepsilon^{-\frac{2n}{n+1}}\frac{\left(\prod\limits_{k=1}^{n}|a_k|\right)^{\frac{2}{n+1}}}{(n+1)\left(r\left(B_0,0\right)r\left(B_{n+1},\infty\right)\right)^{\frac{1}{n+1}}},$$

$$\left(r\left(B_0,0\right)r\left(B_{n+1},\infty\right)\right)^{\frac{n+2}{n+1}}\leqslant\varepsilon^{-\frac{2n}{n+1}}\frac{\left(\prod\limits_{k=1}^{n}|a_k|\right)^{\frac{2}{n+1}}}{n+1},$$

$$r\left(B_0,0\right)r\left(B_{n+1},\infty\right)\leqslant\varepsilon^{-\frac{2n}{n+2}}(n+1)^{-\frac{n+1}{n+2}}\left(\prod_{k=1}^{n}|a_k|\right)^{\frac{2}{n+2}}.$$

And from here the following inequality holds

$$\left[r\left(B_0,0\right)r\left(B_{n+1},\infty\right)\right]^{\gamma}\prod_{k=1}^{n}r\left(B_k,a_k\right)\leqslant$$

$$\leqslant\varepsilon^{-\frac{2n\gamma}{n+2}}(n+1)^{-\frac{\gamma(n+1)}{n+2}}\left(\prod_{k=1}^{n}|a_k|\right)^{\frac{2\gamma}{n+2}}\prod_{k=1}^{n}r\left(B_k,a_k\right),$$

which proves the inequality (5). □

Remark 1. Note that the condition $r\left(B_k,a_k\right)\geqslant\varepsilon$ for $k=\overline{1,n}$ is essential, since it allowed us to obtain the estimates $r\left(B_0,0\right)$ and $r\left(B_{n+1},\infty\right)$ without any additional restrictions on γ.

3 Some consequences

Theorem 4 is quite general, however, if the points satisfy certain conditions, then more specific estimates can be written. The following theorem considers the case when all points a_k, $k=\overline{1,n}$, belong to the unit circle. In particular, the following result is correct.

Theorem 5. *Let $n,m\in\mathbb{N}$, $\gamma\in\mathbb{R}^+$. Then for any fixed system of different points $A_n=\{a_k\}_{k=1}^n\in\mathbb{C}\backslash\{0\}$ and any set of mutually non-overlapping domains B_0, B_{n+1}, $\{B_k\}_{k=1}^n$, such that $a_0=0\in B_0\subset\overline{\mathbb{C}}$, $a_{n+1}=\infty\in B_{n+1}\subset\overline{\mathbb{C}}$, $|a_k|=1$ and $a_k\in B_k\subset\overline{\mathbb{C}}$, $k=\overline{1,n}$, besides there exists $\varepsilon>0$ such that $r\left(B_k,a_k\right)\geqslant\varepsilon$ for an arbitrary $k=\overline{1,n}$, the following inequalities hold:*
if $n=2m$, then

$$J_n(\gamma)\leqslant 2^n\varepsilon^{-\frac{2n\gamma}{n+2}}(n+1)^{-\frac{\gamma(n+1)}{n+2}}\,(n-1)^{-\frac{n}{4}}\left(\prod_{k=1}^{m-1}\sin\frac{k\pi}{n}\right)^{\frac{2n}{n-1}}; \tag{13}$$

if $n=2m+1$, then

$$J_n(\gamma)\leqslant 2^n\varepsilon^{-\frac{2n\gamma}{n+2}}(n+1)^{-\frac{\gamma(n+1)}{n+2}}\,(n-1)^{-\frac{n}{4}}\left(\prod_{k=1}^{m}\sin\frac{k\pi}{n}\right)^{\frac{2n}{n-1}}. \tag{14}$$

Proof. Since the configuration of the domains B_k and the points a_k described in the condition of this theorem satisfies all conditions of the Theorem 4, then the inequality (5) is true for it. Since $|a_k|=1$, $k=\overline{1,n}$, then we note

$$\left(\prod_{k=1}^{n}|a_k|\right)^{\frac{2\gamma}{n+2}}=1. \tag{15}$$

Let us now evaluate the expression $\prod\limits_{k=1}^{n}r\left(B_k,a_k\right)$. Let for concreteness

$$0=\arg a_1<\arg a_2<\ldots<\arg a_n<2\pi.$$

Denote $\alpha_1:=\frac{1}{\pi}(\arg a_2-\arg a_1)$, $\alpha_2:=\frac{1}{\pi}(\arg a_3-\arg a_2)$, $\ldots$, $\alpha_n:=\frac{1}{\pi}(2\pi-\arg a_n)$. From here we get

$$\prod_{1\leqslant p<k\leqslant n}|a_p-a_k|=\prod_{1\leqslant p<k\leqslant n}2\sin\frac{\pi(\alpha_p+\ldots+\alpha_{k-1})}{2}. \tag{16}$$

Taking into account that $\sum\limits_{k=1}^{n}\alpha_k=2$ and using elementary calculations of the product written on the right-hand side of the inequality (16), we obtain that the maximum of this expression is achieved when all α_k are equal, and therefore for $n = 2m$ we obtain

$$\prod_{1\leqslant p<k\leqslant n}\left(2\sin\frac{\pi(\alpha_p+...+\alpha_{k-1})}{2}\right)\leqslant 2^{\frac{n^2-n}{2}}\left(\prod_{k=1}^{m-1}\sin\frac{k\pi}{n}\right)^{n}.$$

Taking the Theorem 3 and the inequality (3) into account, we have:

$$\prod_{k=1}^{n} r(B_k,a_k)\leqslant 2^n\,(n-1)^{-\frac{n}{4}}\left(\prod_{k=1}^{m-1}\sin\frac{k\pi}{n}\right)^{\frac{2n}{n-1}}. \tag{17}$$

Further, by substituting the inequalities (17) and (15) in the inequality (5), we obtain the inequality (13). If $n = 2m + 1$, then we get

$$\prod_{1\leqslant p<k\leqslant n}\left(2\sin\frac{\pi(\alpha_p+...+\alpha_{k-1})}{2}\right)\leqslant 2^{\frac{n^2-n}{2}}\left(\prod_{k=1}^{m}\sin\frac{k\pi}{n}\right)^{n},$$

therefore,

$$\prod_{k=1}^{n} r(B_k,a_k)\leqslant 2^n\,(n-1)^{-\frac{n}{4}}\left(\prod_{k=1}^{m}\sin\frac{k\pi}{n}\right)^{\frac{2n}{n-1}}. \tag{18}$$

Substituting the inequalities (18) and (15) in the inequality (5), we obtain the inequality (14). □

The following theorem considers the case when all points a_k, $k=\overline{1,n}$, belong to some straight line that is parallel to the imaginary axis.

Theorem 6. *Let $n\in\mathbb{N}$, $\gamma\in\mathbb{R}^+$. Then for any fixed system of different points $A_n=\{a_k\}_{k=1}^n\in\mathbb{C}\backslash\{0\}$, such that $a_k=1+iy_k$, where y_k, $k=\overline{1,n}$, are some real numbers, and any set of mutually non-overlapping domains B_0, B_{n+1}, $\{B_k\}_{k=1}^n$, $a_0=0\in B_0\subset\overline{\mathbb{C}}$, $a_{n+1}=\infty\in B_{n+1}\subset\overline{\mathbb{C}}$, $a_k\in B_k\subset\overline{\mathbb{C}}$, $k=\overline{1,n}$, besides there exists $\varepsilon>0$ such that $r\left(B_k,a_k\right)\geqslant\varepsilon$ for an arbitrary $k=\overline{1,n}$, the following inequality holds:*

$$J_n(\gamma)\leqslant\varepsilon^{-\frac{2n\gamma}{n+2}}(n+1)^{-\frac{\gamma(n+1)}{n+2}}\,(n-1)^{-\frac{n}{4}}\left(\prod_{k=1}^{n}(1+y_k^2)\right)^{\frac{\gamma}{n+2}}\left(\prod_{1\leqslant p<k\leqslant n}|y_p-y_k|\right)^{\frac{2}{n-1}}. \tag{19}$$

Proof. First, let us note that for $k = \overline{1,n}$ the equality $|a_k| = \sqrt{1 + y_k^2}$ holds, and therefore

$$\left(\prod_{k=1}^{n} |a_k|\right)^{\frac{2\gamma}{n+2}} = \left(\prod_{k=1}^{n} (1 + y_k^2)\right)^{\frac{\gamma}{n+2}}. \tag{20}$$

Further, taking into account that all points a_k, $k = \overline{1,n}$, belong to one straight line, then from inequality (3) we obtain

$$\prod_{k=1}^{n} r(B_k, a_k) \leqslant (n-1)^{-\frac{n}{4}} \left(\prod_{1 \leqslant p < k \leqslant n} |y_p - y_k|\right)^{\frac{2}{n-1}}. \tag{21}$$

Substituting the expressions (20) and (21) in the inequality (5), we obtain the inequality (19). □

Remark 2. Note that the case when the points a_k, $k = \overline{1,n}$, belong to some arbitrary straight line that does not pass through the origin can always be reduced to the Theorem 6 by some linear transformation.

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