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On some differential inclusions with anti-periodic solutions

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Abstract

In this paper, we investigate a class of second- and first-order differential inclusions, along with an algebraic inclusion, all subject to anti-periodic boundary conditions in a real Hilbert space. These problems, denoted as $(P_{\varepsilon\mu})_{ap}$, $(P_\mu)_{ap}$, and (E_{00}), involve operators that are odd, maximal monotone, and possibly set-valued. The second- and first-order differential inclusions are parameterized by two nonnegative constants, ε and μ, which affect the behavior of the differential terms.

We establish the existence and uniqueness of strong solutions for the problems $(P_{\varepsilon\mu})_{ap}$ and $(P_\mu)_{ap}$, as well as for the algebraic inclusion (E_{00}). Additionally, we prove the continuous dependence of the solution to problem $(P_{\varepsilon\mu})_{ap}$ on parameters ε and μ. We also provide approximation results for the solutions to $(P_\mu)_{ap}$ and (E_{00}) as the parameters ε and μ approach zero. Finally, we discuss some applications of our theoretical results.

1 Introduction

Consider the following pair of second-order and first-order inclusions with anti-periodic boundary conditions in a real Hilbert space H :

$$(P_{\varepsilon\mu})_{ap} \begin{cases} -\varepsilon u'' + \mu u' + A\,u + B\,u \ni f \ \text{ a.e. in } (0,T), & (E_{\varepsilon\mu}) \\ u(0) + u(T) = 0, \quad u'(0) + u'(T) = 0, \end{cases}$$

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along with

$$(P_\mu)_{ap} \begin{cases} \mu u' + A\,u + B\,u \ni f \quad \text{a.e. in } (0,T), & (E_\mu) \\ u(0) + u(T) = 0, \end{cases}$$

where the nonnegative parameters ε and μ satisfy $\varepsilon + \mu > 0$, $T > 0$ is a given time instant, and the following assumptions are fulfilled:

$(\mathbf{H_f})$ $f \in W^{1,2}(0,T;H)$ and $f(0) + f(T) = 0$;

$(\mathbf{H_A})$ The operator $A : D(A) \subset H \to H$ is odd, maximal monotone (possibly set-valued), and satisfies the following strong monotonicity condition:

$$(a - b,\, x - y) \geq \omega_0 \|x - y\|^2 \text{ for all } x,\, y \in D(A) \text{ and } a \in A\,x,\, b \in A\,y,$$

for some positive constant ω_0;

$(\mathbf{H_B})$ The operator $B : H \to H$ is odd, maximal monotone (possibly set-valued) and satisfies the following condition: for each $r > 0$, there is $L_r > 0$ with the property that for all $x \in H$ with $\| x \| \leq r$, it holds that $\| B\,x \| \leq L_r$.

The inclusion (E_μ) is derived from $(E_{\varepsilon\mu})$ by setting ε to zero. It is also noteworthy that the problem $(P_\mu)_{ap}$ only includes the boundary condition $u(0) + u(T) = 0$. In the case when the parameter ε is 'small', the problem $(P_{\varepsilon\mu})_{ap}$ is referred to as a perturbed problem associated with $(P_\mu)_{ap}$, while the latter is called unperturbed (or reduced problem). In this case, we could consider $(P_{\varepsilon\mu})_{ap}$ as a regularization of $(P_\mu)_{ap}$. Sometimes, it can be useful to consider regularizations of $(P_\mu)_{ap}$ that yield solutions that are more regular with respect to t, approximating the solution to $(P_\mu)_{ap}$ for small ε (see Lions [16, pp. VII-IX]). From the additional term that involves the parameter, this regularization method came to be known as the method of artificial viscosity and became widely used in various fields, such as control theory, numerical analysis and partial differential equations.

Finally, we introduce the algebraic inclusion

$$A\,u(t) + B\,u(t) \ni f(t) \text{ for a.e. } t \in (0,T), \tag{E_{00}}$$

which is obtained by taking $\varepsilon = \mu = 0$ in the inclusion $(E_{\varepsilon\mu})$.

Okochi [21] began studying non-linear evolution equations with anti-periodic solutions. The author proved the existence and uniqueness of the solution to problem $(P_\mu)_{ap}$ when A is an odd maximal cyclical monotone operator, and $B = 0$ (refer also to [22] for quasi-linear equations of parabolic type).

Problems $(P_\mu)_{ap}$ and/or $(P_{\varepsilon\mu})_{ap}$ (including different kinds of perturbations) have been extensively studied in numerous papers in the context where A is an odd maximal cyclical monotone operator (see Haraux [14], Aizicovici and Pavel [2], Aizicovici, McKibben and Reich [1], Chen [10] and references therein).

For hyperbolic problems with anti-periodic solutions, we refer to Haraux [14, Section 4], where the author proved that such weak solutions to the semi-linear wave equation with a dumping term exist. Nakao and Okochi [19] also studied the quasilinear wave equation with viscosity (see also [20]) and obtained results regarding its anti-periodic solutions. Chen, Nieto and O'Regan [11] investigated anti-periodic solutions to problem $(P_\mu)_{ap}$, assuming that A satisfies a linear growth condition and its domain $D(A)$ is embedded compactly into H, whereas $B = \partial G$ as a bounded operator, with $G \in C^1(H; H)$.

Our current goal is to institute a working structure in which A is not a subdifferential anymore, in order to facilitate new applications to hyperbolic systems, such as a telegraph system. Compared to [2] and [5] (where A is of subdifferential type), here we impose a stronger condition on f and require the strong positivity of A to show that the solution to $(P_\mu)_{ap}$ exists (see Theorem 2.3 below).

In the following, we review some fundamental definitions essential to this paper. The base of our framework consists in a real Hilbert space H alongside its inner product $(\cdot, \cdot)$ and the induced norm $\| \cdot \|$.

A set-valued operator $G : D(G) \subset H \to H$ is said to be monotone if

$$(a - b,\, x - y) \geq 0 \ \text{ for all } a,\, b \in D(G),\, a \in G\,x,\, b \in G\,y.$$

If the operator does not admit any proper monotone extensions, then it is said to be maximal monotone. According to Minty's Theorem, a monotone operator G is maximal monotone if and only if the range $R(I_H + \lambda G) = H$ for all (equivalently, some) $\lambda > 0$ (here, I_H denotes the identity operator on H). Therefore, if $\lambda > 0$ and G is maximal monotone, it makes sense to define the single-valued operators

$$J_\lambda = (I_H + \lambda G)^{-1},\ G_\lambda = (I_H - J_\lambda)/\lambda \ \text{ with } \ D(J_\lambda) = D(G_\lambda) = H.$$

These operators are called the resolvent and the Yosida approximation of G, respectively. For several properties of these two operators see, e.g., [17, Theorem 1.3, p.21]. For further details on the theory of monotone operators, including the set-valued case, we direct the reader to the relevant monographs [7], [9], [17].

This paper is organized as follows. In Section 2, we present the main result, which addresses the existence and uniqueness of solutions to the previously introduced problems. This result also provides essential uniform estimates with respect to ε and μ, applicable to all solutions to problem $(P_{\varepsilon\mu})_{ap}$, and will be employed in the subsequent sections. In Section 3 we establish that the solution to problem $(P_{\varepsilon\mu})_{ap}$ is continuous with respect to the parameters ε and μ. Additionally, we show that for small ε, the solution $u_{\varepsilon\mu}$ to $(P_{\varepsilon\mu})_{ap}$

approximates the solution to the reduced problem $(P_\mu)_{ap}$. Furthermore, we aim to prove that as $(\varepsilon,\mu)\to(0_+,0_+)$, the solution to $(P_{\varepsilon\mu})_{ap}$ converges to the solution to the algebraic inclusion (E_{00}). Finally, Section 4 applies our abstract theory to specific cases of the semilinear telegraph system and the semilinear heat equation with time anti-periodic conditions.

2 Existence, uniqueness, and uniform estimates of the solutions

The goal of our first result is to prove that the solutions to the problems introduced in Section 1 indeed exist and are unique. Additionally, we obtain some uniform estimates with respect to the parameters ε and μ of these solutions. These estimates will be crucial in proving the results presented in the subsequent sections.

The Hilbert space $L^2(0,T;H)$ alongside its usual scalar product and norm will be denoted by $(\mathcal{H},\ \langle\cdot,\cdot\rangle,\ \|\cdot\|_{\mathcal{H}})$. If G is a set-valued operator in H, then $S_{\mathcal{H}}(G)$ denotes the class of all sections of G that belong to $\mathcal{H}$. Additionally, the usual norm in $C([0,T];H)$ is denoted by $\|\cdot\|_\infty$.

From this point onward, we investigate (strong) solutions to problems $(P_{\varepsilon\mu})_{ap}$ and $(P_\mu)_{ap}$ in the sense presented below.

Definition 2.1. *Under the assumptions outlined in Section 1, a function $u_{\varepsilon\mu}\in W^{2,2}(0,T;H)$ is said to be a (strong) solution to problem $(P_{\varepsilon\mu})_{ap}$ if the following conditions are all satisfied*
(i) $u_{\varepsilon\mu}(t)\in D(A)$ for a.e. $t\in(0,T)$;

$$(ii)\begin{cases} -\varepsilon u''_{\varepsilon\mu}(t)+\mu u'_{\varepsilon\mu}(t)+\xi_{\varepsilon\mu}(t)+\eta_{\varepsilon\mu}(t)=f(t) \text{ for a.e. } t\in(0,T),\\ \xi_{\varepsilon\mu}\in S_{\mathcal{H}}(A\,u_{\varepsilon\mu}(\cdot)),\ \eta_{\varepsilon\mu}\in S_{\mathcal{H}}(B\,u_{\varepsilon\mu}(\cdot)); \end{cases} \tag{1}$$

(iii) $u_{\varepsilon\mu}(0)+u_{\varepsilon\mu}(T)=0$, $u'_{\varepsilon\mu}(0)+u'_{\varepsilon\mu}(T)=0$.
In a similar way, a function $u_\mu\in W^{1,2}(0,T;H)$ is said to be a (strong) solution to problem $(P_\mu)_{ap}$ if u_μ fulfills conditions (i), (ii) (with $\varepsilon=0$), and $u_\mu(0)+u_\mu(T)=0$.

We also recall the following inequality for $H-$valued antiperiodic functions for later reference:

Lemma 2.2. *If $u\in W^{1,2}(0,T;H)$ and satisfies $u(0)+u(T)=0$, then*

$$\|u\|_{C([0,T];H)}\le\frac{\sqrt{T}}{2}\|u'\|_{L^2(0,T;H)}\ . \tag{2}$$

The proof of this lemma follows directly from the equality

$$2u(t)=\int_0^t u'(s)\,ds-\int_t^T u'(s)\,ds \quad \text{for all } t\in[0,T].$$

Theorem 2.3. *(i) Assume that A is an odd maximal monotone operator and (H_B) is fulfilled. Then, for every $\varepsilon > 0$, $\mu \geq 0$, and $f \in L^2(0,T;H)$, the problem $(P_{\varepsilon\mu})_{ap}$ has a unique solution $u_{\varepsilon\mu} \in W^{2,2}(0,T;H)$ which satisfies the following estimate*

$$\varepsilon \parallel u''_{\varepsilon\mu} \parallel_{\mathcal{H}} \leq \parallel f \parallel_{\mathcal{H}} . \tag{3}$$

(ii) Assume that (H_A) is satisfied. Then, for every nonnegative ε and μ such that $\varepsilon + \mu > 0$, and for f satisfying (H_f), both problems $(P_\mu)_{ap}$ and $(P_{\varepsilon\mu})_{ap}$ have unique solutions $u_\mu \in W^{1,2}(0,T;H)$ and $u_{\varepsilon\mu} \in W^{2,2}(0,T;H)$, respectively. Moreover, the following estimates hold

$$\begin{aligned} \parallel u'_\mu \parallel_{\mathcal{H}} &\leq \omega_0^{-1} \parallel f' \parallel_{\mathcal{H}} \ \textit{ for every } \mu > 0, \\ \parallel u'_{\varepsilon\mu} \parallel_{\mathcal{H}} &\leq \omega_0^{-1} \parallel f' \parallel_{\mathcal{H}} \ \textit{ for every } \varepsilon > 0,\ \mu \geq 0. \end{aligned} \tag{4}$$

In addition, the algebraic inclusion (E_{00}) has a unique solution u_{00} belonging to $W^{1,2}(0,T;H)$, satisfying $u_{00}(0) + u_{00}(T) = 0$ and $u(t) \in D(A)$ for all $t \in [0,T]$.

Proof. Define the linear operator $\mathcal{L} : D(\mathcal{L}) \subset \mathcal{H} \to \mathcal{H}$ as

$$\begin{aligned} &D(\mathcal{L}) = \{v \in W^{2,2}(0,T;H); v(0) + v(T) = v'(0) + v'(T) = 0\}, \\ &\quad \mathcal{L}\, v = -\varepsilon v'' + \mu v' \ \text{ if } \ \varepsilon > 0, \text{ and} \\ &D(\mathcal{L}) = \{v \in W^{1,2}(0,T;H); v(0) + v(T) = 0\},\ \mathcal{L}\, v = \mu v' \text{ if } \varepsilon = 0. \end{aligned} \tag{5}$$

On the one hand, if $\varepsilon > 0$, by the $T-$ anti-periodicity of v and v', and by using (2) combined with Hölder's inequality, we have

$$\langle \mathcal{L}\, v, v \rangle = \varepsilon \parallel v' \parallel^2_{\mathcal{H}} \geq \frac{\varepsilon T^2}{4} \parallel v \parallel^2_{\mathcal{H}} \ \text{ for all } v \in D(\mathcal{L}). \tag{6}$$

Thus, $\mathcal{L}$ is strongly monotone with constant $\varepsilon T^2/4$. On the other hand, if $\varepsilon = 0$, the operator $\mathcal{L}$ is only positive. Clearly, $\mathcal{L}$ is maximal monotone in $\mathcal{H}$. Specifically, it can be easily checked that $R(I_{\mathcal{H}} + \mathcal{L}) = \mathcal{H}$.

(i) Let $\varepsilon > 0$, $\mu \geq 0$, and $f \in L^2(0,T;H)$. For $\lambda > 0$, let us introduce the approximating problem below

$$(P_\lambda)_{ap} \begin{cases} \mathcal{L}\, u + \mathcal{A}_\lambda u + \mathcal{B}_\lambda u = f \ \text{ a.e. in } (0,T), & (E_\lambda) \\ u(0) + u(T) = 0,\ u'(0) + u'(T) = 0, & (BC) \end{cases}$$

where $\mathcal{A}$ and $\mathcal{B}$ are the canonical extensions of A, and respectively B to $\mathcal{H}$ (see, e.g., [13, p. 28]), and $\mathcal{A}_\lambda$ and $\mathcal{B}_\lambda$ are their Yosida approximations. The operator $\mathcal{L} + \mathcal{A}_\lambda + \mathcal{B}_\lambda$ is maximal monotone and, in fact, strongly monotone in $\mathcal{H}$, provided that $\mathcal{L}$ is strongly monotone. Consequently, the problem $(P_\lambda)_{ap}$ has a unique solution $u_{\varepsilon\mu,\lambda} \in D(\mathcal{L})$, which we will denote simply as u_λ.

The monotonicity and oddness of both $\mathcal{A}_\lambda$ and $\mathcal{B}_\lambda$, combined with (BC), imply that for any positive λ, we have

$$\langle \mathcal{A}_\lambda u_\lambda, u''_\lambda\rangle = -\langle (\mathcal{A}_\lambda u_\lambda)', u'_\lambda\rangle \leq 0, \quad \langle \mathcal{B}_\lambda u_\lambda, u''_\lambda\rangle = -\langle (\mathcal{B}_\lambda u_\lambda)', u'_\lambda\rangle \leq 0. \tag{7}$$

From (E_λ), (BC), and the above inequalities, one sees that

$$\varepsilon\langle u''_\lambda, u''_\lambda\rangle \underbrace{-\mu\langle u'_\lambda, u''_\lambda\rangle}_{=0} - \underbrace{\langle \mathcal{A}_\lambda u_\lambda, u''_\lambda\rangle}_{\leq 0} - \underbrace{\langle \mathcal{B}_\lambda u_\lambda, u''_\lambda\rangle}_{\leq 0} = -\langle f, u''_\lambda\rangle \text{ for all } \lambda > 0.$$

This, along with (2) (with u replaced by u'_λ and u_λ) and Hölder's inequality, implies that

$$\begin{aligned} &\| u''_\lambda \|_{\mathcal{H}} \leq \varepsilon^{-1} \| f \|_{\mathcal{H}}, \| u'_\lambda \|_{\mathcal{H}} \leq T(2\varepsilon)^{-1} \| f \|_{\mathcal{H}}, \\ &\| u_\lambda \|_\infty \leq T\sqrt{T}(4\varepsilon)^{-1} \| f \|_{\mathcal{H}} \ \text{ for all } \lambda > 0. \end{aligned} \tag{8}$$

As u_λ satisfies estimate $(8)_3$, we infer from (H_B), and subsequently from equation (E_λ) combined with $(8)_{1,2}$, that for all $\lambda > 0$

$$\| \mathcal{B}_\lambda u_\lambda \|_{\mathcal{H}} \leq \sqrt{T} L_\varepsilon, \ \| \mathcal{A}_\lambda u_\lambda \|_{\mathcal{H}} \leq \| f \|_{\mathcal{H}} \left(2 + \mu T(2\varepsilon)^{-1}\right) + \sqrt{T} L_\varepsilon. \tag{9}$$

Here L_ε stands for the positive constant specified in assumption (H_B) corresponding to $r = T\sqrt{T} \| f \|_{\mathcal{H}} /(4\varepsilon)$.
Now, for every positive λ and ν, we derive from equations (E_λ) and (E_ν) that

$$\begin{aligned} -\varepsilon \int_0^T (w''_{\lambda\nu}, w_{\lambda\nu})\, dt + \mu \int_0^T (w'_{\lambda\nu}, w_{\lambda\nu})\, dt \\ +\langle \mathcal{A}_\lambda u_\lambda - \mathcal{A}_\nu u_\nu, w_{\lambda\nu}\rangle + \langle \mathcal{B}_\lambda u_\lambda - \mathcal{B}_\nu u_\nu, w_{\lambda\nu}\rangle = 0, \end{aligned}$$

where we denote $w_{\lambda\nu} = u_\lambda - u_\nu$. This implies, due to the $T-$ anti-periodicity boundary conditions (BC), the definitions of the Yosida approximation, and the fact that $\mathcal{A}$ and $\mathcal{B}$ are monotone (see also [17, Theorem 1.3(ii), p.21]), that

$$\begin{aligned} \varepsilon \| w'_{\lambda\nu} \|^2_{\mathcal{H}} + &\underbrace{\langle \mathcal{A}_\lambda u_\lambda - \mathcal{A}_\nu u_\nu, J^{\mathcal{A}}_\lambda u_\lambda - J^{\mathcal{A}}_\nu u_\nu\rangle}_{\geq 0} \\ &+ \underbrace{\langle \mathcal{B}_\lambda u_\lambda - \mathcal{B}_\nu u_\nu, J^{\mathcal{B}}_\lambda u_\lambda - J^{\mathcal{B}}_\nu u_\nu\rangle}_{\geq 0} \\ &= -\langle \mathcal{A}_\lambda u_\lambda - \mathcal{A}_\nu u_\nu, \lambda\mathcal{A}_\lambda u_\lambda - \nu\mathcal{A}_\nu u_\nu\rangle \\ &\quad - \langle \mathcal{B}_\lambda u_\lambda - \mathcal{B}_\nu u_\nu, \lambda\mathcal{B}_\lambda u_\lambda - \nu\mathcal{B}_\nu u_\nu\rangle. \end{aligned} \tag{10}$$

Here $J^{\mathcal{A}}_\lambda$ and $J^{\mathcal{B}}_\lambda$ denote the resolvent operators $(I_{\mathcal{H}}+\lambda\mathcal{A})^{-1}$ and $(I_{\mathcal{H}}+\lambda\mathcal{B})^{-1}$, respectively. Therefore, combining (10) with (9), we obtain the existence of a positive constant C such that

$$\| w_{\lambda\nu} \|^2_{\mathcal{H}} = \| u'_\lambda - u'_\nu \|^2_{\mathcal{H}} \leq C(\lambda + \nu)/\varepsilon, \tag{11}$$

for all positive λ and ν. This implies, by (2), that for all positive λ and ν

$$\| u_\lambda - u_\nu \|_\infty^2 \leq \frac{T}{4} \| u_\lambda' - u_\nu' \|_{\mathcal{H}}^2 \leq \frac{TC}{4\varepsilon}(\lambda + \nu). \tag{12}$$

Therefore, according to $(8)_{1,2}$, (9), and (12), there exist $u \in W^{2,2}(0,T;H)$ and ξ, $\eta \in \mathcal{H}$, such that, as $\lambda \to 0_+$, the following convergences hold

$$u_\lambda \to u \ \text{ in } C([0,T];H), \tag{13}$$

$$u_\lambda' \to u' \ \text{ in } \mathcal{H}, \tag{14}$$

$$u_\lambda'' \to u'' \ \text{ weakly in } \mathcal{H}, \tag{15}$$

$$\mathcal{A}_\lambda u_\lambda \to \xi,\ \mathcal{B}_\lambda u_\lambda \to \eta, \ \text{weakly in } \mathcal{H}. \tag{16}$$

Next, we will prove that $u_\lambda' \to u'$ in $C([0,T];H)$ as $\lambda \to 0_+$. In this sense, set $w_\lambda(t) := u_\lambda'(t) - u'(t)$, and define $h_\lambda(t) :=\| w_\lambda(t) \|$ for all $\lambda > 0$ and $t \in [0,T]$. Then, for all $\lambda > 0$ and all $t \in [0,T]$,

$$h_\lambda(t) \leq \| u_\lambda'(t) \| + \| u'(t) \| \leq \frac{\sqrt{T}}{2} \| u_\lambda'' \|_{\mathcal{H}} + \| u' \|_\infty < \infty,$$

by (2) (in which u is replaced by u_λ') and $(8)_1$. In addition, from $(8)_1$, the set $\{h_\lambda\}_{\lambda>0}$ is equi-continuous since, for all $\lambda > 0$, t, $s \in [0,T]$,

$$\begin{aligned} | h_\lambda(t) - h_\lambda(s) | &\leq \| w_\lambda(t) - w_\lambda(s) \| \\ &= \left\| \int_s^t w_\lambda(\tau)'\, d\tau \right\| \leq \sqrt{T} \left(\| u_\lambda'' \|_{\mathcal{H}} + \| u'' \|_{\mathcal{H}}\right) \sqrt{| t-s |} \\ &\leq \sqrt{T} \left(\| f \|_{\mathcal{H}} /\varepsilon + \| u'' \|_{\mathcal{H}}\right) \sqrt{| t-s |}. \end{aligned}$$

Therefore, by the the Arzelà-Ascoli Criterion, along with (14) (see also [8, Theorem 4.9]), it follows that $h_\lambda \to 0$ as $\lambda \to 0_+$ in $C[0,T]$; i.e.,

$$u_\lambda' \to u' \text{ in } C([0,T];H) \text{ as } \lambda \to 0_+. \tag{17}$$

Thus, using (13) and (17), we conclude that $u(0)+u(T) = 0$ and $u'(0)+u'(T) = 0$, implying that $u \in D(\mathcal{L})$.

Next, we will verify that u satisfies the inclusion $(E_{\varepsilon\mu})$. Indeed, from (13) and (16), together with [6, Proposition 1.1(iv), p.42], we get that $u \in D(\mathcal{A})$, $u \in D(\mathcal{B})$, and $\xi \in S_{\mathcal{H}}(A\,u(\cdot))$ and $\eta \in S_{\mathcal{H}}(B\,u(\cdot))$. Thus, if we pass to the (weak) limit in (E_λ) and use (14) and (15), we obtain that $u \in D(\mathcal{L}+\mathcal{A}+\mathcal{B})$ and satisfies

$$\begin{aligned} -\varepsilon u''(t) + \mu u(t) + \xi(t) + \eta(t) = f(t) \text{ for a.e. } t \in (0,T) \text{ and} \\ \xi \in S_{\mathcal{H}}(A\,u(\cdot)),\ \eta \in S_{\mathcal{H}}(B\,u(\cdot)). \end{aligned} \tag{18}$$

Hence, $u = u_{\varepsilon\mu}$ is the desired solution to problem $(P_{\varepsilon\mu})_{ap}$. Since $\mathcal{L}$ is strongly monotone, it follows that the solution $u_{\varepsilon\mu}$ is unique.

Finally, using estimate $(8)_1$ obtained above and the weak lower semicontinuity of norms, we derive the estimate (3) by passing to the weak limit as $\lambda \to 0_+$.

(ii) Assume that (H_A), (H_B), and (H_f) hold.

In this framework, we decompose A as $A = \omega_0 I_H + A_1$, where $A_1 := A - \omega_0 I_H$, with $D(A_1) = D(A)$, according to the definition of the sum of set-valued operators and assumption (H_A). Obviously, A_1 is a maximal monotone operator. Next, denote by $\mathcal{A}_1$ and $\mathcal{B}$ the canonical extensions of A_1 and B to $\mathcal{H}$, respectively, and by $\mathcal{A}_{1\lambda}$ and $\mathcal{B}_\lambda$ their Yosida approximations, for $\lambda > 0$. For f satisfying (H_f) and $\lambda > 0$, consider the problems

$$(P_\lambda^0)_{ap} \begin{cases} \mu u' + \omega_0 u + \mathcal{A}_{1\lambda} u + \mathcal{B}_\lambda u = f \ \text{ a.e. in } (0,T), & (E_\lambda^0) \\ u(0) + u(T) = 0, & (BC^0) \end{cases}$$

if $\varepsilon = 0$ and

$$(P_\lambda')_{ap} \begin{cases} -\varepsilon u'' + \mu u' + \omega_0 u + \mathcal{A}_{1\lambda} u + \mathcal{B}_\lambda u = f \ \text{ a.e. in } (0,T), & (E_\lambda') \\ u(0) + u(T) = 0, \ u'(0) + u'(T) = 0, & (BC) \end{cases}$$

if $\varepsilon > 0$.

Clearly, $\mathcal{L} + \omega_0 I + \mathcal{A}_{1\lambda} + \mathcal{B}_\lambda$ is a maximal monotone operator in $\mathcal{H}$. In addition, it is also strongly monotone, since so is $\mathcal{L} + \omega_0 I$. Thus, for every $\lambda > 0$, the problems $(P_\lambda')_{ap}$ and $(P_\lambda^0)_{ap}$ have unique solutions $u_{\varepsilon\mu,\lambda} \in D(\mathcal{L})$ and $u_{\mu,\lambda} \in D(\mathcal{L})$, respectively, denoted, as in the previous case, by u_λ.

First, let us consider the case when $\varepsilon = 0$. To start with, we need to prove that the problem $(P_\mu)_{ap}$ has a unique solution $u_\mu \in W^{1,2}(0,T;H)$.

To do so, since (H_f) is fulfilled and $u_\lambda \in D(\mathcal{L})$, and considering that $\mathcal{A}_{1\lambda}$ and $\mathcal{B}_\lambda$ are odd Lipschitz operators, we obtain from (E_λ^0) that the solution u_λ to problem $(P_\mu)_{ap}$ belongs to $W^{2,2}(0,T;H)$ and $u_\lambda'(0) + u_\lambda'(T) = 0$. First, we differentiate (E_λ^0) with respect to t a.e. on $(0,T)$ and then we multiply the equation we obtain by u_λ' in $\mathcal{H}$ to obtain, for all $\lambda > 0$

$$\omega_0 \parallel u_\lambda' \parallel_{\mathcal{H}}^2 + \langle (\mathcal{A}_{1\lambda} u_\lambda)', u_\lambda' \rangle + \langle (\mathcal{B}_\lambda u_\lambda)', u_\lambda' \rangle = \langle f', u_\lambda' \rangle, \tag{19}$$

from the $T-$ anti-periodicity of u_λ'. This implies that

$$\begin{aligned} &\parallel u_\lambda' \parallel_{\mathcal{H}} \leq \omega_0^{-1} \parallel f_\lambda' \parallel_{\mathcal{H}}, \ \ \parallel u_\lambda \parallel_\infty \leq \sqrt{T}(2\omega_0)^{-1} \parallel f_\lambda' \parallel_{\mathcal{H}}, \\ &\parallel u_\lambda \parallel_{\mathcal{H}} \leq T(2\omega_0)^{-1} \parallel f_\lambda' \parallel_{\mathcal{H}} \ \text{ for all } \lambda > 0, \end{aligned} \tag{20}$$

by (2) along with Hölder's inequality, and (7) (with $\mathcal{A}_\lambda$ replaced by $\mathcal{A}_{1\lambda}$). As in the previous case, from (H_B) and the estimates in (20), together with (E_λ^0), we obtain the existence of positive constants C_1, C_2, both independent of λ, such that

$$\| \mathcal{B}_\lambda u_\lambda \| \leq C_1, \ \ \| \mathcal{A}_{1\lambda} u_\lambda \| \leq C_2, \ \forall \lambda > 0, \tag{21}$$

Following the approach in Case 1, for every positive λ and ν, we derive from equations (E^0_λ), (E^0_ν), and (21) that

$$\begin{aligned} &\omega_0 \| u_\lambda - u_\nu \|^2_{\mathcal{H}} + \underbrace{\langle \mathcal{A}_{1\lambda} u_\lambda - \mathcal{A}_{1\nu} u_\nu, J^{\mathcal{A}_1}_\lambda u_\lambda - J^{\mathcal{A}_1}_\nu u_\nu \rangle}_{\geq 0} \\ &\qquad\qquad + \underbrace{\langle \mathcal{B}_\lambda u_\lambda - \mathcal{B}_\nu u_\nu, J^{\mathcal{B}}_\lambda u_\lambda - J^{\mathcal{B}}_\nu u_\nu \rangle}_{\geq 0} \\ &\quad = -\langle \mathcal{A}_{1\lambda} u_\lambda - \mathcal{A}_{1\nu} u_\nu, \lambda \mathcal{A}_{1\lambda} u_\lambda - \nu \mathcal{A}_{1\nu} u_\nu \rangle \\ &\qquad - \langle \mathcal{B}_\lambda u_\lambda - \mathcal{B}_\nu u_\nu, \lambda \mathcal{B}_\lambda u_\lambda - \nu \mathcal{B}_\nu u_\nu \rangle \leq 2(C_1^2 + C_2^2)(\lambda + \nu), \end{aligned} \tag{22}$$

where $J^{\mathcal{A}_1}_\lambda$ is the resolvent of $\mathcal{A}_1$. Now, by using the information obtained so far, there exist $u \in W^{1,2}(0,T;H)$ and ξ_1, $\eta \in \mathcal{H}$, such that, as $\lambda \to 0_+$, we have

$$u_\lambda \to u \text{ in } \mathcal{H}, \tag{23}$$

$$u'_\lambda \to u' \text{ weakly in } \mathcal{H}, \tag{24}$$

$$\mathcal{A}_{1\lambda} u_\lambda \to \xi_1, \ \mathcal{B}_\lambda u_\lambda \to \eta \ \text{ weakly in } \mathcal{H}. \tag{25}$$

Now, consider the function $g_\lambda : [0,T] \to \mathbb{R}$, $g_\lambda(t) = \| u_\lambda(t) - u(t) \|$ for all $\lambda > 0$ and $t \in [0,T]$. On the one hand, the set $\{g_\lambda\}_{\lambda>0}$ is bounded in $C[0,T]$, by $(20)_2$. On the other hand, according to $(20)_1$, this set is also equi-continuous. Consequently, the Arzelà-Ascoli Criterion ensures that

$$u_\lambda \to u \text{ in } C([0,T];H) \text{ as } \lambda \to 0_+. \tag{26}$$

Consequently, $u(0) + u(T) = 0$, so $u \in D(\mathcal{L})$ (see (5)).

Next, from [6, Proposition 1.1(iv), p.42], combined with (26) and (25), we obtain that $u \in D(\mathcal{A}) \cap D(\mathcal{B})$ for a.e. $t \in (0,T)$, $\xi_1 \in S_{\mathcal{H}}(A_1\, u(\cdot))$ and $\eta \in S_{\mathcal{H}}(B\, u(\cdot))$. Following arguments similar to those used in the proof of Case 1, we take the (weak) limit in (E^0_λ) and obtain that u verifies the equation

$$\mu u'(t) + \xi(t) + \eta(t) = f(t) \ \text{ for a.e. } t \in (0,T),$$

where $\xi(t) = \omega_0 u(t) + \xi_1(t) \in A\, u(t)$ for a.e. $t \in (0,T)$. That is, problem $(P_\mu)_{ap}$ admits solution $u = u_\mu$, as desired. Since A is strongly positive, this means that u_μ is unique.

If we take the weak limit $\lambda \to 0_+$ in $(20)_1$ and use the fact that the norms are weakly lower semicontinuous, we get that estimate $(4)_1$ holds true.

Let $\varepsilon > 0$. From (i), problem $(P_{\varepsilon\mu})_{ap}$ admits a unique solution $u_{\varepsilon\mu}$. In addition, as in the previous case, one obtains that this solution is the limit in $C([0,T];H)$ of the approximating solutions u_λ to problems $(P'_\lambda)_{ap}$, as $\lambda \to 0_+$. To prove estimate $(4)_2$, we start by differentiating (E'_λ) with respect to t, to obtain

$$-\varepsilon u_\lambda^{(3)} + \mu u''_\lambda + \omega_0 u'_\lambda + (\mathcal{A}_{1\lambda} u_\lambda)' + (\mathcal{B}_\lambda u_\lambda)' = f' \text{ a.e. in } (0,T). \tag{27}$$

Due to the oddness of the operators $\mathcal{A}_{1\lambda}$ and $\mathcal{B}_\lambda$, along with $f(0)+f(T)=0$ and (BC), we also get from (E'_λ) that $u''_\lambda(0)+u''_\lambda(T)=0$. We now multiply (27) by u'_λ with respect to the inner product of $\mathcal{H}$, then we integrate by parts and use (BC), as well as $u''_\lambda(0)+u''_\lambda(T)=0$ to obtain

$$\varepsilon \parallel u''_\lambda \parallel^2_{\mathcal{H}} + \omega_0 \parallel u'_\lambda \parallel^2_{\mathcal{H}} \leq \parallel f' \parallel_{\mathcal{H}} \parallel u'_\lambda \parallel_{\mathcal{H}} .$$

Here we have used (7) (with $\mathcal{A}_\lambda$ replaced by $\mathcal{A}_{1\lambda}$). We derive from the above inequality that

$$\parallel u'_\lambda \parallel_{\mathcal{H}} \leq \omega_0^{-1} \parallel f' \parallel_{\mathcal{H}},$$

which implies, by taking the limit as $\lambda \to 0_+$, that the solution $u_{\varepsilon\mu}$ to problem $(P_{\varepsilon\mu})_{ap}$ satisfies $(4)_2$; namely, $\parallel u'_{\varepsilon\mu} \parallel_{\mathcal{H}} \leq \omega_0^{-1} \parallel f' \parallel_{\mathcal{H}}$.

Next, we consider the algebraic inclusion (E_{00}). From [5, Theorem 6] we know that (E_{00}) admits a unique solution $u_{00} \in W^{1,2}(0,T;H)$ given by

$$u_{00}(t) = Q^{-1} f(t) \text{ for all } t \in [0,T], \tag{28}$$

where $Q := A + B$, with $D(Q) = D(A)$. Moreover, the operator $Q^{-1} : H \to D(A)$ is Lipschitz continuous with constant ω_0^{-1}. In particular, as Q is odd and $f(0)+f(T)=0$, we derive from (28) that $u_{00}(0)+u_{00}(T)=0$. Moreover, (28) implies that $u(t) \in D(A)$ for all $t \in [0,T]$.

This completes our proof. □

A similar result to Theorem 2.3 can be derived for the case where the problems introduced in Section 1 are perturbed non-monotonically by a Lipschitz operato. More exactly, we now consider the problems

$$(\overline{P}_{\varepsilon\mu})_{ap} \begin{cases} -\varepsilon u'' + \mu u' + A\,u + B\,u \ni F(u) + f \ \text{ a.e. in } (0,T), & (\overline{E}_{\varepsilon\mu}) \\ u(0)+u(T)=0, \quad u'(0)+u'(T)=0, & (BC) \end{cases}$$

alongside

$$(\overline{P}_0)_{ap} \begin{cases} \mu u' + A\,u + B\,u \ni F(u) + f \quad \text{a.e. in } (0,T), & (\overline{E}_0) \\ u(0)+u(T)=0, & (BC^0) \end{cases}$$

as well as the algebraic inclusion

$$A\,u + B\,u \ni F(u) + f \;\text{ a.e. in }\; (0,T). \tag{$\overline{E}_{00}$}$$

Here, the nonlinear operator F satisfies
$(\mathbf{H_F})$ $F : H \to H$ is a Lipschitz operator with Lipschitz constant $L > 0$.

Theorem 2.4. *Assume that (H_A), (H_B), and (H_F) are fulfilled, with constants L and ω_0 verifying $L < \omega_0$. Then, for every nonnegative ε and μ such that $\varepsilon + \mu > 0$, and f satisfying (H_f), the problems $(P_{\varepsilon\mu})_{ap}$ and $(P_\mu)_{ap}$ have unique solutions $u_{\varepsilon\mu} \in W^{2,2}(0,T;H)$ and $u_\mu \in W^{1,2}(0,T;H)$, respectively. Furthermore, one can derive the following estimates*

$$\begin{aligned} &\| u'_\mu \|_{\mathcal{H}} \le (\omega_0 - L)^{-1} \| f' \|_{\mathcal{H}} \quad \textit{for every } \mu > 0, \\ &\| u'_{\varepsilon\mu} \|_{\mathcal{H}} \le (\omega_0 - L)^{-1} \| f' \|_{\mathcal{H}} \quad \textit{for every } \varepsilon > 0,\ \mu \ge 0. \end{aligned} \tag{29}$$

In addition, the algebraic inclusion $(\overline{E}_{00})$ has a unique solution u_{00} belonging to $W^{1,2}(0,T;H)$ that satisfies $u_{00}(0) + u_{00}(T) = 0$ and $u(t) \in D(A)$ for all $t \in [0,T]$.

Proof. Let $\varepsilon > 0$ and $\mu \ge 0$ be fixed. We prove the statements of the theorem for the problem $(\overline{P}_{\varepsilon\mu})_{ap}$, noting that similar arguments and computations apply to the problem $(\overline{P}_\mu)_{ap}$.

Here, the Banach Contraction Principle is employed to prove that problem $(\overline{P}_{\varepsilon\mu})_{ap}$ has a unique solution. To this end, for $v \in \mathcal{H}$, consider the problem

$$(P^v_{\varepsilon\mu})_{ap} \begin{cases} -\varepsilon u'' + \mu u' + A\,u + B\,u \ni F\,v + f \text{ a.e. in } (0,T), \\ u(0) + u(T) = 0,\ u'(0) + u'(T) = 0. \end{cases} \tag{30}$$

Clearly, for every $v \in \mathcal{H}$, it follows from (H_F) that $F\,v \in \mathcal{H}$. Let $u_v \in W^{2,2}(0,T;H)$ denote the solution to this problem, which is known to exist and be unique by Theorem 2.3. Define the operator

$$\mathcal{P} : \mathcal{H} \to \mathcal{H},\ \mathcal{P}v = u_v.$$

Let $v_1,\ v_2 \in \mathcal{H}$. Then, $w = u_{v_1} - u_{v_2} = \mathcal{P}v_1 - \mathcal{P}v_2$ satisfies

$$\begin{aligned} &-\varepsilon w'' + \mu w' + \mathcal{A}\,u_{v_1} - \mathcal{A}\,u_{v_2} + \mathcal{B}\,u_{v_1} - \mathcal{B}\,u_{v_2} \ni F\,v_1 - F\,v_2 \text{ in } \mathcal{H}, \\ &\ w(0) + w(T) = 0,\ w'(0) + w'(T) = 0. \end{aligned} \tag{31}$$

We now take the scalar product of $(31)_1$ with w in $\mathcal{H}$, use the integration by parts in the first term and $(31)_2$, as well as the assumptions (H_A) and (H_B), to get

$$\varepsilon \| w' \|^2_{\mathcal{H}} + \omega_0 \| w \|^2_{\mathcal{H}} \le L \| v_1 - v_2 \|_{\mathcal{H}} \| w \|_{\mathcal{H}}\ . \tag{32}$$

From (2), together with Hölder's inequality, and (32) it now follows

$$\left(\frac{4\varepsilon}{T^2}+\omega_0\right)\parallel \mathcal{P}v_1-\mathcal{P}v_2\parallel_{\mathcal{H}}^2\leq L\parallel v_1-v_2\parallel_{\mathcal{H}}\parallel \mathcal{P}v_1-\mathcal{P}v_2\parallel_{\mathcal{H}}. \tag{33}$$

Since $L<\omega_0$, we obtain that $\mathcal{P}$ is a contraction on $\mathcal{H}$. Hence, $\mathcal{P}$ has a fixed point that is unique in $\mathcal{H}$. This, in fact, is the unique solution to $(\overline{P}_{\varepsilon\mu})_{ap}$.

Now, from $(4)_2$, where we replace f by $f+F(u_{\varepsilon\mu})$, as $F(0)=0$, we have

$$\omega_0\parallel u'_{\varepsilon\mu}\parallel_{\mathcal{H}}\leq\parallel f'+F'(u_{\varepsilon\mu})u'_{\varepsilon\mu}\parallel_{\mathcal{H}}\leq\parallel f'\parallel_{\mathcal{H}}+L\parallel u'_{\varepsilon\mu}\parallel_{\mathcal{H}},$$

which implies $(29)_2$.

The argument which proves that $(\overline{E}_{00})$ has a unique solution is identical to the argument employed in the proof of Theorem 2.3.

Our proof is thus complete. □

Remark 2.5. *If, in addition, F is an odd operator, and f satisfies $f(t+T)+f(t)=0$ for a.e. $t\in\mathbb{R}$, then the solutions derived in this section can be extended to all of $\mathbb{R}$ imposing $T-$anti-periodicity.*

3 Continuous dependence of the solution to $(P_{\varepsilon\mu})_{ap}$ on ε and μ and approximation results

This section is designated to investigate the continuous dependence of the solution $u_{\varepsilon\mu}$ to problem $(P_{\varepsilon\mu})_{ap}$ on parameters ε and μ. We will also obtain approximating results regarding the solutions to the reduced problem $(P_\mu)_{ap}$ and the algebraic inclusion (E_{00}). In what follows, $\mathcal{O}$ shall denote the usual big Landau symbol.

Theorem 3.1. *Assume that (H_B) is fulfilled.*
(i) Let $\varepsilon_0>0$ and $\mu_0\geq 0$ be fixed. Suppose that A is an odd maximal monotone operator. For every $\varepsilon>0$, $\mu\geq 0$, and $f\in L^2(0,T;H)$, let $u_{\varepsilon\mu}\in W^{2,2}(0,T;H)$ be the unique solution to problem $(P_{\varepsilon\mu})_{ap}$ given by Theorem 2.3 (i). Then the following estimate and convergence hold

$$\begin{aligned}\parallel u_{\varepsilon\mu}-u_{\varepsilon_0\mu_0}\parallel_{C([0,T];H)}&=\mathcal{O}(\mid\varepsilon-\varepsilon_0\mid)+\mathcal{O}(\mid\mu-\mu_0\mid),\\ u_{\varepsilon\mu}\to u_{\varepsilon_0\mu_0}\ &\text{in } C^1([0,T];H)\ \text{ as } (\varepsilon,\mu)\to(\varepsilon_0,\mu_0).\end{aligned}\tag{34}$$

(ii) Let $\mu_0>0$ be fixed. Assume that (H_A) holds. For every nonnegative ε and μ such that $\varepsilon+\mu>0$, and f satisfying (H_f), let $u_{\varepsilon\mu}\in W^{2,2}(0,T;H)$ and $u_\mu\in W^{1,2}(0,T;H)$ be the unique solutions to problems $(P_{\varepsilon\mu})_{ap}$, and respectively $(P_\mu)_{ap}$, given by Theorem 2.3 (ii). Then, the following estimate and

approximation hold

$$\begin{aligned} &\| u_{\varepsilon\mu} - u_{\mu_0} \|_{L^2(0,T;H)} = \mathcal{O}(\sqrt{\varepsilon}) + \mathcal{O}(| \mu - \mu_0 |), \\ &u_{\varepsilon\mu} \to u_{\mu_0} \quad in\ C([0,T];H) \quad as\ (\varepsilon,\mu) \to (0_+,\mu_0). \end{aligned} \tag{35}$$

In addition, the following estimate holds

$$\| u_{\varepsilon\mu} - u_{00} \|_{L^2(0,T;H)} = \mathcal{O}(\sqrt{\varepsilon}) + \mathcal{O}(\mu) \ as\ (\varepsilon,\mu) \to (0_+,0_+), \tag{36}$$

where $u_{00} \in W^{1,2}(0,T;H)$ is the unique solution to the (algebraic) inclusion (E_{00}), given by Theorem 2.3(ii). Furthermore, if $\mu^2/\varepsilon = \mathcal{O}(1)$, then

$$u_{\varepsilon\mu} \to u_{00} \ in\ C([0,T];H) \ as\ (\varepsilon,\mu) \to (0_+,0_+). \tag{37}$$

Proof. **(i)** From $(E_{\varepsilon\mu})$ and $(E_{\varepsilon_0\mu_0})$ we have,

$$\begin{aligned} -\varepsilon(u''_{\varepsilon\mu} - u''_{\varepsilon_0\mu_0}) - (\varepsilon-\varepsilon_0)u''_{\varepsilon_0\mu_0} + \mu(u'_{\varepsilon\mu} - u'_{\varepsilon_0\mu_0}) + (\mu-\mu_0)u'_{\varepsilon_0\mu_0} \\ + A\,u_{\varepsilon\mu} - A\,u_{\varepsilon_0\mu_0} + B\,u_{\varepsilon\mu} - B\,u_{\varepsilon_0\mu_0} \ni 0 \ \text{ a.e. in } (0,T). \end{aligned} \tag{38}$$

Set $w_{\varepsilon\mu} := u_{\varepsilon\mu} - u_{\varepsilon_0\mu_0}$ and take the scalar product of (38) and $w_{\varepsilon\mu}$ in $\mathcal{H}$ to obtain

$$\begin{aligned} \varepsilon \, \|w'_{\varepsilon\mu}\|^2_{\mathcal{H}} &\le \| w_{\varepsilon\mu} \|_{\mathcal{H}} \left(| \varepsilon - \varepsilon_0 | \, \| u''_{\varepsilon_0\mu_0} \|_{\mathcal{H}} + | \mu - \mu_0 | \, \| u'_{\varepsilon_0\mu_0} \|_{\mathcal{H}} \right) \\ &\le \frac{T \, \| f \|_{\mathcal{H}}}{2\varepsilon_0} \, \| w'_{\varepsilon\mu} \|_{\mathcal{H}} \left(| \varepsilon - \varepsilon_0 | + \frac{T}{2} \, | \mu - \mu_0 | \right) \end{aligned} \tag{39}$$

by integration by parts, the monotonicity of A and B, as well as the boundary conditions satisfied by $u_{\varepsilon\mu}$ and $u_{\varepsilon_0\mu_0}$. Also, we have used (3) and (2) (with u replaced by $w_{\varepsilon\mu}$ and $u'_{\varepsilon_0\mu_0}$).

From (39) we derive

$$\| w'_{\varepsilon\mu} \|_{\mathcal{H}} = \| u'_{\varepsilon\mu} - u'_{\varepsilon_0\mu_0} \|_{\mathcal{H}} \le \frac{T \, \| f \|_{\mathcal{H}}}{2\varepsilon_0\varepsilon} \left(| \varepsilon - \varepsilon_0 | + \frac{T}{2} \, | \mu - \mu_0 | \right). \tag{40}$$

From this and (2), we obtain that

$$\| u_{\varepsilon\mu} - u_{\varepsilon_0\mu_0} \|_{\infty} \le \frac{\sqrt{T}}{2} \, \| u'_{\varepsilon\mu} - u'_{\varepsilon_0\mu_0} \|_{\mathcal{H}} = \mathcal{O}(| \varepsilon - \varepsilon_0 |) + \mathcal{O}(| \mu - \mu_0 |). \tag{41}$$

In order to check convergence $(34)_2$, let $\delta_0 \in (0,\varepsilon_0)$ be fixed. Denote $I_0 = (\varepsilon_0 - \delta_0, \varepsilon_0 + \delta_0) \times ((\mu_0 - \delta_0, \mu_0 + \delta_0) \cap \mathbb{R}_+)$. For $(\varepsilon,\mu) \in I_0$, by (3) (see also (2)), we have

$$\| u''_{\varepsilon\mu} \|_{\mathcal{H}} \le (\varepsilon_0 - \delta_0)^{-1} \, \| f \|_{\mathcal{H}}, \ \| u'_{\varepsilon\mu} \|_{\infty} \le \sqrt{T}(2(\varepsilon_0 - \delta_0))^{-1} \, \| f \|_{\mathcal{H}} \, . \tag{42}$$

Following a similar pathway to the arguments employed in the proof of Theorem 2.3, from the Arzelà-Ascoli Criterion, along with (42), (41), and the uniqueness of the solution $u_{\varepsilon_0\mu_0}$ to problem $(P_{\varepsilon_0\mu_0})_{ap}$, we get

$$u'_{\varepsilon\mu} \to u'_{\varepsilon_0\mu_0} \text{ in } C([0,T];H) \text{ as } (\varepsilon,\mu) \to (\varepsilon_0,\mu_0). \tag{43}$$

Finally, this combined with (41), implies $(34)_2$.

(ii) First, we consider that $(\varepsilon,\mu) \to (0_+,\mu_0)$, with $\mu_0 > 0$. Subtracting $(E_{\varepsilon\mu})$ and (E_{μ_0}) we obtain

$$-\varepsilon u''_{\varepsilon\mu} + \mu(u'_{\varepsilon\mu} - u'_{\mu_0}) + (\mu-\mu_0)u'_{\mu_0} + A\, u_{\varepsilon\mu} - A\, u_{\mu_0} + B\, u_{\varepsilon\mu} - B\, u_{\mu_0} \ni 0. \tag{44}$$

Set $\bar{w}_{\varepsilon\mu} := u_{\varepsilon\mu} - u_{\mu_0}$ take the scalar product of (44) and $\bar{w}_{\varepsilon\mu}$ in $\mathcal{H}$, then use the strong monotonicity of A and arguments similar to those in the previous case, we get

$$\begin{aligned} &\varepsilon \parallel u'_{\varepsilon\mu} \parallel^2_{\mathcal{H}} + \omega_0 \parallel \bar{w}_{\varepsilon\mu} \parallel^2_{\mathcal{H}} \\ &\leq \mid \mu - \mu_0 \mid \cdot \parallel u'_{\mu_0} \parallel_{\mathcal{H}} \parallel \bar{w}_{\varepsilon\mu} \parallel_{\mathcal{H}} + \varepsilon \parallel u'_{\varepsilon\mu} \parallel_{\mathcal{H}} \parallel u'_{\mu_0} \parallel_{\mathcal{H}} \\ &\leq \frac{\varepsilon}{2} \parallel u'_{\varepsilon\mu} \parallel^2_{\mathcal{H}} + \frac{\omega_0}{2} \parallel \bar{w}_{\varepsilon\mu} \parallel^2_{\mathcal{H}} + \frac{\parallel f' \parallel^2_{\mathcal{H}} \varepsilon}{2\omega_0^2} + \frac{(\mu-\mu_0)^2 \parallel f' \parallel^2_{\mathcal{H}}}{2\omega_0^3}. \end{aligned} \tag{45}$$

Here we have also used (4) along with the elementary inequality $xy \leq (x^2 + y^2)/2$. Clearly, from (45) we get

$$\varepsilon \parallel u'_{\varepsilon\mu} \parallel^2_{\mathcal{H}} + \omega_0 \parallel \bar{w}_{\varepsilon\mu} \parallel^2_{\mathcal{H}} = \mathcal{O}(\varepsilon) + \mathcal{O}(\mid \mu - \mu_0 \mid^2),$$

which implies

$$\parallel u_{\varepsilon\mu} - u_{\mu_0} \parallel_{\mathcal{H}} = \mathcal{O}(\sqrt{\varepsilon}) + \mathcal{O}(\mid \mu - \mu_0 \mid) \text{ as } (\varepsilon,\mu) \to (0_+,\mu_0). \tag{46}$$

Finally, consider an arbitrary sequence $(\varepsilon_n,\mu_n) \to (0_+,\mu_0)$ as $n \to \infty$. We have (see (4))

$$\parallel u'_{\varepsilon_n\mu_n} \parallel_{\mathcal{H}} \leq \frac{\parallel f' \parallel_{\mathcal{H}}}{\omega_0} \text{ for all } n \in \mathbb{N}, \ \parallel u'_{\mu_0} \parallel_{\mathcal{H}} \leq \frac{\parallel f' \parallel_{\mathcal{H}}}{\omega_0}.$$

Therefore, the sequence $\{\parallel u_{\varepsilon_n\mu_n} - u_{\mu_0} \parallel\}_{n\in\mathbb{N}}$ is bounded and uniformly equicontinuous in $C[0,T]$. From the Arzel-Ascoli Criterion, estimate (46) (see also [8, Theorem 4.9]), and that the solution u_{μ_0} to problem $(P_{\mu_0})_{ap}$ is unique, we obtain $u_{\varepsilon_n\mu_n} \to u_{\mu_0}$ as $n \to \infty$, which implies

$$u_{\varepsilon\mu} \to u_{\mu_0} \text{ in } C([0,T];H) \text{ as } (\varepsilon,\mu) \to (0_+,\mu_0).$$

Next we assume that $(\varepsilon,\, \mu) \to (0_+,\, 0_+)$.

By computations similar to those carried out in order to derive (45), we get

$$\begin{aligned} &\varepsilon \parallel u'_{\varepsilon\mu} - u'_{00} \parallel^2_{\mathcal{H}} + \omega_0 \parallel u_{\varepsilon\mu} - u_{00} \parallel^2_{\mathcal{H}} \\ &\quad \leq \mu \parallel u'_{00} \parallel_{\mathcal{H}} \parallel u_{\varepsilon\mu} - u_{00} \parallel_{\mathcal{H}} + \varepsilon \parallel u'_{00} \parallel_{\mathcal{H}} \parallel u'_{\varepsilon\mu} - u'_{00} \parallel_{\mathcal{H}} . \end{aligned} \tag{47}$$

Using again the elementary inequality $xy \leq (x^2 + y^2)/2$ in (47) we obtain

$$\varepsilon \parallel u'_{\varepsilon\mu} - u'_{00} \parallel^2_{\mathcal{H}} + \omega_0 \parallel u_{\varepsilon\mu} - u_{00} \parallel^2_{\mathcal{H}} = \mathcal{O}(\varepsilon) + \mathcal{O}(\mu^2), \tag{48}$$

as $(\varepsilon, \mu) \to (0_+, 0_+)$, hence

$$\parallel u_{\varepsilon\mu} - u_{00} \parallel_{\mathcal{H}} = \mathcal{O}(\sqrt{\varepsilon}) + \mathcal{O}(\mu) \text{ as } (\varepsilon, \mu) \to (0_+, 0_+). \tag{49}$$

In order to obtain convergence (37), we note that the assumption $\mu^2/\varepsilon = \mathcal{O}(1)$ as $(\varepsilon, \mu) \to (0_+, 0_+)$, combined with (48), yields that we can find two positive constants C_0 and δ_0 that are independent of both ε and μ, with the property that

$$\parallel u'_{\varepsilon\mu} \parallel^2_{\mathcal{H}} \leq C_0 \text{ for all } (\varepsilon, \mu) \in (0, \delta_0) \times (0, \delta_0).$$

This estimate together with (2) implies that the set $\{\parallel u_{\varepsilon\mu} - u_{00} \parallel;\ (\varepsilon, \mu) \in (0, \delta_0) \times (0, \delta_0)\}$ is relatively compact in $C[0, T]$. Thus, by the Arzelà-Ascoli Criterion combined with (49), we obtain that every convergent sequence with positive components $(\varepsilon_n, \mu_n) \to (0_+, 0_+)$ has a convergent subsequence, again denoted $\big(\varepsilon_n, \mu_n\big)_n$, such that

$$u_{\varepsilon_n\mu_n} \to u_{00} \text{ in } C([0, T]; H) \text{ as } (\varepsilon_n, \mu_n) \to (0_+, 0_+).$$

Finally, the uniqueness of the solution u_{00} to algebraic inclusion (E_{00}) implies

$$u_{\varepsilon\mu} \to u_{00} \text{ in } C([0, T]; H) \text{ as } (\varepsilon, \mu) \to (0_+, 0_+).$$

Our proof is thus complete. □

Remark 3.2. *In the case of problems $(\overline{P}_{\varepsilon\mu})_{ap}$, $(\overline{P}_{\mu})_{ap}$, as well as the inclusion $(\overline{E}_{00})$ obtained by considering the Lipschitz perturbation F, the statements of Theorem 3.1 (ii) hold true assuming the framework in Theorem 2.4. This can be easily verified by reviewing the proof above and noting that all estimates in Theorem 3.1 (ii) hold, with the constant ω_0 replaced by $\omega_0 - L$, where L is the Lipschitz constant of F.*

4 Applications

In this section, we will illustrate the relevance of our abstract theorems by applying them to boundary value problems for the telegraph system and the semilinear heat equation.

4.1 Semilinear telegraph system with time anti-periodic solution

We encounter telegraph systems in the theory of integrated circuits, models of arterial networks (such as arterial blood flow), traffic flows on networks, and networks of open channels (see [18]).

Let $D_T := [0,T] \times [0,1]$, where we consider the problem presented below, denoted as $(P^1_{\varepsilon\mu})_{ap}$:

$$\begin{cases} -\varepsilon u_{tt} + \mu u_t + v_x + ru = f_1(t,x), \\ -\varepsilon v_{tt} + cv_t + u_x + gv = f_2(t,x), \ (t,x) \in D_T, \end{cases}$$

$$(-u(t,0), u(t,1)) \in \Gamma(v(t,0), v(t,1)), \quad t \in (0,T),$$

$$\begin{cases} u(0,x) + u(T,x) = 0, \ v(0,x) + v(T,x) = 0, \\ u_t(0,x) + u_t(T,x) = 0, \ v_t(0,x) + v_t(T,x) = 0, \ x \in (0,1). \end{cases}$$

The above problem can be considered a Lions-type regularization (for small ε) of the reduced problem linked with the telegraph differential system, denoted by $(P^1_\mu)_{ap}$:

$$\begin{cases} \mu u_t + v_x + ru = f_1(t,x), \\ cv_t + u_x + gv = f_2(t,x), \ (t,x) \in D_T, \end{cases}$$

$$(-u(t,0), u(t,1)) \in \Gamma(v(t,0), v(t,1)), \quad t \in (0,T), \tag{50}$$

$$u(0,x) + u(T,x) = 0, \ v(0,x) + v(T,x) = 0, \ x \in (0,1). \tag{51}$$

Here, the positive constants r, g, μ, c represent the resistance, conductance, inductance and capacitance per unit length of an electrical circuit (long line) and the functions $u(t,x)$, $v(t,x)$ represent the current and voltage at time instant t and point x (see, e.g., [12] and [17, Chapter III]).

We assume that

(h_{rgc}) r, g, c are strictly positive constants;

(h_{f_i}) $f_i \in W^{1,1}(0,T; L^2(0,1))$, and $f_i(0) + f_i(T) = 0, \ i = 1,2$;

(h_Γ) $\Gamma \subset \mathbb{R}^2$ (possibly set-valued) is an odd maximal monotone operator.

The boundary condition at the ends of the circuit is very general, which means that many other classical boundary conditions may emerge from it as isolated cases. For example, if Γ is linear, this condition turns into equalities that represent Ohm's law at the ends of the circuit where $x = 0$ and $x = 1$. On the other hand, if Γ is the subdifferential of the function $j : \mathbb{R}^2 \to (-\infty, +\infty]$, defined by $j(x,y) = 0$ if $x = y$, and $j(x,y) = +\infty$, otherwise, the condition corresponds to space periodic boundary conditions

$$u(t,0) = u(t,1), \ v(t,0) = v(t,1) \ \ t \in (0,T).$$

Notice that in both problems $(P^1_{\varepsilon\mu})_{ap}$ and $(P^1_\mu)_{ap}$, the parameter μ appears exclusively in one equation of the system. We will consider the inductance, represented by μ, to be a small parameter. If the corresponding frequency is small, then the inductance of the line is small as well (see [4, Chapter 3]).

In the Hilbert space $H = L^2(0,1) \times L^2(0,1)$, with the standard inner product and norm, problems $(P^1_{\varepsilon\mu})_{ap}$ and $(P^1_{\mu})_{ap}$ can be expressed as abstract boundary value problems.
Indeed, define $A : D(A) \subset H \to H$ by

$$D(A) = \big\{ \mathbf{v} = [v_1, v] \in H^1(0,1)^2;\ [-v_1(0), v_1(1)] \in \Gamma([v_2(0), v_2(1)]) \big\},$$
$$A\,\mathbf{v} = [v_2' + rv_1, v_1' + gv_2] \quad \forall\ \mathbf{v} \in D(A).$$

Setting $\mathbf{w} := [u, v]$, $\mathbf{f} := [f_1, f_2]$, we can write the two problems above as

$$\begin{cases} -\varepsilon \mathbf{w}(t) + [\mu u', cv'](t) + A\,\mathbf{w}(t) \ni \mathbf{f}(t) \text{ a.e. } t \in (0,T), \\ \mathbf{w}(0) + \mathbf{w}(T) = [0,0],\ \ \mathbf{w}'(0) + \mathbf{w}'(T) = [0,0], \end{cases} \tag{52}$$

and

$$\begin{cases} [\mu u', cv'](t) + A\,\mathbf{w}(t) \ni \mathbf{f}(t) \text{ a.e. } t \in (0,T), \\ \mathbf{w}(0) + \mathbf{w}(T) = [0,0], \end{cases} \tag{53}$$

where $\mathbf{w}(t) = [u(t,\cdot), v(t,\cdot)]$ and $\mathbf{f}(t) := [f_1(t,\cdot), f_2(t,\cdot)]$.

The above definition of the operator A renders it to be maximal monotone in H (see [15, Section 5.1]). On the other hand, since A is not a subdifferential, that means that it is also not cyclically monotone. A simple calculation reveals that A is strongly monotone with constant $\omega_0 = \min\{r, g\}$. Therefore, we obtain the following result which is very similar to Theorem 2.3 and Theorem 3.1 above.

Theorem 4.1. *Assume that (h_{rcg}), (h_Γ), and (h_{f_i}), $i = 1,2$, are fulfilled. Then, for every nonnegative ε and μ such that $\varepsilon + \mu > 0$, and f_i satisfying (H_{f_i}), $i = 1,2$, the problems $(P^1_{\varepsilon\mu})_{ap}$ and $(P^1_{\mu})_{ap}$ have unique solutions*

$$[u_{\varepsilon\mu}, v_{\varepsilon\mu}] \in (W^{2,2}(0,T; L^2(0,1)) \cap L^2(0,T; H^1(0,1))^2,$$

and

$$[u_{\mu}, v_{\mu}] \in (W^{1,2}(0,T; L^2(0,1)) \cap L^2(0,T; H^1(0,1))^2.$$

In addition, the following estimates hold

$$\begin{aligned} &\| [u'_{\varepsilon\mu}, v'_{\varepsilon\mu}] \|_{\mathcal{H}^2} \le \omega_0^{-1} \| [f_1', f_2'] \|_{(L^2(D_T))^2} \quad \textit{for every } \varepsilon > 0,\ \mu \ge 0, \\ &\| [u'_{\mu}, v'_{\mu}] \|_{(L^2(D_T))^2} \le \omega_0^{-1} \| [f_1', f_2'] \|_{\mathcal{H}^2} \ \ \textit{for every } \mu > 0. \end{aligned} \tag{54}$$

Moreover, for nonnegative ε_0 and μ_0 such that $\varepsilon_0 + \mu_0 > 0$, the following *estimates and approximations hold*

$$\begin{aligned}
&\| [u_{\varepsilon\mu} - u_{\varepsilon_0\mu_0}, v_{\varepsilon\mu} - v_{\varepsilon_0\mu_0}] \|_{C([0,T];H)} = \mathcal{O}(| \varepsilon - \varepsilon_0 |) + \mathcal{O}(| \mu - \mu_0 |), \\
&[u_{\varepsilon\mu}, v_{\varepsilon\mu}] \to [u_{\varepsilon_0\mu_0}, v_{\varepsilon_0\mu_0}] \quad \text{in } C^1([0,T];H) \quad \text{as } (\varepsilon,\mu) \to (\varepsilon_0, \mu_0), \\
&\| [u_{\varepsilon\mu} - u_{\mu_0}, v_{\varepsilon\mu} - v_{\mu_0}] \|_{L^2(D_T)^2} = \mathcal{O}(\sqrt{\varepsilon}) + \mathcal{O}(| \mu - \mu_0 |) \text{ and} \\
&[u_{\varepsilon\mu}, v_{\varepsilon\mu}] \to [u_{\mu_0}, v_{\mu_0}] \quad \text{in } C([0,T];H) \quad \text{as } (\varepsilon,\mu) \to (0_+, \mu_0),
\end{aligned} \tag{55}$$

where $H = L^2(0,1) \times L^2(0,1)$.

Proof. Upon reviewing the proof of Theorem 2.3, it becomes evident that all the arguments and computations used there remain valid within this framework, as the coefficients μ and c of the term $[u', v']$ are irrelevant. Consequently, the solutions to $(P^1_{\varepsilon\mu})_{ap}$ and $(P^1_{\mu})_{ap}$ both exist and are unique; we also obtain the uniform estimates (54).

In order to derive (55), we look at Theorem 3.1 and apply similar arguments to those employed in its proof. However, we do need some tiny adjustments that we highlight below. To proceed with, consider the case $(\varepsilon,\mu) \to (0_+, \mu_0)$; the other case, $(\varepsilon,\mu) \to (\varepsilon_0, \mu_0)$, is similar. Obviously, from $(52)_1$ and $(53)_1$, we obtain

$$\begin{aligned}
-\varepsilon[u''_{\varepsilon\mu}, v''_{\varepsilon\mu}] + [\mu(u'_{\varepsilon\mu} - u'_{\mu_0}), c(v'_{\varepsilon\mu} - v'_{\mu_0})] + [(\mu - \mu_0)u'_{\mu_0}, 0] \\
+ A\,[u_{\varepsilon\mu}, v_{\varepsilon\mu}] - A\,[u_{\mu_0}, v_{\mu_0}] \ni [0, 0].
\end{aligned} \tag{56}$$

Taking the scalar product in $\mathcal{H}^2$ of the above inclusion and $[u_{\varepsilon\mu} - u_{\mu_0}, v_{\varepsilon\mu} - v_{\mu_0}]$ using the strong monotonicity of A and the antiperiodic conditions, we derive

$$\begin{aligned}
\varepsilon \| [u'_{\varepsilon\mu}, v'_{\varepsilon\mu}] \|^2_{\mathcal{H}^2} &+ \omega_0 \| [u_{\varepsilon\mu} - u_{\mu_0}, v_{\varepsilon\mu} - v_{\mu_0}] \|^2_{\mathcal{H}^2} \\
&\le | \mu - \mu_0 | \cdot \| u'_{\mu_0} \|_{\mathcal{H}} \| u_{\varepsilon\mu} - u_{\mu_0} \|_{\mathcal{H}} \\
&+ \varepsilon \big(\| u'_{\varepsilon\mu} \|_{\mathcal{H}} \| u'_{\mu_0} \|_{\mathcal{H}} + \| v'_{\varepsilon\mu} \|_{\mathcal{H}} \| v'_{\mu_0} \|_{\mathcal{H}} \big) \\
&\le K \left(| \mu - \mu_0 | \| u_{\varepsilon\mu} - u_{\mu_0} \|_{\mathcal{H}} + \varepsilon \| [u'_{\varepsilon\mu}, v'_{\varepsilon\mu}] \|_{\mathcal{H}^2} \right),
\end{aligned} \tag{57}$$

by (54) (here $K = \omega_0^{-1} \| [f'_1, f'_2] \|_{\mathcal{H}^2}$). From the elementary inequality $xy \le (x^2 + y^2)/2$, this implies

$$\frac{\varepsilon}{2} \| [u'_{\varepsilon\mu}, v'_{\varepsilon\mu}] \|^2_{\mathcal{H}^2} + \frac{\omega_0}{2} \| [u_{\varepsilon\mu} - u_{\mu_0}, v_{\varepsilon\mu} - v_{\mu_0}] \|^2_{\mathcal{H}^2} \le \frac{K^2}{2} \left(\frac{(\mu - \mu_0)^2}{\omega_0} + \varepsilon \right).$$

Thus,

$$\| [u_{\varepsilon\mu} - u_{\mu_0}, v_{\varepsilon\mu} - v_{\mu_0}] \|_{\mathcal{H}} = \mathcal{O}(\sqrt{\varepsilon}) + \mathcal{O}(| \mu - \mu_0 |) \text{ as } (\varepsilon,\mu) \to (0_+, \mu_0). \tag{58}$$

Finally, the Arzel-Ascoli Criterion along with (54) and the uniqueness of the solution $[u_{\mu_0}, v_{\mu_0}]$ to problem $(P^1_{\mu_0})_{ap}$, ensures the convergence

$$[u_{\varepsilon\mu}, v_{\varepsilon\mu}] \to [u_{\mu_0}, v_{\mu_0}] \text{ in } C([0,T];H) \text{ as } (\varepsilon,\mu) \to (0_+, \mu_0).$$

Our proof is thus complete. □

4.2 Semilinear heat equation with anti-periodic solutions

Let N be a positive integer, and let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. In this subsection, we set $\mu = 1$, to derive a Lions regularization for the anti-periodic solution of a semilinear heat equation presented below. Thus, for $\varepsilon > 0$, we consider the following pair of problems

$$(P^2)_{ap} \begin{cases} u_t - \Delta_x u + \theta\, u = \left(\int_\Omega u^2 dx\right)^p u + f \text{ a.e. in } \mathbb{R}\times\Omega, \\ u(t,x) = 0 \text{ for a.e. } t\in\mathbb{R},\ x\in\partial\Omega, \\ u(0,x) + u(T,x) = 0,\ x\in\Omega, \end{cases}$$

$$(P^2_\varepsilon)_{ap} \begin{cases} -\varepsilon u_{tt} + u_t - \Delta_x u + \theta\, u = \left(\int_\Omega u^2 dx\right)^p u + f \text{ a.e. in } \mathbb{R}\times\Omega, \\ u(t,x) = 0 \text{ or } \frac{\partial u}{\partial \nu}(t,x) = 0 \text{ for a.e. } t\in\mathbb{R},\ x\in\partial\Omega, \\ u(0,x) + u(T,x) = 0,\ u_t(0,x) + u_t(T,x) = 0,\ x\in\Omega, \end{cases}$$

where $p \geq 0$ and Δ_x represents the $N-$dimensional Laplacian with respect to the variable x. Suppose that the following assumptions are met

(h_f) $f \in W^{1,2}(0,T;L^2(\Omega))$, and $f_i(0) + f_i(T) = 0,\ i = 1,2$;
(h_θ) $\theta \subset \mathbb{R}^2$ is an odd, maximal monotone (potentially set-valued) operator.

We choose the Hilbert spaces $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$, and define the even, proper, convex and lower semicontinuos function $\phi : H \to [0, +\infty]$

$$\phi(v) = \begin{cases} \frac{1}{2} \parallel \nabla v \parallel^2_{L^2(\Omega;\mathbb{R}^N)} + \parallel j(v) \parallel^2_{L^1(\Omega)} & \text{if } v\in V \text{ and } j(v)\in L^1(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

where $j : \mathbb{R} \to (-\infty, +\infty]$ is such that $\partial j = \theta$ (here and throughout ∂ denotes the subdifferential).

Next, we will derive the abstract formulations of the aforementioned problems in H. To this end, we define the operators $A : D(A) \to H$ and $F : H \to H$ as follows

$$D(A) = \{u \in H^2(\Omega)\cap V;\ \Delta u + \bar\theta u \in H\},\ A\, v = -\Delta\, v + \bar\theta\, v \text{ for all } v\in D(A),$$
$$F : H \to F,\ F\, v = \parallel u \parallel^p v \text{ for all } v \in H,$$

where $\bar\theta$ is the canonical extension to the operator θ to the Hilbert space $L^2(0,T;H)$. It is known that $A = \partial\psi$ (see, e.g., [6, Proposition 3.8, p. 89]). Therefore, A is maximal cyclically monotone and also strongly monotone with constant $\omega_0 = \lambda_1$, where λ_1 is the smallest eigenvalue of the Laplace operator on $H_0^1(\Omega)$. On the other hand, F is the Fréchet derivative of the even, proper, and convex function

$$\psi : H \to \mathbb{R},\ \psi(v) = \frac{1}{2p+2} \parallel v \parallel^{2p+2} \quad \text{for all } v \in H.$$

Also, the operator F is local Lipschitz continuous being continuous Fréchet differentiable on H. Indeed, for all $r > 0$

$$\| F\, v_1 - F\, v_2 \| \leq (2p+1) r^{2p} \| v_1 - v_2 \| \ \text{ if } v_i \in H, \| v_i \| \leq r, i = 1, 2. \quad (59)$$

Clearly, $(P_\varepsilon^2)_{ap}$ can be expressed in abstract form in H as follows

$$\begin{cases} -\varepsilon u''(t) + u'(t) + A\, u(t) \ni F\, u(t) + f(t) \text{ a.e. } t \in (0, T), \\ u(0) + u(T) = 0, \ u'(0) + u'(T) = 0, \end{cases} \quad (60)$$

where $u(t) = u(t, \cdot)$ and $f(t) = f(t, \cdot)$. Consider $u_\varepsilon \in W^{2,2}(0, T; H)$ as a solution to $(P_\varepsilon^2)_{ap}$. Since A and F are odd operators of subdifferential type, taking the scalar product in $L^2(0, T; H)$ of $(60)_1$ (with u_ε instead of u) and u_ε', applying the chain rule for subdifferential (see, e.g., [9, Lemma 3.3, p. 73]), and the $T-$ antiperiodicity of u_ε', we obtain $\| u_\varepsilon' \|_{\mathcal{H}} \leq \| f \|_{L^2((0,T)\times\Omega)}$.

This implies, by (2), that

$$\| u_\varepsilon \|_\infty \leq \underbrace{\sqrt{T} \, \| f \|_{L^2(0,T)} / 2}_{= \mathcal{R}_0}. \quad (61)$$

Moreover, using a similar technique, we can show that the solution to $(P^2)_{ap}$ satisfies the same estimate. To apply Theorem 2.4, we define the operator $F_{\mathcal{R}_0} = F \circ h_{\mathcal{R}_0}$, where $h_{\mathcal{R}_0}$ is the radial retraction given by $h_{\mathcal{R}_0}(x) = x$ if $\| x \| \leq \mathcal{R}_0$, and $h_{\mathcal{R}_0}(x) = \mathcal{R}_0 x / \| x \|$ otherwise. Since $h_{\mathcal{R}_0}$ is Lipschitz with a constant in the range $[1, 2]$ (see, e.g., [3, p. 55]), it follows that F_r is Lipschitz on H with a Lipschitz constant no greater that $L_{\mathcal{R}_0} := 2(2p+1)\mathcal{R}_0^{2p}$ (see (59)). If $L_{\mathcal{R}_0} < \omega_0 = \lambda_1$, that is

$$2^{1-2p}(2p+1) T^p \| f \|_{L^2((0,T)\times\Omega)}^{2p} < \lambda_1, \quad (62)$$

then Theorem 2.4 applies to our problems with $F_{\mathcal{R}_0}$ in place of F, ensuring the existence and uniqueness of solutions in this case. As can be seen from (61) and the definition of $F_{\mathcal{R}_0}$, these solutions are, in fact, the unique solutions of the original problems.

Consequently, from Theorem 2.4 and Theorem 3.1 (see also Remark 3.2), we have the following result.

Theorem 4.2. *Let* $p \geq 0$. *Assume that* (h_θ) *and* (62) *are satisfied. Then, for every* $\varepsilon > 0$ *and* f *satisfying* (h_f), *the problems* $(P_\varepsilon^2)_{ap}$ *and* $(P^2)_{ap}$ *have unique solutions* $u_\varepsilon \in W^{2,2}(0, T; L^2(\Omega)) \cap C(0, T; H_0^1(\Omega))$ *and* $u \in W^{1,2}(0, T; L^2(\Omega))$ $\cap\, C(0, T; H_0^1(\Omega))$, *respectively. Moreover, for any fixed* $\varepsilon_0 \geq 0$, *the following* *estimates and approximations hold true*

$$\begin{aligned}
&\| u_\varepsilon - u_{\varepsilon_0} \|_{C([0,T];L^2(\Omega))} = \mathcal{O}(| \varepsilon - \varepsilon_0 |),\\
&\| u'_\varepsilon - u'_{\varepsilon_0} \|_{L^2((0,T);H_0^1(\Omega))} = \mathcal{O}(| \varepsilon - \varepsilon_0 |) \quad \textit{and}\\
&u_\varepsilon \to u_{\varepsilon_0} \quad \textit{in} \quad C^1([0,T];L^2(\Omega) \quad \textit{as} \quad \varepsilon \to \varepsilon_0 > 0,\\
&\| u_\varepsilon - u \|_{L^2((0,T)\times\Omega))} = \mathcal{O}(\sqrt{\varepsilon}),\\
&u_\varepsilon \to u \quad \textit{in} \quad C([0,T];L^2(\Omega)) \quad \textit{as } \varepsilon \to 0_+.
\end{aligned} \tag{63}$$

In particular, the last convergence in (63) confirms that $(P^2_\varepsilon)_{ap}$ provides a genuine Lions elliptic regularization for the problem $(P^2)_{ap}$.

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