DOI: 10.2478/auom-2025-0020
An. Şt. Univ. Ovidius Constanţa **Vol. 33(2),2025, 89–105**

Necessary conditions and sufficient conditions for h-dichotomy of skew-evolution cocycles in Banach spaces

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Abstract

This study focuses on the problem of h-dichotomy for skew-evolution cocycles within Banach spaces. It outlines the necessary conditions and sufficient conditions for this framework, along with those for the notable concepts like exponential dichotomy, polynomial dichotomy and uniform h-dichotomy. These conditions are established through the use of strongly invariant families of projectors.

1 Introduction

Recently, the theory of asymptotic behavior in dynamical systems has experienced remarkable advancements, marking a significant evolution in this field. Researchers have made substantial progress in addressing a variety of previously unresolved problems, leading to important breakthroughs that have enhanced our understanding of dynamical systems. As a result, the theory has reached a new level of maturity, providing a more robust framework for analyzing the long-term behavior of these systems. This development not only deepens theoretical insights but also opens new avenues for practical applications across various scientific disciplines, including physics, biology, and engineering. The ongoing exploration of these concepts continues to yield fruitful results, demonstrating the vitality and relevance of asymptotic analysis in contemporary dynamical systems research.

Key Words: evolution semiflow, skew-evolution cocycle, h-dichotomy, strong h-growth.
2010 Mathematics Subject Classification: Primary 34D09; Secondary 34D05.
Received: 01.08.2024
Accepted: 29.11.2024

This paper investigates the general concept of h-dichotomy or dichotomy with growth rates in the context of skew-evolution cocycles within Banach spaces. Skew-evolution cocycles are particularly advantageous for studying evolution equations in nonuniform cases due to their dependence on three variables, a feature that distinguishes them from skew-product semiflows and evolution operators, that only depend on two elements. This added complexity allows for a more comprehensive analysis of the dynamics involved, making it possible to capture a broader spectrum of behaviors. We refer to the works of Boruga (Toma) and Megan [2], Chow and Leiva [5], Elaydi and Sacker [8], Latushkin, Montgomery-Smith and Randolph [12], Megan, Sasu and Sasu [18] for skew-product semiflows or evolution operators and for more recent contributions for skew-evolution cocycles we refer to the papers of Găină, Megan and Boruga (Toma) [10], Lupa and Popa [13], Megan, Stoica and Buliga [20], Mihiţ, Borlea and Megan [21], Mihiţ and Megan [22], Stoica and Megan [25].

The concept of dichotomy is a significant factor in the asymptotic behaviors of dynamical systems. It has been comprehensively examined from the perspectives of uniform and nonuniform behavior, whether exponential or polynomial by Barreira and Valls [1], Bento and Silva [3], Dragievi [6], Dragievi, Sasu and Sasu [7], Lupa and Megan [14], Megan and Stoica [19], Megan, Sasu and Sasu [17], Sasu and Sasu [24], Popa, Megan and Ceauşu [23].

As a generalization of exponential and polynomial dichotomy, we naturally focus on a kind of dichotomy known as h-dichotomy or dichotomy with growth rates, in which the bijective and nondecreasing function $h : \mathbb{R}_+ \to [1, \infty)$ represents a growth rate. We mention some papers where the concept of (uniform or nonuniform) dichotomy with growth rates is studied: [4] by Bento, Lupa, Megan and Silva, [9] by Găină, [11] by Kovacs, Megan and Mihiţ, [15] by Megan.

Megan, Găină and Boruga (Toma) presented in [16] the necessary conditions and sufficient conditions for dichotomy with growth rates in the context of skew-evolution cocycles using invariant families of projectors. So, in this paper, the authors continued their work and obtained necessary conditions and sufficient conditions for h-dichotomy with respect to strongly invariant families of projectors. Also, in both papers, the particular cases of exponential dichotomy, polynomial dichotomy and uniform h-dichotomy are given for skew-evolution cocycles in Banach spaces using invariant and strongly invariant families of projectors. The growth property proves useful for obtaining sufficient conditions, so it is used the strong h-growth concept, which is a special one because of the growth rate that is differentiable.

2 Preliminaries

Consider a metric space denoted as Y a Banach space referred to as W. The notation $\mathcal{B}(W)$ represents the Banach algebra of all bounded linear operators that act on W. The norms on W and on $\mathcal{B}(W)$, are represented using $||\cdot||$.

The following two sets are introduced:

$$\Delta = \{(t,s) \in \mathbb{R}_+^2 : t \geq s\}$$

$$T = \{(t,s,t_0) \in \mathbb{R}_+^3 : t \geq s \geq t_0\}.$$

The definitions of **evolution semiflow, skew-evolution semiflow**, **skew-evolution cocycle**, **strongly measurable skew-evolution cocycle** and **family of projectors** used in this paper are not only well-established but also elaborated upon in [16]. This alignment highlights the continuity and relevance of these concepts within the broader context of our research.

Invariant families of projectors are defined in the subsequent definition, which elucidates their role and significance in the context of dynamical systems.

Definition 1. The mapping $P : \mathbb{R}_+ \times Y \to \mathcal{B}(W)$ is referred to as an *invariant family of projectors* for C as long as the next equation takes place for every $(t,s) \in \Delta$ and $y \in X$:

$$\Phi(t,s,y)P(s,y) = P(t,\varphi(t,s,y))\Phi(t,s,y).$$

Remark 1. In the case where $P : \mathbb{R}_+ \times Y \to \mathcal{B}(W)$ is an invariant family of projectors for C, its complementary family of projectors $Q : \mathbb{R}_+ \times Y \to \mathcal{B}(W)$ is also invariant for C.

The following definition outlines the idea of a strongly invariant family of projectors.

Definition 2. The mapping $P : \mathbb{R}_+ \times Y \to \mathcal{B}(W)$ is defined as a *strongly invariant family of projectors* for C if, for every $(t,s) \in \Delta$ and $y \in Y$, it is invariant to C, the mapping $\Phi(t,s,y)$ acts as an isomorphism from *Range* $Q(s,y)$ to *Range* $Q(t,\varphi(t,s,y))$.

Remark 2. In the case that $P : \mathbb{R}_+ \times Y \to \mathcal{B}(W)$ is a strongly invariant family of projectors for the skew-evolution cocycle C, then there exists an isomorphism $\Psi : \Delta \times Y \to \mathcal{B}(W)$ from *Range* $Q(t,\varphi(t,s,y))$ to *Range* $Q(s,y)$, for each $(t,s) \in \Delta$ and $y \in Y$.

From this point forward, the following notations will be utilized:

$$\Phi_P(t,s,y) = \Phi(t,s,y)P(s,y),$$

$$\Phi_Q(t,s,y) = \Phi(t,s,y)Q(s,y),$$
$$\Psi_Q(t,s,y) = \Psi(t,s,y)Q(t,\varphi(t,s,y)),$$

for every $(t,s) \in \Delta$, $y \in Y$.

Proposition 1. Provided that $P : \mathbb{R}_+ \times Y \to \mathcal{B}(W)$ is a strongly invariant family of projectors for the skew-evolution cocycle C, then this establishes:

(I) $\Phi_Q(t,s,y)\Psi_Q(t,s,y) = Q(t,\varphi(t,s,y))$;

(II) $\Psi_Q(t,s,y)\Phi_Q(t,s,y) = Q(s,y)$;

(III) $\Psi_Q(t,s,y) = Q(s,y)\Psi_Q(t,s,y)$;

(IV) $\Psi_Q(t,t_0,y_0) = \Psi_Q(s,t_0,y_0)\Psi_Q(t,s,\varphi(s,t_0,y_0))$,

for each $(t,s,t_0) \in T$, $(y,y_0) \in Y^2$.

Proof. It results from [21].

□

Definition 3. A nondecreasing function $h : \mathbb{R}_+ \to [1,\infty)$ with $\lim_{t\to\infty} h(t) = \infty$ is said to be a *growth rate.*

Let us consider the next classes of function:

- $\mathcal{H}$ the set of differentiable growth rates $h : \mathbb{R}_+ \to [1,\infty)$ with the property that there exists $H > 1$ so that $h'(t) \leq Hh(t)$, for every $t \geq 0$.
- $\mathcal{H}_1$ the set of differentiable growth rates $h : \mathbb{R}_+ \to [1,\infty)$ with the propertythere exist $m > 0$ so that $h'(t) \geq mh(t)$, for every $t \geq 0$.

Proposition 2. If $h \in \mathcal{H}$, then:

(i) $h(t) \leq h(s)e^{H(t-s)}$, for every $(t,s) \in \Delta$;

(ii) $h(t+1) \leq e^H h(t)$, for every $t \geq 0$;

Proof. See [16]. □

Definition 4. We define the pair (C,P) as being *h-dichotomic* if there exist three constants $N \geq 1, \nu > 0$ and $\varepsilon \geq 0$ satisfying the following conditions:

(hd_1) $h(t)^\nu||\Phi_P(t,t_0,y_0)w_0|| \leq Nh(s)^\nu h(s)^\varepsilon||\Phi_P(s,t_0,y_0)w_0||$;

(hd_2) $h(t)^\nu||\Phi_Q(s,t_0,y_0)w_0|| \leq Nh(s)^\nu h(t)^\varepsilon||\Phi_Q(t,t_0,y_0)w_0|$,

for each $(t,s,t_0) \in T$, $(y_0,w_0) \in Y \times W$.

Remark 3. In the previous definition it is possible to let $\nu \in (0,1)$.

Remark 4. We call the pair (C,P) h-dichotomic iff there are $N \geq 1, \nu > 0$ and $\varepsilon \geq 0$ that satisfy:

(hd_1') $h(t)^\nu ||\Phi_P(t,s,y)w|| \leq Nh(s)^\nu h(s)^\varepsilon ||P(s,y)w||$;

(hd_2') $h(t)^\nu ||Q(s,y)w|| \leq Nh(s)^\nu h(t)^\varepsilon ||\Phi_Q(t,s,y)w||$,

for every $(t,s) \in \Delta$ and $(y,w) \in Y \times W$.

Proposition 3. We characterize the pair (C,P) as h-dichotomic iff there exist the constants $N \geq 1, \nu > 0$ and $\varepsilon \geq 0$ that satisfy the following requirements:

(hd_1'') $h(t)^\nu ||\Phi_P(t,s,y)w|| \leq Nh(s)^{\varepsilon+\nu} ||P(s,y)w||$;

(hd_2'') $h(t)^\nu ||\Psi_Q(t,s,y)w|| \leq Nh(t)^\varepsilon h(s)^\nu ||Q(t,\varphi(t,s,y))w||$,

for every $(t,s) \in \Delta$, $(y,w) \in Y \times W$.

Proof. This represents a specific case of *Theorem 1* found in [21]. □

Proposition 4. We refer to the pair (C,P) as h-dichotomic iff there are $N \geq 1, \nu > 0$ and $\varepsilon \geq 0$ with:

(hd_1''') $h(t)^\nu ||\Phi_P(t,t_0,y_0)w_0|| \leq Nh(s)^{\varepsilon+\nu} ||\Phi_P(s,t_0,y_0)w_0||$;

(hd_2''') $h(s)^\nu ||\Psi_Q(t,t_0,y_0)w_0|| \leq Nh(s)^\varepsilon h(t_0)^\nu ||\Psi_Q(t,s,\varphi(s,t_0,y_0))w_0||$,

for each $(t,s,t_0) \in T$, $(y_0,w_0) \in Y \times W$.

Proof. This statement is a specific case of *Proposition 3* from [21].

□

Remark 5. Setting $h(t) = e^t$ results in exponential dichotomy, while $h(t) = t+1$ gives us polynomial dichotomy. Additionally, when $\varepsilon = 0$, both cases represent their uniform variants and also the uniform h-dichotomy concept.

Remark 6. Uniform h-dichotomy implies h-dichotomy, while the reverse implication does not hold, as *Example 4.1.* from [16] shows.

Definition 5. The pair (C,P) has *h-growth* if there exist three constants $M \geq 1, \omega > 0$ and $\delta \geq 0$ that fulfill the conditions outlined below:

(hg_1) $h(s)^\omega ||\Phi_P(t,t_0,y_0)w_0|| \leq Mh(s)^\delta h(t)^\omega ||\Phi_P(s,t_0,y_0)w_0||$;

(hg_2) $h(s)^\omega ||\Phi_Q(s,t_0,y_0)w_0|| \leq Mh(t)^{\omega+\delta} ||\Phi_Q(t,t_0,y_0)w_0||$, for each $(t, s, t_0) \in T$, $(y_0, w_0) \in Y \times W$.

Definition 6. The pair (C, P) has *strong h-growth* if there exist the following constants $M \geq 1, \omega > 0$ and $\delta \geq 0$ that meet the criteria from below:

(shg_1) $h(s)^\omega ||\Phi_P(t, t_0, y_0)w_0|| \leq M \frac{h'(s)}{h(s)} h(s)^\delta h(t)^\omega ||\Phi_P(s, t_0, y_0)w_0||$;

(shg_2) $h(s)^\omega ||\Phi_Q(s, t_0, y_0)w_0|| \leq M \frac{h'(t)}{h(t)} h(t)^{\omega+\delta} ||\Phi_Q(t, t_0, y_0)w_0||$,

for every $(t, s, t_0) \in T$ and $(y_0, w_0) \in Y \times W$.

Remark 7. The pair (C, P) is considered to have strong h-growth iff there exist three constants $M \geq 1, \omega > 0$ and $\delta \geq 0$ with:

(shg'_1) $h(s)^\omega ||\Phi_P(t, s, y)w|| \leq M h(t)^\omega \frac{h'(s)}{h(s)} h(s)^\delta ||P(s, y)w||$;

(shg'_2) $h(s)^\omega ||Q(s, y)w|| \leq M h(t)^\omega \frac{h'(t)}{h(t)} h(t)^\delta ||\Phi_Q(t, s, y)w||$,

for every $(t, s) \in \Delta$ and $(y, w) \in Y \times W$.

Remark 8. When $\delta = 0$, we obtain the concept of *uniform strong h-growth*. Furthermore, for $h(t) = e^t$ we achieve the *(uniform) exponential growth* concept and for $h(t) = t + 1$, the *(uniform) strong polynomial growth* concept.

Remark 9. If $h \in \mathcal{H}$ and the pair (C, P) has strong h-growth, then it also has h-growth.

Remark 10. If the pair (C, P) has h-growth, then it also has uniform strong h-growth. However, the next example illustrates that the converse statement is false.

Example 1. Consider Y as a metric space and W as a Banach space, $h : \mathbb{R}_+ \to [1, \infty)$ a differentiable growth rate, the evolution semiflow $\varphi : \Delta \times Y \to X$, the skew-evolution semiflow $\Phi : \Delta \times Y \to \mathcal{B}(W)$ defined by

$$\Phi(t, s, y) = \frac{h(t)^{3-\sin \ln h(t)}}{h(t)^{3-\sin \ln h(s)}} P(s, y) + \frac{h(s)^{3-\sin \ln h(s)}}{h(t)^{3-\sin \ln h(t)}} Q(s, y)$$

and the families of projectors $P, Q : \mathbb{R}_+ \times Y \to \mathcal{B}(W)$ that have the property $P(t, \varphi(t, t_0, y_0)) = P(t_0, y_0)$. Hence, $C = (\Phi, \varphi)$ is a skew-evolution cocycle.

Hence, the pair (C, P) has h-growth for $\omega = 4, \delta = 2$ and $M = 1$.

Supposing that (C, P) has uniform strong h-growth with $h(t) = e^t$, then there exist $M \geq 1$ and $\omega > 0$ fulfilling the following:

$$\frac{(e^t)^{3-\sin t}}{(e^s)^{3-\sin s}} ||P(s, y)w|| \leq M e^\omega (t - s) ||P(s, y)w||.$$

Specifically, by setting $t = 2n\pi + \pi$ and $s = 2n\pi + \frac{\pi}{2}$, we find

$$e^{2n\pi+2\pi} \leq Me^{\frac{\pi}{2}\omega},$$

which is absurd for $n \to \infty$. This means that (C, P) has not uniform strong h-growth.

Remark 11. While the h-dichotomy concept implies the h-growth property, the reverse implication does not take place, as shown in *Example 5.1.* from [16].

Proposition 5. The pair (C, P) is characterized by strong h-growth iff there exist $M \geq 1, \omega > 0$ and $\delta \geq 0$ for which the following holds:

(shg_1''') $h(s)^\omega ||\Phi_P(t,s,y)w|| \leq M\frac{h'(s)}{h(s)}h(s)^\delta h(t)^\omega ||P(s,y)w||$;

(shg_2''') $h(s)^\omega ||\Psi_Q(t,s,y)w|| \leq M\frac{h'(t)}{h(t)}h(t)^{\omega+\delta} ||Q(t,\varphi(t,s,y))w||$,

for every $(t, s) \in \Delta$ and $(y, w) \in Y \times W$.

Proof. Necessity. From *Remark 7* we obtain $(shg_1) \implies (shg_1''')$ and from *Proposition 1* we have

$$\begin{aligned} h(s)^\omega ||\Psi_Q(t,s,y)w|| &= h(s)^\omega ||Q(s,y)\Psi_Q(t,s,y)w|| \leq \\ &\leq Mh(t)^{\omega+\delta}\frac{h'(t)}{h(t)}||\Phi_Q(t,s,y)\Psi_Q(t,s,y)w|| = \\ &= M\frac{h'(t)}{h(t)}h(t)^{\omega+\delta}||Q(t,\varphi(t,s,y))w||, \end{aligned}$$

for every $(t, s) \in \Delta$ and $(y, w) \in Y \times W$.

Sufficiency. From *Remark 7* we obtain $(shg_1''') \implies (shg_1)$. For $(shg_2''') \implies (shg_2)$ we have that

$$\begin{aligned} h(s)^\omega ||Q(s,y)w|| &= h(s)^\omega ||\Psi_Q(t,s,y)\Phi_Q(t,s,y)w|| \leq \\ &\leq Mh(t)^{\omega+\delta}\frac{h'(t)}{h(t)}||\Phi_Q(t,s,y)w||, \end{aligned}$$

for every $(t, s) \in \Delta$ and $(y, w) \in Y \times W$. □

Proposition 6. The pair (C, P) has strong h-growth iff there are $M \geq 1, \omega > 0$ and $\delta \geq 0$ relevant to the following statements:

(shg_1^{IV}) $h(s)^\omega ||\Phi_P(t,t_0,y_0)w_0|| \leq M\frac{h'(s)}{h(s)}h(s)^\delta h(t)^\omega ||\Phi_P(s,t_0,y_0)w_0||$;

(shg_2^{IV}) $h(t_0)^\omega||\Psi_Q(t,t_0,y_0)w_0|| \leq M\frac{h'(s)}{h(s)}h(s)^{\omega+\delta}||\Psi_Q(t,s,\varphi(s,t_0,y_0))w_0||$,

for every $(t,s,t_0) \in T$, $(y_0,w_0) \in Y \times W$.

Proof. Necessity. The authors assume that (C,P) has strong h-growth. From *Remark 7* it results $(shg_1^{IV}) \iff (shg_1)$.

From *Proposition 5* we obtain

$$\begin{aligned} h(t_0)^\omega||\Psi_Q(t,t_0,y_0)w_0|| &= h(t_0)^\omega||\Psi_Q(s,t_0,y_0)\Psi_Q(t,s,\varphi(s,t_0,y_0))w_0|| \leq \\ &\leq M\frac{h'(s)}{h(s)}h(s)^{\omega+\delta}||\Psi_Q(t,s,\varphi(s,t_0,y_0))w_0||, \end{aligned}$$

for each $(t,s,t_0) \in T$, $(y_0,w_0) \in Y \times W$.

Sufficiency. This is obtained from *Proposition 5* with $s = t_0$, respectively $s = t$.

□

3 The main results

Let us take $C = (\Phi,\varphi)$ a strongly measurable skew-evolution cocycle, P a strongly invariant family of projectors, $\Psi : \Delta \times Y \to \mathcal{B}(W)$ with the properties from *Proposition 1* while $h : \mathbb{R}_+ \to [1,\infty)$ is defined as a differentiable growth rate.

Under these circumstances, we establish the dichotomy for necessary conditions and sufficient conditions concerning growth rates for skew-evolution cocycles within Banach spaces:

Theorem 1. If the pair (C,P) is h-dichotomic, then there exist three constants $D \geq 1, \varepsilon \geq 0$ and $d > 0$ with:

(hD_1) $\int\limits_t^\infty \frac{h'(\zeta)}{h(\zeta)}h(\zeta)^d||\Phi_P(\zeta,t_0,y_0)w_0||d\zeta \leq Dh(t)^{\varepsilon+d}||\Phi_P(t,t_0,y_0)w_0||$,

(hD_2) $\int\limits_{t_0}^t \frac{h'(\zeta)}{h(\zeta)}h(\zeta)^{-d}||\Psi_Q(t,\zeta,\varphi(\zeta,t_0,y_0))w_0||d\zeta \leq Dh(t)^{\varepsilon-d}||Q(t,\varphi(t,t_0,y_0))w_0||$,

for each $(t,t_0) \in \Delta$ and $(y_0,w_0) \in Y \times W$.

Proof. Let be $d \in (0,\nu)$.

The relation $(hd_1) \implies (hD_1)$ is immediate from [16] and the relation $(hd_2) \implies (hD_2)$ is proved as follows:

$$\begin{aligned}
& \int_{t_0}^{t} \frac{h'(\zeta)}{h(\zeta)} h(\zeta)^{-d} ||\Psi_Q(t,\zeta,\varphi(\zeta,t_0,y_0))w_0|| d\zeta \leq \\
\leq \quad & \int_{t_0}^{t} \frac{h'(\zeta)}{h(\zeta)} h(\zeta)^{-d} N h(t)^{\varepsilon} \left(\frac{h(\zeta)}{h(t)}\right)^{\nu} ||Q(t,\varphi(t,t_0,y_0))w_0|| d\zeta = \\
= \quad & N h(t)^{-\nu+\varepsilon} ||Q(t,\varphi(t,t_0,y_0))w_0|| \int_{t_0}^{t} h'(\zeta) h(\zeta)^{-d+\nu-1} d\zeta \leq \\
\leq \quad & N h(t)^{-\nu+\varepsilon} ||Q(t,\varphi(t,t_0,y_0))w_0|| \frac{h(t)^{-d+\nu}}{\nu - d} = \\
= \quad & \frac{N h(t)^{\varepsilon-d}}{\nu - d} ||Q(t,\varphi(t,t_0,y_0))w_0|| \leq \\
\leq \quad & D h(t)^{\varepsilon-d} ||Q(t,\varphi(t,t_0,y_0))w_0||,
\end{aligned}$$

for each $(t,t_0) \in \Delta$, $(y_0,w_0) \in Y \times W$, where $D = 1 + \frac{N}{\nu-d}$.

□

Theorem 2. Consider a pair (C,P) with strong h-growth for some $h \in \mathcal{H}$. Suppose there exist $D \geq 1, \varepsilon \geq 0, \delta \geq 0$ and $d > \delta$ such that (hD_1) and (hD_2) are fulfilled. Under this circumstances, the pair (C,P) satisfies the h-dichotomy conditions.

Proof. From [16] we obtain the relation $(hD_1) \implies (hd_1)$. For $(hD_2) \implies (hd_2)$ we obtain the following results:

Case 1: We consider $t \geq s+1$.

$$
\begin{aligned}
||\Psi_Q(t,s,y)w|| = & \int_s^{s+1} ||\Psi_Q(t,s,y)w|| d\zeta \leq \\
\leq & \int_s^{s+1} Mh(\zeta)^{\delta}\left(\frac{h(\zeta)}{h(s)}\right)^{\omega}\frac{h'(\zeta)}{h(\zeta)}||\Psi_Q(t,\zeta,\varphi(\zeta,s,y))w||d\zeta \leq \\
\leq & Mh(t)^{\delta}\left(\frac{h(s)}{h(t)}\right)^{d}\int_s^{s+1}\frac{h'(\zeta)}{h(\zeta)}\left(\frac{h(\zeta)}{h(s)}\right)^{\omega+d}\left(\frac{h(\zeta)}{h(t)}\right)^{-d}||\Psi_Q(t,\zeta,\varphi(\zeta,s,y))w||d\zeta \leq \\
\leq & Mh(t)^{\delta}e^{H(\omega+d)}\left(\frac{h(s)}{h(t)}\right)^{d}\int_{t_0}^{t}\frac{h'(\zeta)}{h(\zeta)}\left(\frac{h(\zeta)}{h(t)}\right)^{-d}||\Psi_Q(t,\zeta,\varphi(\zeta,s,y))w||d\zeta \leq \\
\leq & Mh(t)^{\delta}e^{H(\omega+d)}\left(\frac{h(s)}{h(t)}\right)^{d}Dh(t)^{\varepsilon}||Q(t,\varphi(t,s,y))w|| \leq \\
\leq & DMe^{H(\omega+d)}h(t)^{\varepsilon+\delta}\left(\frac{h(s)}{h(t)}\right)^{d}||Q(t,\varphi(t,s,y))w|| \leq \\
\leq & DMe^{H(\omega+d)}h(t)^{\varepsilon+\delta}\left(\frac{h(t)}{h(s)}\right)^{\delta-d}||Q(t,\varphi(t,s,y))w||,
\end{aligned}
$$

for every $(t,s)\in\Delta$ and $(y,w)\in Y\times W$.

Case 2: Let $t\in[s,s+1)$.

$$
\begin{aligned}
& ||\Psi_Q(t,s,y)w|| \leq \\
\leq & Mh(t)^{\delta}\left(\frac{h(t)}{h(s)}\right)^{\omega}\frac{h'(t)}{h(t)}||Q(t,\varphi(t,s,y))w|| \leq \\
\leq & MH\left(\frac{h(s)}{h(t)}\right)^{d-\delta}\left(\frac{h(t)}{h(s)}\right)^{\omega+d-\delta}h(t)^{\delta}||Q(t,\varphi(t,s,y))w|| \leq \\
\leq & MHe^{H(\omega+d-\delta)}h(t)^{\varepsilon+\delta}\left(\frac{h(s)}{h(t)}\right)^{d-\delta}||Q(t,\varphi(t,s,y))w||,
\end{aligned}
$$

for every $(t,s)\in\Delta$ and $(y,w)\in Y\times W$.

□

Corollary 1. Let the pair (C,P) have uniform strong h-growth with $h\in\mathcal{H}$. The pair is uniformly h-dichotomic iff there exist constants $D\geq 1$ and $d>0$ satisfying the next properties:

(uhD_1) $\int_t^{\infty}\frac{h'(\zeta)}{h(\zeta)}h(\zeta)^d||\Phi_P(\zeta,t_0,y_0)w_0||d\zeta \leq Dh(t)^d||\Phi_P(t,t_0,y_0)w_0||$,

(uhD_2) $\int\limits_{t_0}^{t} \frac{h'(\zeta)}{h(\zeta)} h(\zeta)^{-d} ||\Psi_Q(t,\zeta,\varphi(\zeta,t_0,y_0))w_0||d\zeta \leq \frac{D||Q(t,\varphi(t,t_0,y_0))w_0||}{h(t)^d}$,

for any $(t,t_0) \in \Delta$ and $(y_0,w_0) \in Y \times W$.

Proof. This is a special case corresponding to $\varepsilon = 0$, in *Theorem 1* and *Theorem 2* . □

Remark 12. In *Theorem 1* and *Theorem 2* if $h(t) = e^t$ or $h(t) = t+1$, then necessary conditions and sufficient conditions are obtained for exponential dichotomy, respectively polynomial dichotomy.

In addition, if $\varepsilon = 0$ in *Corollary 1*, then integral characterizations in the uniform case for both exponential dichotomy and polynomial dichotomy are established.

The last results also present necessary conditions and sufficient conditions for h-dichotomy:

Theorem 3. If (C,P) is h-dichotomic with $\nu > \varepsilon$, then there exist $D \geq 1, d_1, d_2 \in (0, \nu - \varepsilon)$ with:

(hD_1') $\int\limits_{t_0}^{t} \frac{h'(\zeta)}{h(\zeta)} \frac{h(\zeta)^{-d_1}}{||\Phi_P(\zeta,t_0,y_0)w_0||} d\zeta \leq \frac{Dh(t)^{\varepsilon+d_1}}{||\Phi_P(t,t_0,y_0)w_0||}$,

for any $(t,t_0) \in \Delta$ and $(y_0,w_0) \in Y \times W$ with $\Phi_P(t,t_0,y_0)w_0 \neq 0$;

(hD_2') $\int\limits_{t_0}^{\infty} \frac{h'(\zeta)}{h(\zeta)} \frac{h(\zeta)^{d_2}}{||\Psi_Q(t,\zeta,\varphi(\zeta,t_0,y_0))w_0||} d\zeta \leq \frac{Dh(t_0)^{\varepsilon+d_2}}{||\Psi_Q(t,t_0,y_0)w_0||}$,

for any $(t,t_0) \in \Delta$ and $(y_0,w_0) \in Y \times W$ with $\Psi_Q(t,t_0,y_0)w_0 \neq 0$.

Proof. The relation $(hd_1) \implies (hD_1')$ results immediately from [16]. For $(hd_2) \implies (hD_2')$ we conclude:

$$\begin{aligned}
&\int_{t_0}^{\infty} \frac{h'(\zeta)}{h(\zeta)} \frac{h(\zeta)^{d_2}}{||\Psi_Q(t,\zeta,\varphi(\zeta,t_0,y_0))w_0||} d\zeta \leq \\
\leq \quad & \int_{t_0}^{\infty} \frac{h'(\zeta)}{h(\zeta)} N h(\zeta)^{\varepsilon} \left(\frac{h(\zeta)}{h(t_0)}\right)^{-\nu} \frac{h(\zeta)^{d_2}}{||\Psi_Q(t,t_0,y_0)w_0||} d\zeta = \\
= \quad & \frac{N h(t_0)^{\nu}}{||\Psi_Q(t,t_0,y_0)w_0||} \int_{t_0}^{\infty} h'(\zeta) h(\zeta)^{d_2-\nu+\varepsilon-1} d\zeta \leq \\
\leq \quad & \frac{N h(t_0)^{\nu}}{||\Psi_Q(t,t_0,y_0)w_0||} \frac{h(t_0)^{d_2-\nu+\varepsilon}}{-d_2+\nu-\varepsilon} = \\
= \quad & \frac{N h(t_0)^{\varepsilon+d_2}}{-d_2+\nu-\varepsilon} \frac{1}{||\Psi_Q(t,t_0,y_0)w_0||} \leq \\
\leq \quad & \frac{D h(t_0)^{\varepsilon+d_2}}{||\Psi_Q(t,t_0,y_0)w_0||},
\end{aligned}$$

for any $(t,t_0) \in \Delta$, $(y_0,w_0) \in Y \times W$ with $\Psi_Q(t,t_0,y_0)w_0 \neq 0$, where $D = 1 + \frac{N}{-d_2+\nu-\varepsilon}$.

□

Theorem 4. Let us consider $h \in \mathcal{H} \cap \mathcal{H}_1$ and (C,P) with strong h-growth. If there exist $D \geq 1, \delta \geq 0$, $d_1 > \varepsilon$ and $d_2 > \delta$ with (hD_1') and (hD_2'), then (C,P) is h-dichotomic.

Proof. The relation $(hD_1') \implies (hd_1)$ is verified from [16]. For the relation $(hD_2') \implies (hd_2)$ the following can be stated:

Case 1: Let $t \geq t_0 + 1$ with $\Psi_Q(t,t_0,y_0)w_0 \neq 0$.

$$\begin{aligned}
&\frac{1}{||Q(t,\varphi(t,t_0,y_0))w_0||} = \int_{t-1}^{t} \frac{1}{||\Phi_Q(t,t_0,y_0)\Psi_Q(t,t_0,y_0)w_0||}d\zeta \leq \\
\leq\ & \int_{t-1}^{t} Mh(t)^{\delta}\left(\frac{h(t)}{h(\zeta)}\right)^{\omega}\frac{h'(t)}{h(t)}\frac{1}{||\Phi_Q(\zeta,t_0,y_0)\Psi_Q(t,t_0,y_0)w_0||}d\zeta \leq \\
\leq\ & \int_{t-1}^{t} \frac{MH}{m}h(t)^{\delta-d_2}\left(\frac{h(t)}{h(\zeta)}\right)^{\omega+d_2}\frac{h'(\zeta)}{h(\zeta)}\frac{h(\zeta)^{d_2}}{||\Psi_Q(t,\zeta,\varphi(\zeta,t_0,y_0))w_0||}d\zeta \leq \\
\leq\ & \frac{MH}{m}e^{H(\omega+d_2)}h(t)^{\delta-d_2}\frac{Dh(t_0)^{\varepsilon+d_2}}{||\Psi_Q(t,t_0,y_0)w_0||} \leq \\
\leq\ & \frac{DMH}{m}e^{H(\omega+d_2)}\left(\frac{h(t)}{h(t_0)}\right)^{\delta-d_2}\frac{h(t_0)^{\delta-d_2}h(t_0)^{\varepsilon+d_2}}{||\Psi_Q(t,t_0,y_0)w_0||} \leq \\
\leq\ & \frac{DMH}{m}e^{H(\omega+d_2)}\left(\frac{h(t)}{h(t_0)}\right)^{\delta-d_2}\frac{h(t)^{\varepsilon+\delta}}{||\Psi_Q(t,t_0,y_0)w_0||},
\end{aligned}$$

for all $(t,t_0)\in\Delta$ and $(y_0,w_0)\in Y\times W$.

Case 2: We consider $t\in[t_0,t_0+1)$.

$$\begin{aligned}
&||\Psi_Q(t,t_0,y_0)w_0|| \leq \\
\leq\ & Mh(t)^{\delta}\left(\frac{h(t)}{h(t_0)}\right)^{\omega}\frac{h'(t_0)}{h(t_0)}||Q(t,\varphi(t,t_0,y_0))w_0|| \leq \\
\leq\ & MHh(t)^{\delta}\left(\frac{h(t)}{h(t_0)}\right)^{\omega+d_2-\delta}\left(\frac{h(t)}{h(t_0)}\right)^{\delta-d_2}||Q(t,\varphi(t,t_0,y_0))w_0|| \leq \\
\leq\ & MHe^{H(\omega+d_2-\delta)}h(t)^{\varepsilon+d_2}\left(\frac{h(t)}{h(t_0)}\right)^{\delta-d_2}||Q(t,\varphi(t,t_0,y_0))w_0||,
\end{aligned}$$

for all $(t,t_0)\in\Delta$ and $(y_0,w_0)\in Y\times W$.

□

In particular, we present the following consequences

Corollary 2. If $h\in\mathcal{H}\cap\mathcal{H}_1$ and the pair (C,P) has uniform strong h-growth, then it is uniformly h-dichotomic iff there exist $D\geq 1, d_1, d_2\in(0,1)$ with:

(uhD_1') $\int_{t_0}^{t}\frac{h'(\zeta)}{h(\zeta)}\frac{h(\zeta)^{-d_1}}{||\Phi_P(\zeta,t_0,y_0)w_0||}d\zeta \leq \frac{Dh(t)^{d_1}}{||\Phi_P(t,t_0,y_0)w_0||}$, for each $(t, t_0) \in \Delta$ and $(y_0, w_0) \in Y \times W$ with $\Phi_P(t, t_0, y_0)w_0 \neq 0$;

(uhD_2') $\int\limits_{t_0}^{\infty} \frac{h'(\zeta)}{h(\zeta)} \frac{h(\zeta)^{d_2}}{||\Psi_Q(t,\zeta,\varphi(\zeta,t_0,y_0))w_0||} d\zeta \leq \frac{Dh(t_0)^{d_2}}{||\Psi_Q(t,t_0,y_0)w_0||}$,

for each $(t, t_0) \in \Delta$ and $(y_0, w_0) \in Y \times W$ with $\Psi_Q(t, t_0, y_0)w_0 \neq 0$.

Proof. This is established based on *Theorem 3* and *Theorem 4* when $\varepsilon = 0$. □

Remark 13. If we consider in *Theorem 3* and *Theorem 4* $h(t) = e^t$, then we establish necessary conditions and sufficient conditions for the concept of exponential dichotomy.

Furthermore, if we consider in *Corollary* **??***cort2uhdcupsi* the same given case, then we outline integral characterization for uniform exponential dichotomy.

4 Conclusion

In this paper, we have extended the work presented in [17] by deriving necessary conditions and sufficient conditions for dichotomy with growth rates for strongly invariant families of projectors. Our findings provide a deeper understanding of the dynamics of skew-evolution cocycles within Banach spaces.

Moreover, we have specifically utilized the concept of differential growth rates to define and analyze the h-growth concept. This approach not only reinforces the applicability of growth rate functions in the study of dynamical systems but also highlights their significance in establishing the conditions for dichotomy.

These results contribute to the existing body of knowledge in the field, offering new insights and frameworks for future research concerning the asymptotic behavior of dynamical systems. Further investigations could explore the implications of these conditions in broader contexts or apply them to specific classes of evolution equations.

Acknowledgements

The authors are highly indebted to the editors and reviewers for their insightful suggestions, which have considerably improved the overall quality of this manuscript.

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