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An alternate measure of the cumulative residual Sharma-Taneja-Mittal entropy

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Abstract

We define a new alternate measure of the cumulative residual Sharma-Taneja-Mittal entropy. For this measure, there are given upper and lower bounds, is introduced a consistent test based on the uniform distribution and some concrete numerical examples are formulated.

1 Introduction

In order to analyse some physical phenomena, Tsallis introduced Tsallis entropy in [57], which is a generalization of Shannon entropy (see [47]). The idea was to work with another formula instead of classical logarithm, which is used in Shannon entropy. Tsallis entropy is applied especially in physics, the reader can find concrete applications in: superstatistics (see [7]), spectral statistics (see [58]), earthquakes (see [3], [9], [12]), plasma (see [23]), non-coding human DNA (see [28]), income distribution (see [50]), statistical mechanics (see [31], [35], [43], [57]), internet (see [2]), stock exchanges (see [17], [18]).

There are also other generalizations of Shannon entropy: Rényi entropy, Varma entropy, Abe entropy, Kaniadakis entropy, cumulative entropy, weighted entropy, relative entropy, with applications in: finance (see [30], [55], [56], [61]), Markov chains (see [4], [5], [6]), combinatorics (see [36], [37]), Lie symmetries (see [14], [32]), model selection (see [52], [53]), survival analysis

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(see [42], [46], [51]), nonlinear equations (see [13], [54]), statistical mechanics (see [44], [45], [49]), machine learning (see [16], [59]).

Sharma and Taneja [48] and Mittal [24] introduced, in the framework of the information theory, Sharma-Taneja-Mittal entropy, a generalized entropy with two parameters. Afterwards, Kaniadakis et al. [20] reconsidered this entropy from a physical point of view. It was proved that many aspects of the statistical mechanics based on Boltzmann-Gibbs entropy remain valid in the case of statistical mechanics based on Sharma-Taneja-Mittal entropy. Scarfone and Wada [41] studied the thermo-statistics properties of this theory in the microcanonical picture.

We remark that Sharma-Taneja-Mittal entropy includes some of the one-parameter entropies mentioned above, namely Abe entropy [1], Kaniadakis entropy [19] and Tsallis entropy [57]. Consequently, we are able to consider all these one-parameter entropies in an unified scheme.

Sharma-Taneja-Mittal entropy is useful in: analysis of record values (see [29]), investigations of diffusion processes (see [10]), modeling holographic dark energy (see [40], [63]), investigation of the different phenomenon of black holes (see [11], [38]), modeling uncertainty in the theory of human cognition (see [8]) and estimating the performance of clustering models in data analysis (see [21]).

Tsallis entropy plays an important role in the measurement uncertainty of random variables and is the basis of the nonextensive statistical mechanics, which generalizes the Boltzmann-Gibbs theory. Sati and Gupta [39] introduced the cumulative residual Tsallis entropy (CRTE) and studied it from the point of view of reliability modeling. Rajesh and Sunoj [33] defined an alternate measure of CRTE and showed that it has some additional features and simple relationships with other important information and reliability measures.

By considering Sharma-Taneja-Mittal entropy instead of Tsallis entropy in the definition of CRTE from [48], we obtain the cumulative residual Sharma-Taneja-Mittal entropy (CRSTME). In this paper, we work with an alternate measure of CRSTME, defined in a similar way like the alternate measure of CRTE from [33]. The difference is that we have two parameters θ_1 and θ_2, instead of one parameter θ.

The paper is organized as follows. After this *Introduction*, in Section 2, named *Preliminaries*, we present the main notions and notations used throughout the article, including the definition of the alternate measure of the cumulative residual Sharma-Taneja-Mittal entropy, denoted via $CSTM_{\theta_1,\theta_2}(X)$, which will be used in the next sections. In Section 3, named *Bounds for* $CSTM_{\theta_1,\theta_2}(X)$, we give an upper bound for $CSTM_{\theta_1,\theta_2}(X)$ (Theorem 3.4) and a lower bound for $CSTM_{\theta_1,\theta_2}(X)$ (Theorem 3.7). In Section 4, named *Test based on uniform distribution*, we introduce a consistent test with the help of the uniform distribution. In section 5, named *Concrete examples*, there are given some examples considering fixed values for the parameter θ_1. Finally, we formulate *Conclusions* concerning the results obtained in the paper.

2 Preliminaries

We use the notations $(\xrightarrow[n]{p})$ for convergence in probability and $(\xrightarrow[n]{a.s.})$ for almost surely convergence (as $n \to \infty$).

Let X be a random variable having absolutely continuous cumulative distribution function F_X, survival function $\overline{F}_X \stackrel{def}{=} 1-F_X$ and probability density function f_X.

Shannon entropy of X is given by

$$H_X = -\int_{-\infty}^{\infty} f_X(x) \log\left(f_X(x)\right) dx,$$

where "log" is the natural logarithm function.

Rao et al. [34] and Wang et al. [60] defined a non-negative measure of uncertainty, namely the cumulative residual entropy (CRE). This measure is obtained by replacing the probability density function f_X in the expression of Shannon entropy by the survival function $\overline{F}_{|X|}$. More exactly, CRE is defined by

$$CRE(X) = CRE(F) = -\int_0^{\infty} \overline{F}_{|X|}(x) \log\left(\overline{F}_{|X|}(x)\right) dx,$$

where $\overline{F}_{|X|}(x) = P\left(|X| > x\right)$ for any $x \geq 0$.

Using CRE, Noughabi [27] developed a test for uniformity and compared the percentage points and power of seven alternative distributions. In order to test the uniformity, Mohamed et al. [25] used the fractional and weighted CRE measures.

Let $\theta \in (0, \infty) \setminus \{1\}$. Tsallis entropy of X is given via

$$H_X^T = \frac{1}{\theta - 1} \int_{-\infty}^{\infty} \left[f_X(x) - \left(f_X(x)\right)^{\theta} \right] dx = \frac{1}{\theta - 1} \left(1 - \int_{-\infty}^{\infty} \left(f_X(x)\right)^{\theta} dx \right).$$

Sati and Gupta [39] introduced the cumulative residual Tsallis entropy (CRTE), given via

$$CT_{\theta}^{*}(X) = CT_{\theta}^{*}(F) = \frac{1}{\theta - 1} \int_0^{\infty} \left[f_{|X|}(x) - \left(\overline{F}_{|X|}(x)\right)^{\theta} \right] dx =$$
$$\frac{1}{\theta - 1} \left(1 - \int_0^{\infty} \left(\overline{F}_{|X|}(x)\right)^{\theta} dx \right).$$

Afterwards, Rajesh and Sunoj [33] introduced an alternate measure of CRTE, as

$$CT_\theta(X) = CT_\theta(F) = \frac{1}{\theta - 1} \int_0^\infty \left[\overline{F}_{|X|}(x) - \left(\overline{F}_{|X|}(x) \right)^\theta \right] dx.$$

If $\theta \to 1$, then $CT_\theta(X) \to CRE(X)$, but $CT^*_\theta(X) \not\to CRE(X)$.

According to Rajesh and Sunoj [33], $CT_\theta(X)$ has more interesting mathematical features than $CRE(X)$. More exactly, it can be easily estimated from sample data and these estimates converge asymptotically to the true values, for the standard uniform distribution, denoted by $U(0,1)$ (i.e. $X \sim U(0,1)$), the value of $CT_\theta(X)$ is $\dfrac{1}{2(1+\theta)}$ and $CT_\theta(X)$ handles the information in residual life.

Let $\theta_1, \theta_2 \in (0, \infty)$ such that $\theta_2 > \theta_1$.

Sharma-Taneja-Mittal entropy of X is given via

$$H_X^{STM} = \frac{1}{\theta_2 - \theta_1} \int_{-\infty}^{\infty} \left((f_X(x))^{\theta_1} - (f_X(x))^{\theta_2} \right) dx.$$

Like in [39] we define the cumulative residual Sharma-Taneja-Mitall entropy (CRSTME) via

$$CSTM^*_{\theta_1,\theta_2}(X) = CSTM^*_{\theta_1,\theta_2}(F) = \frac{1}{\theta_2 - \theta_1} \int_0^\infty \left[\left(f_{|X|}(x) \right)^{\theta_1} - \left(\overline{F}_{|X|}(x) \right)^{\theta_2} \right] dx$$

and, as in [33], we define an alternate measure of the cumulative residual Sharma-Taneja-Mittal entropy by

$$CSTM_{\theta_1,\theta_2}(X) = CSTM_{\theta_1,\theta_2}(F) = \frac{1}{\theta_2 - \theta_1} \int_0^\infty \left[\left(\overline{F}_{|X|}(x) \right)^{\theta_1} - \left(\overline{F}_{|X|}(x) \right)^{\theta_2} \right] dx.$$

Remark 2.1. We can easily see that $CSTM_{\theta_1,\theta_2}(X) \geq 0$.

Proposition 2.2. *The following relationship is hold:*

$$CSTM_{\theta_1,\theta_2}(X) = \frac{\theta_2 - 1}{\theta_2 - \theta_1} CT_{\theta_2}(X) - \frac{\theta_1 - 1}{\theta_2 - \theta_1} CT_{\theta_1}(X).$$

Proof. We have

$$CSTM_{\theta_1,\theta_2}(X) = \frac{1}{\theta_2 - \theta_1} \left(\int_0^\infty \left(\left(\overline{F}_{|X|}(x) \right)^{\theta_1} - \overline{F}_{|X|}(x) + \overline{F}_{|X|}(x) - \left(\overline{F}_{|X|}(x) \right)^{\theta_2} \right) dx \right) =$$

$$-\frac{1}{\theta_2-\theta_1}\int_0^\infty(\theta_1-1)\frac{1}{\theta_1-1}\left[\overline{F}_{|X|}(x)-\left(\overline{F}_{|X|}(x)\right)^{\theta_1}\right]dx+$$
$$\frac{1}{\theta_2-\theta_1}\int_0^\infty(\theta_2-1)\frac{1}{\theta_2-1}\left[\overline{F}_{|X|}(x)-\left(\overline{F}_{|X|}(x)\right)^{\theta_2}\right]dx=$$
$$\frac{\theta_2-1}{\theta_2-\theta_1}CT_{\theta_2}(X)-\frac{\theta_1-1}{\theta_2-\theta_1}CT_{\theta_1}(X).$$

□

3 Bounds for $CSTM_{\theta_1,\theta_2}(X)$

Theorem 3.1. *Assume that $\theta_2>\theta_1>1$. If $X\in L^2$ (i.e. $\mathbb{E}(X^2)<\infty$), then $CSTM_{\theta_1,\theta_2}(X)<\infty$.*

Proof. From Proposition 2.2 we have

$$CSTM_{\theta_1,\theta_2}(X)\le\frac{\theta_2-1}{\theta_2-\theta_1}CT_{\theta_2}(X).$$

We apply Theorem 1 from [26] and obtain the conclusion. □

Remark 3.2. Assume that $\theta_2>\theta_1>1$. Then the existence of $Var(X)$ assures that $CSTM_{\theta_1,\theta_2}(X)<\infty$.

Lemma 3.3. *We consider $g_{\theta_1,\theta_2}:[0,1]\to\mathbb{R}$,*

$$g_{\theta_1,\theta_2}(t)=\frac{1}{\theta_2-\theta_1}\left(t^{\theta_1}-t^{\theta_2}\right).$$

Then, for any $t\in[0,1]$,

$$0\le g_{\theta_1,\theta_2}(t)\le\frac{1}{\theta_2-\theta_1}\left[\left(\frac{\theta_1}{\theta_2}\right)^{\frac{\theta_1}{\theta_2-\theta_1}}-\left(\frac{\theta_1}{\theta_2}\right)^{\frac{\theta_2}{\theta_2-\theta_1}}\right].$$

Proof. Let $t\in[0,1]$. We have

$$g'_{\theta_1,\theta_2}(t)=\frac{1}{\theta_2-\theta_1}\left[\theta_1t^{\theta_1-1}-\theta_2t^{\theta_2-1}\right]$$

and

$$g'_{\theta_1,\theta_2}(t)=0\Longleftrightarrow t=\left(\frac{\theta_1}{\theta_2}\right)^{\frac{1}{\theta_2-\theta_1}}.$$

It follows that

$$0 \leq g_{\theta_1,\theta_2}(t) \leq g_{\theta_1,\theta_2}\left(\left(\frac{\theta_1}{\theta_2}\right)^{\dfrac{1}{\theta_2-\theta_1}}\right) =$$

$$\frac{1}{\theta_2-\theta_1}\left[\left(\frac{\theta_1}{\theta_2}\right)^{\dfrac{\theta_1}{\theta_2-\theta_1}} - \left(\frac{\theta_1}{\theta_2}\right)^{\dfrac{\theta_2}{\theta_2-\theta_1}}\right].$$

□

Theorem 3.4. *Assume that X has support $[0,1]$. Then*

$$CSTM_{\theta_1,\theta_2}(X) \leq \frac{1}{\theta_2-\theta_1}\left[\left(\frac{\theta_1}{\theta_2}\right)^{\dfrac{\theta_1}{\theta_2-\theta_1}} - \left(\frac{\theta_1}{\theta_2}\right)^{\dfrac{\theta_2}{\theta_2-\theta_1}}\right].$$

Proof. By Lemma 3.3 we get

$$CSTM_{\theta_1,\theta_2}(X) = \frac{1}{\theta_2-\theta_1}\int_0^1 \left[\left(\overline{F}_{|X|}(x)\right)^{\theta_1} - \left(\overline{F}_{|X|}(x)\right)^{\theta_2}\right] dx \leq$$

$$\int_0^1 \left\{\frac{1}{\theta_2-\theta_1}\left[\left(\frac{\theta_1}{\theta_2}\right)^{\dfrac{\theta_1}{\theta_2-\theta_1}} - \left(\frac{\theta_1}{\theta_2}\right)^{\dfrac{\theta_2}{\theta_2-\theta_1}}\right]\right\} dx =$$

$$\frac{1}{\theta_2-\theta_1}\left[\left(\frac{\theta_1}{\theta_2}\right)^{\dfrac{\theta_1}{\theta_2-\theta_1}} - \left(\frac{\theta_1}{\theta_2}\right)^{\dfrac{\theta_2}{\theta_2-\theta_1}}\right].$$

□

Theorem 3.5. *Let $(X_n)_n$ be a sequence of random variables with absolutely continuous cumulative distribution functions, which converges in distribution to the random variable X and such that, for any n, $X_n \in L^2$. Then*

$$\lim_{n\to\infty} CSTM_{\theta_1,\theta_2}(X_n) = CSTM_{\theta_1,\theta_2}(X).$$

Proof. From Proposition 2.2 we have, for any n,

$$CSTM_{\theta_1,\theta_2}(X_n) = \frac{\theta_2-1}{\theta_2-\theta_1}CT_{\theta_2}(X_n) - \frac{\theta_1-1}{\theta_2-\theta_1}CT_{\theta_1}(X_n).$$

Theorem 2 from [26] assures us that

$$\lim_{n\to\infty} CT_{\theta_1}(X_n) = CT_{\theta_1}(X) \text{ and } \lim_{n\to\infty} CT_{\theta_2}(X_n) = CT_{\theta_2}(X).$$

Hence

$$\lim_{n\to\infty} CSTM_{\theta_1,\theta_2}(X_n) = CSTM_{\theta_1,\theta_2}(X).$$

□

Lemma 3.6. *We consider*

$$C(\theta_1,\theta_2) = e^{\displaystyle\int_0^1 \log\left(\frac{1}{\theta_2-\theta_1}\left(t^{\theta_1}-t^{\theta_2}\right)\right)dt}.$$

Then

$$0 < C(\theta_1,\theta_2) \le \frac{1}{\theta_2-\theta_1}\left[\left(\frac{\theta_1}{\theta_2}\right)^{\dfrac{\theta_1}{\theta_2-\theta_1}} - \left(\frac{\theta_1}{\theta_2}\right)^{\dfrac{\theta_2}{\theta_2-\theta_1}}\right] < \infty.$$

Proof. It is obvious that $0 < C(\theta_1,\theta_2)$.

From Lemma 3.3 we get

$$C(\theta_1,\theta_2) \le e^{\displaystyle\int_0^1 \log\left\{\frac{1}{\theta_2-\theta_1}\left[\left(\frac{\theta_1}{\theta_2}\right)^{\dfrac{\theta_1}{\theta_2-\theta_1}} - \left(\frac{\theta_1}{\theta_2}\right)^{\dfrac{\theta_2}{\theta_2-\theta_1}}\right]\right\}dt} =$$

$$\frac{1}{\theta_2-\theta_1}\left[\left(\frac{\theta_1}{\theta_2}\right)^{\dfrac{\theta_1}{\theta_2-\theta_1}} - \left(\frac{\theta_1}{\theta_2}\right)^{\dfrac{\theta_2}{\theta_2-\theta_1}}\right] < \infty.$$

□

Theorem 3.7. *Assume that the random variable X is nonnegative. Then*

$$CSTM_{\theta_1,\theta_2}(X) \ge C(\theta_1,\theta_2)e^{H^S(X)},$$

where

$$H^S(X) = -\int_0^\infty f_X(x)\log\left(f_X(x)\right)dx.$$

Proof. By the log-sum inequality, we have

$$-\log\left(CSTM_{\theta_1,\theta_2}(X)\right) =$$

$$\log\left(\frac{1}{\displaystyle\int_0^\infty \frac{1}{\theta_2-\theta_1}\left(\left(\overline{F}_X(x)\right)^{\theta_1} - \left(\overline{F}_X(x)\right)^{\theta_2}\right)dx}\right) \le$$

$$\int_0^\infty f_X(x)\log\left(\frac{f_X(x)}{\frac{1}{\theta_2-\theta_1}\left(\left(\overline{F}_X(x)\right)^{\theta_1}-\left(\overline{F}_X(x)\right)^{\theta_2}\right)}\right)dx =$$
$$\int_0^\infty f_X(x)\log\left(f_X(x)\right)dx -$$
$$\int_0^\infty f_X(x)\log\left[\frac{1}{\theta_2-\theta_1}\left(\left(\overline{F}_X(x)\right)^{\theta_1}-\left(\overline{F}_X(x)\right)^{\theta_2}\right)\right]dx =$$
$$-H^S(X)-\int_0^\infty \log\left[\frac{1}{\theta_2-\theta_1}\left(\left(\overline{F}_X(x)\right)^{\theta_1}-\left(\overline{F}_X(x)\right)^{\theta_2}\right)\right]dF_X(x) =$$
$$-H^S(X)-\int_0^1 \log\left(\frac{1}{\theta_2-\theta_1}\left(t^{\theta_1}-t^{\theta_2}\right)\right)dt.$$

It follows that

$$\log\left(CSTM_{\theta_1,\theta_2}(X)\right) \geq H^S(X)+\int_0^1 \log\left(\frac{1}{\theta_2-\theta_1}\left(t^{\theta_1}-t^{\theta_2}\right)\right)dt.$$

Hence

$$CSTM_{\theta_1,\theta_2}(X) \geq e^{\left[H^S(X)+\int_0^1 \log\left(\frac{1}{\theta_2-\theta_1}\left(t^{\theta_1}-t^{\theta_2}\right)\right)dt\right]} =$$
$$C(\theta_1,\theta_2)e^{H^S(X)}.$$

□

4 Test based on uniform distribution

Proposition 4.1. *If $X \sim U(0,1)$ then*

$$CSTM_{\theta_1,\theta_2}(X) = \frac{1}{(1+\theta_1)(1+\theta_2)}.$$

Proof. We have

$$CSTM_{\theta_1,\theta_2}(X) = \frac{1}{\theta_2-\theta_1}\int_0^\infty \left[\left(\overline{F}_{|X|}(x)\right)^{\theta_1}-\left(\overline{F}_{|X|}(x)\right)^{\theta_2}\right]dx =$$
$$\frac{1}{\theta_2-\theta_1}\int_0^1 (x^{\theta_1}-x^{\theta_2})dx = \frac{1}{\theta_2-\theta_1}\left(\frac{1}{\theta_1+1}-\frac{1}{\theta_2+1}\right) =$$
$$\frac{1}{\theta_2-\theta_1}\cdot\frac{\theta_2-\theta_1}{(1+\theta_1)(1+\theta_2)} = \frac{1}{(1+\theta_1)(1+\theta_2)}.$$

□

We consider X_1,...,X_n a random sample defined on $[0,1]$ and with an absolutely continuous cumulative distribution function denoted by F. Let $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$ be the corresponding order statistics.

We take the following estimator of $CSTM_{\theta_1,\theta_2}(F)$, namely

$$CSTM_{\theta_1,\theta_2}(F_n) = \int_0^{\infty} g_{\theta_1,\theta_2}\left(\overline{F}_n(x)\right) dx,$$

where $\theta_1, \theta_2 \in (0,\infty)$, $\theta_2 > \theta_1$, $g_{\theta_1,\theta_2} : [0,1] \to \mathbb{R}$, $g_{\theta_1,\theta_2}(t) = \frac{1}{\theta_2-\theta_1}\left(t^{\theta_1} - t^{\theta_2}\right)$, F_n is the empirical cumulative distribution function, given via (1_A is the characteristic function of the set A)

$$F_n(x) = \sum_{i=1}^{n-1} \frac{i}{n} 1_{[X_{(i)},X_{(i+1)})}(x) + 1_{[X_{(n)},\infty)}(x) \text{ for any } x \in \mathbb{R}$$

and $\overline{F}_n(x) = 1 - F_n(x)$ for any $x \in \mathbb{R}$.

In order to obtain a consistent test of the hypothesis of uniformity, we consider the statistic test

$$T_n(\theta_1,\theta_2) = \frac{1}{\theta_2-\theta_1} \sum_{i=1}^{n-1} \left[\left(1-\frac{i}{n}\right)^{\theta_1} - \left(1-\frac{i}{n}\right)^{\theta_2}\right] \left(X_{(i+1)} - X_{(i)}\right),$$

with the hypotheses:

$H_0 : F \in U(0,1)$.

$H_1 : F \notin U(0,1)$.

Theorem 4.2. *a) The test based on the sample estimate $T_n(\theta_1,\theta_2)$ is consistent.*

b) In the hypothesis H_0 we have $CSTM_{\theta_1,\theta_2}(F_n) \xrightarrow[n]{p} \frac{1}{(1+\theta_1)(1+\theta_2)}$.

Proof. a) From the Glivenko-Cantelli Theorem (see [15]) we have

$$\sup_{x\in\mathbb{R}} |F_n(x) - F(x)| \xrightarrow[n]{a.s.} 0.$$

Moreover, it is easy to show that

$$CSTM_{\theta_1,\theta_2}(F_n) \xrightarrow[n]{a.s.} CSTM_{\theta_1,\theta_2}(F).$$

Hence

$$CSTM_{\theta_1,\theta_2}(F_n) \xrightarrow[n]{p} CSTM_{\theta_1,\theta_2}(F),$$

which gives the conclusion.

b) From a) we have

$$CSTM_{\theta_1,\theta_2}(F_n) \xrightarrow[n]{p} CSTM_{\theta_1,\theta_2}(F).$$

But, if $F \in U(0,1)$, then

$$CSTM_{\theta_1,\theta_2}(F) = \frac{1}{(1+\theta_1)(1+\theta_2)}.$$

Hence

$$CSTM_{\theta_1,\theta_2}(F_n) \xrightarrow[n]{p} \frac{1}{(1+\theta_1)(1+\theta_2)}.$$

□

Theorem 4.3. *We have*

$$0 \leq T_n(\theta_1,\theta_2) \leq \frac{1}{\theta_2-\theta_1}\left[\left(\frac{\theta_1}{\theta_2}\right)^{\dfrac{\theta_1}{\theta_2-\theta_1}} - \left(\frac{\theta_1}{\theta_2}\right)^{\dfrac{\theta_2}{\theta_2-\theta_1}}\right].$$

Proof. Because $\theta_2 > \theta_1$ we get $T_n(\theta_1,\theta_2) \geq 0$.

By Lemma 3.3 we obtain that

$$T_n(\theta_1,\theta_2) \leq \frac{1}{\theta_2-\theta_1}\left[\left(\frac{\theta_1}{\theta_2}\right)^{\dfrac{\theta_1}{\theta_2-\theta_1}} - \left(\frac{\theta_1}{\theta_2}\right)^{\dfrac{\theta_2}{\theta_2-\theta_1}}\right]\left(X_{(n)} - X_{(1)}\right).$$

Because we have a selection on $[0,1]$ it follows that $X_{(n)} - X_{(1)} \leq 1$, hence

$$T_n(\theta_1,\theta_2) \leq \frac{1}{\theta_2-\theta_1}\left[\left(\frac{\theta_1}{\theta_2}\right)^{\dfrac{\theta_1}{\theta_2-\theta_1}} - \left(\frac{\theta_1}{\theta_2}\right)^{\dfrac{\theta_2}{\theta_2-\theta_1}}\right].$$

□

Theorem 4.4. *Under the hypothesis H_0, the mean and the variance of $T_n(\theta_1,\theta_2)$ are given by:*

a) $E\left(T_n(\theta_1,\theta_2)\right) = \frac{1}{n+1} \cdot \frac{1}{\theta_2-\theta_1} \sum_{i=1}^{n-1}\left[\left(1-\frac{i}{n}\right)^{\theta_1} - \left(1-\frac{i}{n}\right)^{\theta_2}\right].$

b) $Var\left(T_n(\theta_1,\theta_2)\right) = \frac{n}{(n+1)^2(n+2)} \cdot \frac{1}{(\theta_2-\theta_1)^2} \sum_{i=1}^{n-1}\left[\left(1-\frac{i}{n}\right)^{\theta_1} - \left(1-\frac{i}{n}\right)^{\theta_2}\right]^2.$

Proof. The random variable $X_{(i+1)} - X_{(i)}$ has a beta distribution with parameter-vector $(1, n)$ for any $i = \overline{1, n-1}$ (see [15]). Taking into account the expression of $T_n(\theta_1, \theta_2)$ the conclusion follows immediately. □

Theorem 4.5. *Under the hypothesis H_0 we have*

$$\lim_{n\to\infty} E\left(T_n(\theta_1,\theta_2)\right) = \frac{1}{(1+\theta_1)(1+\theta_2)}$$

and

$$\lim_{n\to\infty} Var\left(T_n(\theta_1,\theta_2)\right) = 0.$$

Proof. We have

$$\lim_{n\to\infty} E\left(T_n(\theta_1,\theta_2)\right) = \frac{1}{\theta_2-\theta_1}\int_0^1 \left(x^{\theta_1} - x^{\theta_2}\right) dx = \frac{1}{(1+\theta_1)(1+\theta_2)}$$

and

$$\lim_{n\to\infty} Var\left(T_n(\theta_1,\theta_2)\right) =$$
$$\lim_{n\to\infty} \frac{n}{(n+1)(n+2)} \cdot \frac{1}{(\theta_2-\theta_1)^2}\int_0^1 \left(x^{\theta_1} - x^{\theta_2}\right)^2 dx = 0.$$

□

5 Concrete examples

1. Assume that $X \sim U(0,1)$. We have

$$CSTM_{\theta_1,\theta_2}(X) = \frac{1}{\theta_2-\theta_1}\int_0^1 \left[(1-x)^{\theta_1} - (1-x)^{\theta_2}\right] dx =$$
$$\frac{1}{(\theta_1+1)(\theta_2-\theta_1)} - \frac{1}{(\theta_2+1)(\theta_2-\theta_1)} = \frac{1}{(1+\theta_1)(1+\theta_2)},$$

$$CSTM_{1,\theta_2}(X) = \frac{1}{2(1+\theta_2)} \text{ (black line)}$$

and

$$CSTM_{1.01,\theta_2}(X) = \frac{1}{2.01(1+\theta_2)} \text{ (red line).}$$

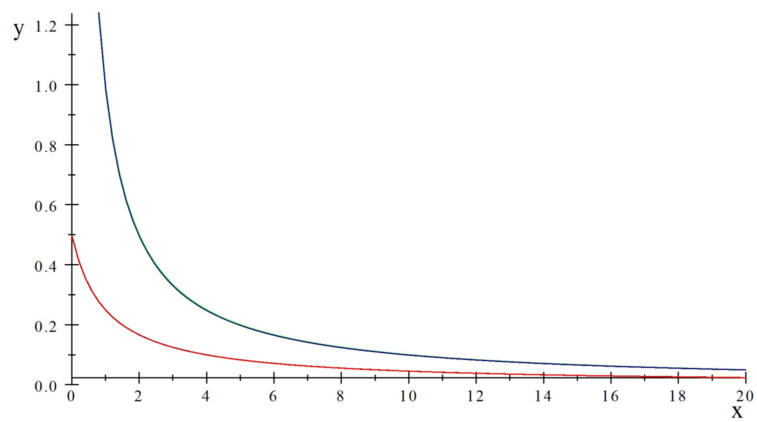
2. Assume that $X \sim Exp(1)$. We have

$$CSTM_{\theta_1,\theta_2}(X) = \frac{1}{\theta_2-\theta_1}\int_0^\infty \left(e^{-\theta_1 x} - e^{-\theta_2 x}\right) dx = \frac{1}{\theta_1(\theta_2-\theta_1)} - \frac{1}{\theta_2(\theta_2-\theta_1)} = \frac{1}{\theta_1\theta_2},$$

$$CSTM_{1,\theta_2}(X) = \frac{1}{\theta_2} \text{ (green line)}$$

and

$$CSTM_{1.01,\theta_2}(X) = \frac{1}{1.01\cdot\theta_2} \text{ (blue line).}$$



3. Using R we generated the following 50 statistical dates which are uniformly distributed on $[0,1]$:

[1] 0.02461368 0.04205953 0.04555650 0.10292468 0.13880606 0.14280002
[7] 0.14711365 0.15244475 0.21640794 0.23162579 0.23303410 0.24608773
[13] 0.26597264 0.28757752 0.28915974 0.31818101 0.32792072 0.36884545
[19] 0.40897692 0.41372433 0.41454634 0.45333416 0.45661474 0.46596245
[25] 0.47779597 0.52810549 0.54406602 0.55143501 0.57263340 0.59414202
[31] 0.64050681 0.65570580 0.67757064 0.69070528 0.69280341 0.70853047
[37] 0.75845954 0.78830514 0.79546742 0.85782772 0.88301740 0.88953932
[43] 0.89241904 0.89982497 0.90229905 0.94046728 0.95450365 0.95683335
[49] 0.96302423 0.99426978

For $\theta_1 = 1.1$ and $\theta_2 = 1.2$, we obtain $T_{50}(1.1, 1.2) = 0.001368169$. Considering the power estimates of the tests at the level $\alpha = 0.05$, we get that the hypothesis H_0 is accepted.

4. Also using R we generated the following 50 statistical dates which are not uniformly distributed on $[0,1]$:

[1] -11.29609472 -7.17727142 -6.13462697 -4.30609056 -3.28200191
[6] -2.11635506 -1.87858682 -1.60265434 -1.46446501 -1.44393500
[11] -1.30155556 -0.94821132 -0.78424322 -0.74852752 -0.47892684
[16] -0.38500324 -0.36165788 -0.35146072 -0.32552773 -0.31695307
[21] -0.18923919 -0.14391235 -0.13644954 -0.11668813 -0.01800397
[26] 0.07748066 0.13290832 0.14410523 0.33510858 0.46599091
[31] 0.48135357 0.49815579 0.51923801 0.80843908 0.89073714
[36] 0.89870327 0.97571576 1.10575455 1.26912955 1.28218896
[41] 1.55602166 1.66597364 2.28805841 3.40118112 3.59865790
[46] 3.63502154 6.77210788 7.29133088 9.31605615 14.31242194

For $\theta_1 = 1.1$ and $\theta_2 = 1.2$, we obtain $T_{50}(1.1, 1.2) = 0.2187792$. Considering the power estimates of the tests at the level $\alpha = 0.05$, we get that the hypothesis H_1 is accepted.

Conclusions

We introduced an alternate measure of the cumulative residual Sharma-Taneja-Mittal residual entropy (which depends on two parameters), generalizing the alternate measure of the cumulative residual Tsallis entropy from [33] (which depends on one parameter). For this new measure we found bounds, defined a consistent test based on the uniform distribution and gave some concrete examples by taking numerical cases for one parameter. For future works, we plan to link this measure with the theory of fractals (see [22], [62], [64]).

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