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Is there a polynomial $D(2X+1)$-quadruple?

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Abstract

In this paper, we show that there does not exist a polynomial $D(2X+1)$-quadruple $\{a,b,c,d\}$, such that $0<a<b<c<d$ and $\deg d=\deg b$.

1 Introduction and motivation

Since Diophantus [3] noted that the product of any two elements of the set $\left\{\frac{1}{16},\frac{33}{16},\frac{17}{4},\frac{105}{16}\right\}$ increased by 1 is a square of rational number, many generalizations of his original problem were also studied. The following definition describes a more general problem:

Definition 1. *Let $\mathcal{R}$ be a commutative ring and $n\in\mathcal{R}\backslash\{0\}$. The set of n (distinct) elements $\{a_1,\dots,a_m\}$ in $\mathcal{R}\backslash\{0\}$ is a* Diophantine m-tuple with the property $D(n)$ *or simply a* $D(n)$-m-tuple *in $\mathcal{R}$ if a_ia_j+n is a perfect square in $\mathcal{R}$, for all $1\le i<j\le m$.*

If zero or equal elements are allowed in such m-tuple then it is called an improper *$D(n)$-m-tuple.*

A polynomial variant of the problem of Diophantus was firstly studied by Jones [24, 23] for the case $\mathcal{R}=\mathbb{Z}[X]$ and $n=1$. Since then, many other variants of such a polynomial problem have also been considered (for example [8, 9, 10, 11, 13, 15, 14]). In polynomial variants of this problem, it is usually assumed that not all polynomials in such a $D(n)$-tuple are constant and the term polynomial $D(n)$-tuple is used.

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Here, we consider $D(n)$-m-tuples in a ring of polynomials with integer coefficients – $\mathbb{Z}[X]$. The representation $2X+1=(X+1)^2-X^2$ might suggest the existence of a $D(2X+1)$-quadruple. Indeed, such an improper $D(2X+1)$-quadruple exists, for example $\{X,X,4X+2,9X+6\}$. In [21] we showed that there is no polynomial $D(n)$-quadruple in $\mathbb{Z}[X]$ for some polynomials $n\in\mathbb{Z}[X]$ that are not representable as a difference of squares of two polynomials in $\mathbb{Z}[X]$. Namely, $D(n)$-quadruples are related to the representations of n by the binary quadratic form x^2-y^2. More precisely, the claim that a $D(n)$-quadruple exists if and only if n can be written as the difference of two squares (up to finitely many exceptions) has been proved for the ring of integers and for rings of integers of certain number fields: [5, 7, 18, 16, 19, 17, 20, 22, 25] (although by a recent result [2] in certain rings of the form $\mathbb{Z}[\sqrt{4k+2}]$ there are elements z which are not difference of two squares but there exists a $D(z)$-quadruple). It is interesting that the existence of $D(n)$-quadruples in these rings is obtained due to two-parameter polynomial sets, for example the set

$$\{m, mk^2-2k-2, m(k+1)^2-2k, m(2k+1)^2-8k-4\}$$

which has the property $D(2m(2k+1)+1)$. This set and several similar ones were constructed by Dujella in [6]. Furthermore, an important "tool" for proving the existence of $D(n)$-quadruples is the following implication:

$$\{a,b,c,d\} \text{ is a } D(n)\text{-quadruple} \ \Rightarrow\ \{aw,bw,cw,dw\} \text{ is a } D(nw^2)\text{-quadruple.}$$

What motivated us to ask whether there is a $D(2X+1)$-quadruple in $\mathbb{Z}[X]$ is precisely the proof of the existence of a $D(2n+1)$-quadruple in $\mathbb{Z}$, for $n\in\mathbb{Z}$, which arises from three cases/sets:

- $\{1,k^2-2k-2,k^2+1,4k^2-4k-3\}$ - a set with the property $D(4k+3)$,
- $\{4,k^2-3k,k^2+k+2,4k^2-4k\}$ - a set with the property $D(8k+1)$,
- $\{2,2k^2-2k-2,2k^2+2k+2,8k^2-2\}$ - a set with the property $D(8k+5)$.

Thus, so far no formula for the $D(2n+1)$-quadruple in $\mathbb{Z}$ has been found. However, it should be kept in mind that Dujella's polynomial formulas are obtained under the condition of so-called *regularity*, i.e. they are of the "very regular" form $\{a,b,a+b+2r,a+4b+4r\}$, where $ab+n=r^2$ and $a(a+4b+4r)+n=\square$. Recall that a $D(n)$-triple $\{a,b,c\}$ is called *regular* if $c=a+b\pm 2r$.

The next reason for our motivation is the (finite) set of exceptions that appear in some rings in which the equivalence between the existence of a $D(n)$-quadruple and representability of n as the difference of two squares holds. For instance, a $D(n)$-quadruple in $\mathbb{Z}$ exists if and only if $n=x^2-y^2$ for $x,y\in\mathbb{Z}$ and $n\notin S=\{-4,-3,-1,3,5,8,12,20\}$. The conjecture is that for $n\in S$ there does not exist a $D(n)$-quadruple. In fact, it is now known that there is no $D(-1)$-quadruple ([1]), which implies the nonexistence of a $D(-4)$-quadruple, but this was preceded by the laborious work of a number of mathematicians (Abu Muriefah, Al-Rashed, Bonciocat, Brown, Cipu, Dujella, Filipin, Fuchs, Fujita, He, Kedlaya, Kihel, Liqun, Mignotte, Mohanty, Ramasamy, Tamura, Togbé and Zheng, according to [4]). However, it should be emphasized that in the case of $\mathbb{Z}[X]$ we do not know whether the equivalence between $D(n)$-quadruples and representability of n as a difference of squares is valid, even though we have a partial result on the topic ([21]). Also, known results and conjectures provide certain restrictions on a potential $D(2X+1)$-quadruple. For example, the non-existence of $D(-1)$-quadruples proved in [1] implies that by inserting $X=-1$ in a $D(2X+1)$-quadruple, we should get two equal elements. Similar conditions follow from the conjecture of non-existence of $D(-3)$, $D(3)$ and $D(5)$-quadruples. The existence of a $D(2X+1)$-quadruple will probably solve the open question of existence of at least three $D(n)$-quadruples, with $n=(4k-1)(4k+1)$ (see Section 3 of [12]).

Here we show the following result.

Theorem 1. *A $D(2X+1)$-pair $\{a,b\}$ in $\mathbb{Z}[X]$ such that $0<a<b$ cannot be extended to a $D(2X+1)$-quadruple $\{a,b,c,d\}$ in $\mathbb{Z}[X]$, such that $0<a<b<c<d$ and $\deg d=\deg b$.*

In addition, we prove that there is no $D(2X+1)$-quadruple $\{a,b,c,d\}$ in $\mathbb{Z}[X]$ such that a is a negative integer and $a<b<c<d$ (Proposition 3).

For the proof of Theorem 1, we rely on the results obtained in [10, 9] where the authors set the upper bound for the size of $D(n)$-m-tuple in $\mathbb{Z}[X]$, where n is a linear polynomial. More precisely, they proved the following:

Theorem 2 ([10], Corollary 1 and Theorem 3)**.** *Let $n=n_1X+n_0$ be a linear polynomial in $\mathbb{Z}[X]$ (with $n_1\neq 0$) and*

$$L=\sup\{|S| \ : \ S \textit{ is a } D(n)\textit{-tuple in } \mathbb{Z}[X]\}.$$

Then

$$L\leq 12.$$

Also, if L_k denotes the number of polynomials of degree k in a polynomial $D(n)$-m-tuple S, then

$$L_1\leq 4,\ L_k\leq 3,\ \forall k\geq 2.$$

In the last part (Section 4) we give some examples of irregular $D(2X+1)$-triples that are obtained as extensions of $D(2X+1)$-pairs due to the solution of the corresponding Pellian equation. For some of them, we can show that they cannot be extended to a $D(2X+1)$-quadruple.

2 Preliminaries

Let $\{a,b,c\}$ be a $D(n)$-triple in $\mathbb{Z}[X]$, where $n = 2X+1$ and $r,s,t \in \mathbb{Z}^+[X]$ such that

$$ab+n = r^2, \quad ac+n = s^2, \quad bc+n = t^2. \tag{1}$$

By Greek letters $\alpha,\beta,\gamma,\dots$ we will denote degrees of polynomials and with capital letters $A,B,C,\dots$ mainly the leading coefficients of polynomials. For $a,b \in \mathbb{Z}[X]$, $a<b$ means that $b-a \in \mathbb{Z}^+[X]$, where $Z^+[X]$ denotes the set of all polynomials in $\mathbb{Z}[X]$ with a positive leading coefficient.

If $a<b<c$, then $\alpha \geq 0$, $\beta,\gamma \geq 1$ because for $a,b \in \mathbb{Z}$ a polynomial $ab+n$ cannot be a square.

Proposition 3. *Let a be a negative integer. Then there is no $D(2X+1)$-quadruple $\{a,b,c,d\}$ in $\mathbb{Z}[X]$ such that $a<b<c<d$.*

Proof. For $a \in \mathbb{Z}$, $a<0$ and $b \in \mathbb{Z}[X]$, the set $\{a,b\}$ is a $D(2X+1)$-pair if and only if one of the following two possibilities holds:

(i) $a=-1$, $b = 2X+1-k^2$, $k \in \mathbb{Z}$,

(ii) $a=-2$, $b = X + \frac{1-k^2}{2}$, $k \in \mathbb{Z}$ and k is odd.

In both cases, we show that there is a unique $c \in \mathbb{Z}[X]$, such that $b<c$ and $\{a,b,c\}$ is a $D(2X+1)$-triple.

(i) If $\{-1,b,c\}$ is a $D(2X+1)$-triple, then

$$b = 2X+1-k^2, \ c = 2X+1-\ell^2,$$

for $k,\ell \in \mathbb{Z}$ and

$$(2X+1-k^2)(2X+1-\ell^2)+2X+1 = (2X+c_0)^2, \ c_0 \in \mathbb{Z}.$$

By substituting $Y = 2X+1$, we get

$$Y^2 + (1-k^2-\ell^2)Y + k^2\ell^2 = (Y+c_0-1)^2$$

which implies $1-k^2-\ell^2 = \pm 2k\ell$, i.e. $|k \pm \ell| = 1$. So, $\ell = \pm k \pm 1$. Since $b<c$, the only possibility is $c = 2X+1-(k-1)^2$ and $k>0$.

(ii) If $\{-2,b,c\}$ is a $D(2X+1)$-triple, then

$$b = X + \frac{1-k^2}{2}, \ c = X + \frac{1-\ell^2}{2},$$

k, ℓ are odd integers and

$$\left(X + \frac{1-k^2}{2}\right)\left(X + \frac{1-\ell^2}{2}\right) + 2X + 1 = (X + c_0)^2,\ c_0 \in \mathbb{Z}.$$

Multiplying the previous equation by 4 and substituting $Y = 2X + 1$, gives

$$Y^2 + (-k^2 - \ell^2 + 4)Y + k^2\ell^2 = (Y + 2c_0 - 1)^2$$

which implies $(k \pm \ell)^2 = 4$, i.e. $\ell = \pm k \pm 2$. Therefore, with the assumptions $b < c$ and $k \geq 3$, we have $c = X + \frac{1-(k-2)^2}{2}$. For $k = 1$, we get $c = X$ and $c = X - 4$, but none of them satisfies $b < c$. □

Note that all $D(2X + 1)$-triples in $Z[X]$ with $a \in \mathbb{Z}$, $a < 0$ and $a < b < c$ are regular ones, i.e. $c = a + b + 2r$.

In light of Proposition 3, from now on we assume that $a, b, c \in \mathbb{Z}^+[X]$ and $0 < a < b < c$, because $A, B, C \in \mathbb{Z}$ must have the same sign. Also, α, β and γ are of the same parity, so

$$\alpha + \beta \geq 2. \tag{2}$$

The following lemma will be used for the classification of possible forms of $D(n)$-triples.

Lemma 4 ([9], Lemma 1). *Let $\{a, b, c\}$ be a $D(n)$-triple in $\mathbb{Z}[X]$ for which (1) holds. Then there exist polynomials $e, u, v, w \in \mathbb{Z}[X]$ such that*

$$ae + n^2 = u^2, \quad be + n^2 = v^2, \quad ce + n^2 = w^2.$$

More precisely,

$$e = n(a + b + c) + 2abc - 2rst. \tag{3}$$

Furthermore, it holds

$$c = a + b + \frac{e}{n} + \frac{2}{n^2}(abe + ruv),$$

where $u = at - rs$, $v = bs - rt$. Also, $w = cr - st$.

We also have

$$\overline{e} = n(a + b + c) + 2abc + 2rst$$

and

$$\overline{u} = at + rs,\ \overline{v} = bs + rt,\ \overline{w} = cr + st.$$

Hence, it holds

$$a\overline{e} + n^2 = \overline{u}^2, \quad b\overline{e} + n^2 = \overline{v}^2, \quad c\overline{e} + n^2 = \overline{w}^2.$$

Also,

$$c = a + b + \frac{\overline{e}}{n} + \frac{2}{n^2}(ab\overline{e} - r\overline{uv}).$$

Since $\deg(n(a+b+c)) = \gamma + 1$ and $\deg(abc) = \alpha + \beta + \gamma > \gamma + 1$, we have $\overline{\varepsilon} = \alpha + \beta + \gamma$ and $\overline{E} = 4ABC$.

We need to point out the useful relation (also used in [9])

$$e \cdot \overline{e} = n^2(c - a - b - 2r)(c - a - b + 2r) \tag{4}$$

and the relation

$$e = n(a + b - c) + 2rw, \tag{5}$$

obtained from (3), (4) and (1).

Note that $u, v < 0$. Indeed, since $0 < a < b < c$, it holds

$$a^2t^2 = a^2bc + a^2n < a^2bc + abn + acn + n^2 = (ab + n)(ac + n) = r^2s^2$$

and $at < rs$. Analogously,

$$b^2s^2 = ab^2c + b^2n < ab^2c + abn + bcn + n^2 = (ab + n)(bc + n) = r^2t^2$$

and $bs < rt$.

We want to find all possible c's for a fixed pair $\{a, b\}$ such that $0 < a < b < c$. Firstly, for $e = 0$, we have a possible triple $\{a, b, c_+\}$, where $c_+ = a + b + 2r$ and $\alpha \leq \beta = \gamma$. If $a + b - 2r \geq b$, then $\alpha = \beta$ and $A > B$, which is a contradiction. Therefore, $c_- = a + b - 2r < b$.

If $e \neq 0$, from (4), we have

$$\varepsilon \leq 2 + \gamma - \alpha - \beta.$$

Let us consider two cases:

(I) $\varepsilon < 2 + \gamma - \alpha - \beta$,

(II) $\varepsilon = 2 + \gamma - \alpha - \beta$.

(I) This case holds if and only if

$$\deg((c - a - b - 2r)(c - a - b + 2r)) < 2\gamma. \tag{6}$$

Obviously, the previous condition does not hold if $\gamma > \beta$ and therefore we consider the following cases:

(i) $\alpha = \beta = \gamma$,

(ii) $\alpha < \beta = \gamma$.

Case (i) is possible only for $\alpha = \beta = \gamma = 1$, $\varepsilon = 0$ and $C = A + B + 2R$. Indeed, according to (2), we have $\varepsilon < 2+\gamma-\alpha-\beta \leq \gamma = \alpha$. Since $\varepsilon < 2-\alpha$, we get

$$2\varepsilon < \alpha + \varepsilon < 2$$

and $\varepsilon = 0$, $\alpha = 1$. Also, (6) yields $C = A + B + 2R$ or $C = A + B - 2R$, but since $B < C$, $C = A + B - 2R$ cannot hold (because it is equivalent to $C = (\sqrt{B} - \sqrt{A})^2 < B$).

If (ii) holds, then

$$\alpha + \varepsilon < 2 \text{ and } \varepsilon < \beta.$$

Therefore $(\alpha, \varepsilon) \in \{(0,0), (0,1), (1,0)\}$. Since $\alpha \equiv \beta \pmod 2$ and $\beta \equiv \varepsilon \pmod 2$ (because $be + n^2 = v^2$), we have $\alpha \equiv \varepsilon \pmod 2$ that excludes the last two possibilities. So, we conclude that only $\alpha = \varepsilon = 0$ with $C = B$ can hold.

(II) may be valid in one of the following cases:

(i) $\alpha \leq \beta < \gamma$,

(ii) $\alpha < \beta = \gamma$ and $C \neq B$,

(iii) $\alpha = \beta = \gamma$ and $C \neq A + B + 2R$.

In (i), $\varepsilon + \alpha = 2 + \gamma - \beta > 2$, while in (ii) and (iii), we have $\varepsilon + \alpha = 2$.

Lemma 5. *Let the assumptions of Lemma 4 hold and $e \neq 0$. Then the following cases arise:*

(1) $\alpha = \beta = \gamma = 1$, $\varepsilon = 0$, $C = A + B + 2R$,

(2) $\alpha < \beta = \gamma$, $\alpha = \varepsilon = 0$, $C = B$,

(3) $\alpha \leq \beta < \gamma$, $\alpha + \beta + \varepsilon - \gamma = 2$, $\alpha + \varepsilon > 2$,

(4) $\alpha < \beta = \gamma$, $\alpha + \varepsilon = 2$, $C \neq B$,

(5) $\alpha = \beta = \gamma$, $\alpha + \varepsilon = 2$, $C \neq A + B + 2R$.

3 Proof of Theorem 1

The idea of proof is to show that there is no proper $D(2X+1)$-triple of types (1), (2) and (4). For the $D(2X+1)$-triple of type (5), we prove that it cannot be a part of a $D(2X+1)$-quadruple $\{a, b, c, d\}$ with $\deg(d) = \alpha = \beta = \gamma$.

Case (1):

We show that the only $D(2X+1)$-triple that satisfies conditions (1) in Lemma 5 is not proper.

Since $C = A + B + 2R$, we put

$$c - a - b - 2r = P \in \mathbb{Z},$$

and compare the leading coefficients in (4):

$$2E(ABC + \underbrace{RST}_{=ABC}) = 4P(\underbrace{-A - B + C}_{=2R} + 2R).$$

From the above and due to $AB = R^2$, we get

$$P = \frac{ERC}{4} = \frac{eRC}{4} \text{ and } c = a + b + 2r + \frac{eRC}{4}.$$

Taking into account that $\alpha = \beta = \gamma = 1$, $\varepsilon = 0$, and $u, v < 0$, we conclude that

$$u = -2X + u_0,\ v = -2X + v_0,\ w = \pm 2X + w_0$$

and

$$ae = -4(u_0 + 1)X + u_0^2 - 1, \tag{7}$$

$$be = -4(v_0 + 1)X + v_0^2 - 1, \tag{8}$$

$$ce = 4(\pm w_0 - 1)X + w_0^2 - 1. \tag{9}$$

Also, plugging $a = AX + a_0$, $b = BX + b_0$, $r = RX + r_0$ and $c = (A + B + 2R)X + a_0 + b_0 + 2r_0 + \frac{eRC}{4}$ into (5) yields

$$e = (2X + 1)\left(-2RX - 2r_0 - \frac{eRC}{4}\right) \pm 2(RX + r_0)(2X + w_0). \tag{10}$$

Since e is a constant, both the leading and the linear coefficients of the polynomial on the right-hand side in (10) are zero. So,

$$0 = -4R \pm 4R \ \Rightarrow\ w = 2X + w_0, \text{ i.e. } w > 0$$

and

$$0 = -4r_0 - \frac{eRC}{2} - 2R + 2Rw_0 + 4r_0 \ \Rightarrow\ Ce = -4 + 4w_0,$$

because $R \neq 0$. Finally, by comparing the constant coefficients in (10), we get

$$e = -2r_0 - \frac{RCe}{4} + 2r_0w_0 = (\underbrace{-4 + 4w_0}_{Ce})\frac{r_0}{2} - \frac{RCe}{4} = \frac{1}{4}(2r_0 - R)Ce.$$

Since $e \neq 0$ and $A, B, R \geq 1$, we have

$$4 = (2r_0 - R)C = (2r_0 - R)(\underbrace{A + B + 2R}_{\geq 1+1+2}) \;\Rightarrow\; A + B + 2R = 4$$

and therefore

$$A = B = R = r_0 = 1 \;\Rightarrow\; ab + 2X + 1 = (X+1)^2 \;\Rightarrow\; a = b = X,$$

which is a contradiction.

Note that from (7)-(9), we have $e = -4(u_0+1)$, $e = -4(v_0+1)$, $e = w_0 - 1$. Also, we obtain $e = -8$ and $\{a, b, c\} = \{X, X, 4X-6\}$ - an improper $D(2X+1)$-triple.

Case (2):

Since $a, e \in \mathbb{Z}$, we have $ae + (2X+1)^2 = (-2X + u_0)^2$, i.e.

$$ae = (-4X + u_0 - 1)(u_0 + 1).$$

The previous relation is possible only for $u_0 = -1$ and $a = 0$ or $e = 0$, which is a contradiction.

Case (4):

According to (5), $\varepsilon \leq \max\{\deg(n(a+b-c)), \deg(2rw)\}$. Observe that

$$\alpha < \beta = \gamma \text{ and } C \neq B \;\Rightarrow\; \deg(n(a+b-c)) = \gamma + 1,$$

and

$$\alpha < \beta = \gamma \text{ and } \alpha + \varepsilon = 2 \;\Rightarrow\; \deg(2rw) = \frac{\alpha+\beta}{2} + \frac{\gamma+\varepsilon}{2} = \gamma + 1.$$

If $\varepsilon = \gamma+1$, then $2 = \alpha+\varepsilon = \alpha+\gamma+1$ implies $\alpha+\gamma = 1$ which is not possible. Therefore, $\varepsilon < \gamma + 1$ which is true if the leading coefficient in (5) is zero, i.e. $2(B - C) + 2RW = 0$. So,

$$C - B = RW = \pm\sqrt{ABCE},$$

because $\deg(ab) \geq 2$ and $ab+n = r^2$ imply $AB = R^2$ and $\deg(ce) = \gamma+\varepsilon > 2$ and $ce + n^2 = w^2$ imply $CE = W^2$. On the other hand, $ae + (2X+1)^2 = u^2$ and $\deg(ae) = 2$ give $AE = U^2 - 4$ and $bc + n = s^2$ and $\deg(bc) \geq 2$ give $BC = S^2$. That means that $\sqrt{ABCE} = S\sqrt{U^2 - 4}$. Since $\sqrt{ABCE}$ should be an integer, it is fulfilled only for $U^2 - 4 = 0$, i.e. for $AE = 0$, and that is not possible.

Case (5):

We will approach this case in a different way than we did with the previous cases. Instead of proving that such a $D(2X+1)$-triple is not possible, we will show that if it is possible, it cannot be a part of a $D(2X+1)$-quadruple $\{a,b,c,d\}$, with $\deg(d)=\delta=\beta$. According to assumptions $\alpha=\beta=\gamma$ and $\alpha+\varepsilon=2$, there are only two possibilities: $\alpha=\beta=\gamma=\varepsilon=1$ or $\alpha=\beta=\gamma=2$ and $\varepsilon=0$. So,

$$\alpha=\beta=\gamma=\delta=1 \quad \text{or} \quad \alpha=\beta=\gamma=\delta=2.$$

The last possibility can be rejected since Theorem 2 implies that there is no $D(n_1X+n_0)$-quadruple ($n_1\neq 0$) in $\mathbb{Z}[X]$ whose elements are quadratic polynomials. Therefore, let us consider the case in which the elements of the quadruple are linear polynomials.

Let

$$\{AX+a_0, BX+b_0, CX+c_0, DX+d_0\} \tag{11}$$

be a $D(2X+1)$-quadruple such that leading coefficients are positive. Then either $\gcd(A,B)>1$ or $\gcd(A,B)=1$ can be valid. We consider each of the cases separately.

Case $\gcd(A,B)=d>1$:

Since

$$\underbrace{AB}_{d^2|}X^2+(\underbrace{a_0B+Ab_0}_{d|}+2)X+a_0b_0+1=\underbrace{R^2}_{d^2|}X^2+\underbrace{2Rr_0}_{d|}X+r_0^2,$$

we have $d=2$ and $A=2A_1$, $B=2B_1$, $\gcd(A_1,B_1)=1$. So,

$$4A_1B_1=R^2 \;\Rightarrow\; A_1=A_2^2,\ B_1=B_2^2,\ R=2R_1.$$

After rewriting ($A=A_2$, $B=B_2$, $R=R_1$ etc.), we conclude that a $D(2X+1)$-quadruple is of the form

$$\{2A^2X+a_0, 2B^2X+b_0, 2C^2X+c_0, 2D^2X+d_0\},$$

because $2A^2C=\square$ and $2A^2D=\square$ imply $C=2\square$, $D=2\square$. Furthermore, we have

$$2a_0B^2+2A^2b_0+2=2(2AB)r_0,\ \ a_0b_0=r_0^2-1,$$

which after substitution $b_0=(r_0^2-1)/a_0$ gives a quadratic equation in r_0

$$A^2r_0^2-2ABa_0r_0-A^2+a_0+a_0^2B^2=0.$$

Since $r_0 \in \mathbb{Z}$, the discriminant $4(A^4 - A^2 a_0) = \square$ yields $a_0 = A^2 - K^2$, for $K \in \mathbb{Z}$. This gives that a $D(2X+1)$-quadruple is of the form

$$\{2A^2X + A^2 - K^2, 2B^2X + B^2 - L^2, 2C^2X + C^2 - M^2, 2D^2X + D^2 - N^2\}.$$

With $Y = 2X+1$, we have a $D(Y)$-quadruple

$$\{A^2Y - K^2, B^2Y - L^2, C^2Y - M^2, D^2Y - N^2\}, \tag{12}$$

where $A, B, C, D \in \mathbb{Z}$ are relatively prime in pairs, and $K, L, M, N \in \mathbb{Z}$. Therefore, $-B^2K^2 - A^2L^2 + 1 = \pm 2ABKL$ and similarly for others, i.e.

$$|AL \pm BK| = |AM \pm CK| = |AN \pm DK| =$$

$$|BM \pm CL| = |BN \pm DL| = |CN \pm DM| = 1.$$

But this is not possible modulo 2, so there is no $D(Y)$-quadruple of the form (12), i.e. there is no $D(2X+1)$-quadruple of the form (11) with $\gcd(A,B) = 2$.

Case $\gcd(A,B) = 1$:

Similar to the previous case, we conclude that $D(2X+1)$-quadruple is of the form

$$\{A^2X + a_0, B^2X + b_0, C^2X + c_0, D^2X + d_0\}.$$

Indeed, from (11), $AB = R^2$ and since $\gcd(A,B) = 1$, we get $A = \square$, $B = \square$. Furthermore, if $A^2C = \square$, then $C = \square$. Also,

$$a_0B^2 + A^2b_0 + 2 = 2ABr_0,\ a_0b_0 = r_0^2 - 1,$$

give a quadratic equation in r_0:

$$A^2r_0^2 - 2ABa_0r_0 - A^2 + 2a_0 + a_0^2B^2 = 0$$

whose discriminant should be equal to a perfect square in $\mathbb{Z}$, i.e. $4(A^4 - 2A^2a_0) = \square$. So, $a_0 = (A^2 - K^2)/2$, for some $K \in \mathbb{Z}$. Therefore, in this case, a $D(2X+1)$-quadruple is of the form

$$\left\{A^2X + \frac{A^2 - K^2}{2}, B^2X + \frac{B^2 - L^2}{2}, C^2X + \frac{C^2 - M^2}{2}, D^2X + \frac{D^2 - N^2}{2}\right\},$$

where $A, B, C, D > 0$ and K, L, M, N are integers of the same parity respectively (i.e. $A \equiv K \pmod 2$, $B \equiv L \pmod 2$ etc.) and $\gcd(A,B) = 1$. Multiplying the given quadruple by 2 and substituting $Y = 2X+1$, yields a $D(4Y)$-quadruple of the form (12).

Lemma 6. *There is no $D(4Y)$-quadruple in $\mathbb{Z}[Y]$ of the form*

$$\{A^2Y-K^2, B^2Y-L^2, C^2Y-M^2, D^2Y-N^2\},$$

where $A,B,C,D,K,L,M,N \in \mathbb{Z}$.

Proof. Since $(A^2Y-K^2)(B^2Y-L^2)+4Y$ is a perfect square in $\mathbb{Z}[X]$, discriminant equals zero so, taking into account all the others, we get

$$|AL \pm BK| = |AM \pm CK| = |AN \pm DK| =$$

$$|BM \pm CL| = |BN \pm DL| = |CN \pm DM| = 2.$$

The previous relations are fulfilled if the set $\{(A,K),(B,L),(C,M),(D,N)\}$ modulo 2 equals:

(i) $\{(0,0),(0,0),(0,0),(0,0)\}$,

(ii) $\{(0,0),(0,0),(0,0),(0,1)\}$,

(iii) $\{(0,0),(0,0),(0,0),(1,0)\}$,

(iv) $\{(0,0),(0,0),(0,0),(1,1)\}$,

(v) $\{(0,0),(0,0),(0,1),(0,1)\}$,

(vi) $\{(0,0),(0,0),(1,0),(1,0)\}$,

(vii) $\{(0,0),(0,0),(1,1),(1,1)\}$,

(viii) $\{(0,0),(0,1),(0,1),(0,1)\}$,

(ix) $\{(0,0),(1,0),(1,0),(1,0)\}$,

(x) $\{(0,0),(1,1),(1,1),(1,1)\}$,

(xi) $\{(0,1),(0,1),(0,1),(0,1)\}$,

(xii) $\{(1,0),(1,0),(1,0),(1,0)\}$,

(xiii) $\{(1,1),(1,1),(1,1),(1,1)\}$.

We observe the coefficients of Y in $(A^2Y-K^2)(B^2Y-L^2)+4Y = (ABY \pm KL)^2$, i.e. the validity of equality

$$-B^2K^2 - A^2L^2 + 4 = 2ABKL. \tag{13}$$

Cases (i)-(vii): If $(A,K),(B,L) \mod 2 = (0,0)$, then in (13) we get a contradiction modulo 8.

Cases (viii) and (xi): Let

$$(A,K),(B,L),(C,M) = (2A_1, 2K_1+1),(2B_1, 2L_1+1),(2C_1, 2M_1+1),$$

where $A_1,\ldots,M_1 \in \mathbb{Z}$. Then (13) modulo 8 yields

$$4+4A_1^2+4B_1^2 \equiv 0 \pmod 8, \text{ i.e. } A_1^2+B_1^2 \equiv 1 \pmod 2.$$

Analogous statements (modulo 8) to the one in (13) give

$$A_1^2+C_1^2 \equiv 1 \pmod 2,\ B_1^2+C_1^2 \equiv 1 \pmod 2.$$

So, $2(A_1^2+B_1^2+C_1^2) \equiv 1 \pmod 2$ - a contradiction! (Or simply, two of the three numbers must have the same parity!)

Cases (ix) and (xii): If

$$(A,K),(B,L),(C,M) = (2A_1+1, 2K_1),(2B_1+1, 2L_1),(2C_1+1, 2M_1),$$

where $A_1,\ldots,M_1 \in \mathbb{Z}$, then (13) modulo 8 gives $A_1^2+B_1^2 \equiv 1 \pmod 2$ etc. and everything is analogous to the previous case.

Cases (x) and (xiii): Let $(A,K),(B,L),(C,M) \equiv (1,1) \pmod 2$. Without loss of generality, let $A,K,B,\ldots$ be positive integers. Observe that

$$AL+BK=2$$

is possible only for $(A,K)=(B,L)=(1,1)$, which means that the first two members of the quadruple are equal. So, we have

$$AL-BK,\ AM-CK,\ BM-CL \in \{2,-2\}.$$

First, let

$$AL-BK=2,\ AM-CK=2,\ BM-CL=2.$$

By multiplying the first equation by M, the second by L and subtracting them, we get

$$K(\underbrace{CL-BM}_{-2}) = 2M-2L,$$

which means that $L=M+K$ is an even number. A contradiction! (For other combinations of signs, we also get $M=K+L$ and $K=M+L$.) □

4 On case (3)

At the moment, we do not know if there are $D(2X+1)$-quadruples containing triples of type (3) (or (5)) from Lemma 5, but we can give some examples of pairs that can be extended to triples in infinitely many ways.

So, let us assume that $\{a, b\}$ is a polynomial $D(2X+1)$-pair, such that $a < b$ and
$$ab + n = r^2,$$
where $r \in \mathbb{Z}^+[X]$. Expanding the pair $\{a, b\}$ means that we are looking for a polynomial $c > b$ for which

$$ac + n = s^2 \quad \text{and} \quad bc + n = t^2$$

hold for some $s, t \in \mathbb{Z}^+[X]$. By eliminating c from these two equations, we obtain
$$bs^2 - at^2 = n(b - a) \tag{14}$$
and, by multiplying (14) with b and substituting $p := bs$, we further obtain

$$p^2 - abt^2 = nb(b - a). \tag{15}$$

If ab is not a perfect square, equation (15) is a *Pellian* equation and we can observe its solutions (p, t) in order to find possible extensions of the pair $\{a, b\}$ with the element c. On the other hand, if ab is a perfect square, equation (15) is not a Pellian equation and we have to observe it separately.

Case 1: ab is a perfect square.
Assume that $ab = q^2$. So, we have

$$2X + 1 = r^2 - q^2,$$

which is only possible for

$$(r, q) = (X + 1, X) \quad \text{and} \quad ab = X^2.$$

Indeed, $2X + 1 = (r - q)(r + q)$ implies that $2X + 1 \mid r \pm q$. Hence, $r \pm q = k(2X + 1)$ and then $1 = k(r \mp q)$, where $k \in \mathbb{Z}[X]$. We conclude that $k = \pm 1$ and easily get $r = X + 1$ and $q = \pm X$. Therefore,

$$\{1, X^2\}$$

is the only pair for which we have to check if there exists c such that the triple $\{1, X^2, c\}$ is of type (3) i.e. such that $c \neq X^2 + 2X + 3$.

Proposition 7. *If $\{1, X^2, c\}$ is a $D(2X+1)$-triple in $\mathbb{Z}[X]$, then it is regular, i.e. $c \in \{X^2 - 2X - 1, X^2 + 2X + 3\}$.*

Proof. Since
$$X^2c + 2X + 1 = t^2, \tag{16}$$

$t(0)^2 = 1$ and $t(0)t'(0) = 1$ (where t' is the derivative of t). So, $t(0) = t'(0) = \pm 1$ and

$$t(X) = X^2 q \pm X \pm 1,$$

for some $q \in \mathbb{Z}[X]$. By plugging it into (16), we get

$$c(X) = X^2 q^2 \pm 2Xq \pm 2q + 1 = (Xq \pm 1)^2 \pm 2q.$$

We still need to check the condition

$$c + 2X + 1 = s^2,$$

which gives

$$(Xq \pm 1)^2 \pm 2q + 2X + 1 = s^2, \tag{17}$$

i.e.

$$\pm 2q + 2X + 1 = (s - Xq \mp 1)(s + Xq \pm 1).$$

Comparing the degrees of polynomials on both sides of the previous relation, we get a contradiction under the assumption $\deg q \geq 1$ ($\deg q \geq \deg q + 1$). Hence, $\deg q = 0$ and $q(X) = q_0 \in \mathbb{Z}$. In this case, the discriminant of the expression on the left in (17) is

$$\mp 4(q_0 - 1)(q_0 + 1)(2q_0 \pm 1).$$

So, $q_0 = \pm 1$ and the corresponding c's are those created by regular extensions, i.e. $c = X^2 + 2X + 3 = a + b + 2r$ or $c = X^2 - 2X - 1 = a + b - 2r$. □

Case 2: ab is not a perfect square.

For $e = 0$, we have $s = a + r$ and $t = b + r$ and in that case

$$(p_0, t_0) = (b(a + r), b + r)$$

is a solution of Pellian equation (15). The existence of more solutions depends on the solvability of the associated Pell's equation

$$P^2 - abT^2 = 1. \tag{18}$$

So, if $P_1 + T_1\sqrt{ab}$ is a (fundamental) solution of Pell's equation (18) and $p_0 + t_0\sqrt{ab}$ is a (fundamental) solution of equation (15), more solutions are obtained by

$$p_n + t_n\sqrt{ab} = (p_0 + t_0\sqrt{ab})(P_1 + T_1\sqrt{ab})^n, n \geq 0. \tag{19}$$

With $(P_0, T_0) = (1, 0)$ we denote the trivial solution of equation (18). From (19) we have

$$t_n = t_0 P_n + p_0 T_n,$$

where $(p_0, t_0) = (b(a+r), b+r)$ and (P_n, T_n) can be obtained by recursions

$$\begin{aligned} P_n &= 2P_1P_{n-1} + P_{n-2}, \\ T_n &= 2P_1T_{n-1} + T_{n-2}, \end{aligned} \tag{20}$$

with initial conditions $(P_0, T_0) = (1, 0)$ and (P_1, T_1). Hence, the following proposition holds:

Proposition 8. *Let $\{a, b\}$ be a $D(2X+1)$-pair in $\mathbb{Z}[X]$, $0 < a < b$, such that Pell's equation* (18) *is solvable. Then it can be extended to a $D(2X+1)$-triple of type (3) or (5) (from Lemma 5) by adding*

$$c = c_n = \frac{t_n^2 - 2X - 1}{b} = \frac{(t_0P_n + p_0T_n)^2 - 2X - 1}{b} \tag{21}$$

for some $n \geq 1$, where $(p_0, t_0) = (b(a+r), b+r)$ and (P_n, T_n) are given by (20).

In what follows, we give examples of pairs for which Pell's equation has solutions. Let us assume that $2X+1 \mid ab$. Hence, $2X+1 \mid r$ and

$$r = (2X+1)q, \ q \in \mathbb{Z}[X].$$

Note that in this case Pell's equation (18) is solvable. Indeed,

$$(P, T) = (4q^2X + 2q^2 - 1, 2q)$$

is a solution of (18).

Corollary 9. *Let $\{a, b\}$ be a $D(2X+1)$-pair in $\mathbb{Z}[X]$ such that $0 < a < b$ and $2X+1 \mid ab$. Then there exists $c \in \mathbb{Z}[X]$ such that $\{a, b, c\}$ is a $D(2X+1)$-triple of type (3) (from Lemma 5).*

Proof. It follows straight from Proposition 8 for

$$(P_0, T_0) = (1, 0), \quad (P_1, T_1) = (4q^2X + 2q^2 - 1, 2q),$$

where $ab + 2X + 1 = (2X+1)^2q^2$. □

The natural question arises: can $\{a, b, c_n, c_m\}$ be a $D(2X+1)$-quadruple for some $m, n \in \mathbb{N}_0$? Well, we don't know for sure, but in some cases we can show that it cannot be. For instance, if there exists an integer X_0 such that

$$c_nc_m(X_0) + 2X_0 + 1 \not\equiv 0, 1 \pmod 4,$$

then $c_n c_m + 2X + 1 \neq \square$ in $\mathbb{Z}[X]$. It is not difficult to show that the following congruence holds for the numerator of c_n given in (21):

$$(t_0 P_n + p_0 T_n)^2 - (2X+1) \equiv b^2 + 2bq + \underbrace{(2X+1)^2 q^2 - (2X+1)}_{ab} \pmod 4.$$

Hence,

$$c_n \equiv a + b + 2q \pmod 4$$

or

$$c_n \equiv a + b + 2r = c_0 \pmod 4,$$

because $2q \equiv 2(2X+1)q = 2r \pmod 4$. Let us formulate the result.

Lemma 10. *Let $\{a, b\}$ be a $D(2X+1)$-pair in $\mathbb{Z}[X]$ such that*

$$ab + 2X + 1 = (2X+1)^2 q^2,$$

for some $q \in \mathbb{Z}[X]$, and let (c_n) be given by (21) for $(P_0, T_0) = (1, 0)$ and $(P_1, T_1) = (4q^2 X + 2q^2 - 1, 2q)$. If there exists an integer X_0 such that

$$(a+b)^2(X_0) + 2X_0 + 1 \equiv 2, 3 \pmod 4,$$

then $c_n c_m + 2X + 1$ is not a perfect square in $\mathbb{Z}[X]$.

Proposition 11. *In terms of Lemma 10 and if the polynomial q has an odd integer root, then $c_n c_m + 2X + 1$ is not a perfect square in $\mathbb{Z}[X]$ for $n, m \in \mathbb{N}_0$ (where c_n is given by (21)).*

Proof. Assume that $q(X_0) = 0$ for some odd integer X_0. Then

$$ab(X_0) + 2X_0 + 1 = 0$$

implies that $a(X_0), b(X_0) \equiv 1 \pmod 2$. Hence,

$$(a+b)^2(X_0) + 2X_0 + 1 \equiv 2 \pmod 4$$

and the assertion follows according to Lemma 10.

Also, note that in this case we have $T_n(X_0) = 0$, $P_n(X_0) = (-1)^n$ and

$$c_n(X_0) = \frac{b^2(X_0) - 2X_0 - 1}{b(X_0)} = \frac{b^2(X_0) + ab(X_0)}{b(X_0)} = (a+b)(X_0) = const,$$

for all $n \geq 0$, so the conclusion can be drawn immediately. $\square$

ab	r	T_1
$2(2X+1)(25X+12)$	$5(2X+1)$	10
$(2X+1)\left(2X^3-19X^2+40X+24\right)$	$(X-5)(2X+1)$	$2(X-5)$
$2(2X+1)\left(4X^3-18X^2+15X+12\right)$	$(2X-5)(2X+1)$	$2(2X-5)$
$(2X-3)(2X+1)\left(9X^2-12X-8\right)$	$(2X+1)(3X-5)$	$2(3X-5)$
$2(2X+1)\left(16X^3-32X^2+5X+12\right)$	$(2X+1)(4X-5)$	$2(4X-5)$
$(2X+1)\left(50X^3-75X^2+24\right)$	$5(X-1)(2X+1)$	$10(X-1)$
$(2X+1)(32X+15)$	$4(2X+1)$	8
$(2X+1)\left(2X^3-15X^2+24X+15\right)$	$(X-4)(2X+1)$	$2(X-4)$
$(2X+1)\left(8X^3-28X^2+16X+15\right)$	$2(X-2)(2X+1)$	$4(X-2)$
$(2X-3)(2X+1)\left(9X^2-6X-5\right)$	$(2X+1)(3X-4)$	$2(3X-4)$
$(2X+1)\left(32X^3-48X^2+15\right)$	$4(X-1)(2X+1)$	$8(X-1)$
$(2X+1)\left(50X^3-55X^2-8X+15\right)$	$(2X+1)(5X-4)$	$2(5X-4)$
$2(2X+1)(9X+4)$	$3(2X+1)$	6
$(2X+1)\left(2X^3-11X^2+12X+8\right)$	$(X-3)(2X+1)$	$2(X-3)$
$2(2X+1)\left(4X^3-10X^2+3X+4\right)$	$(2X-3)(2X+1)$	$2(2X-3)$
$(2X+1)\left(18X^3-27X^2+8\right)$	$3(X-1)(2X+1)$	$6(X-1)$
$2(2X+1)\left(16X^3-16X^2-3X+4\right)$	$(2X+1)(4X-3)$	$2(4X-3)$
$(2X+1)\left(50X^3-35X^2-12X+8\right)$	$(2X+1)(5X-3)$	$2(5X-3)$
$(2X+1)(8X+3)$	$2(2X+1)$	4
$(2X-3)(2X+1)\left(X^2-2X-1\right)$	$(X-2)(2X+1)$	$2(X-2)$
$(2X+1)\left(8X^3-12X^2+3\right)$	$2(X-1)(2X+1)$	$4(X-1)$
$(*)\quad X^2+2X+3$	$X+2$	$X+1$
$(2X+1)\left(18X^3-15X^2-4X+3\right)$	$(2X+1)(3X-2)$	$2(3X-2)$
$(2X+1)\left(32X^3-16X^2-8X+3\right)$	$2(2X-1)(2X+1)$	$4(2X-1)$
$(2X+1)\left(50X^3-15X^2-12X+3\right)$	$(2X+1)(5X-2)$	$2(5X-2)$
$2X(2X+1)$	$2X+1$	2
$(*)\quad X^2\left(X^2-2X-1\right)$	X^2-X-1	$2(X-2)(X-1)$
$X^2(2X-3)(2X+1)$	$(X-1)(2X+1)$	$2(X-1)$
$2X(2X+1)\left(4X^2-2X-1\right)$	$(2X-1)(2X+1)$	$2(2X-1)$
$X(2X+1)\left(18X^2-3X-4\right)$	$(2X+1)(3X-1)$	$2(3X-1)$
$2X(2X+1)\left(16X^2-3\right)$	$(2X+1)(4X-1)$	$2(4X-1)$
$X(2X+1)\left(50X^2+5X-8\right)$	$(2X+1)(5X-1)$	$2(5X-1)$
$(*)\quad X^2-2X-1$	X	$X-1$
$(2X+1)\left(2X^3+X^2-1\right)$	$X(2X+1)$	$2X$
$(2X+1)\left(8X^3+4X^2-1\right)$	$2X(2X+1)$	$4X$
$(2X+1)\left(18X^3+9X^2-1\right)$	$3X(2X+1)$	$6X$
$(2X+1)\left(32X^3+16X^2-1\right)$	$4X(2X+1)$	$8X$
$(2X+1)\left(50X^3+25X^2-1\right)$	$5X(2X+1)$	$10X$
$X(2X+1)\left(2X^2+5X+4\right)$	$(X+1)(2X+1)$	$2(X+1)$
$2X(2X+1)\left(4X^2+6X+3\right)$	$(2X+1)^2$	$2(2X+1)$
$X(2X+1)\left(18X^2+21X+8\right)$	$(2X+1)(3X+1)$	$2(3X+1)$
$2X(2X+1)\left(16X^2+16X+5\right)$	$(2X+1)(4X+1)$	$2(4X+1)$
$X(2X+1)\left(50X^2+45X+12\right)$	$(2X+1)(5X+1)$	$2(5X+1)$
$(2X+1)\left(2X^3+9X^2+12X+3\right)$	$(X+2)(2X+1)$	$2(X+2)$
$(2X+1)\left(8X^3+20X^2+16X+3\right)$	$2(X+1)(2X+1)$	$4(X+1)$
$(2X+1)\left(18X^3+33X^2+20X+3\right)$	$(2X+1)(3X+2)$	$2(3X+2)$
$(2X+1)\left(32X^3+48X^2+24X+3\right)$	$2(2X+1)^2$	$4(2X+1)$
$(2X+1)\left(50X^3+65X^2+28X+3\right)$	$(2X+1)(5X+2)$	$2(5X+2)$
$(2X+1)\left(2X^3+13X^2+24X+8\right)$	$(X+3)(2X+1)$	$2(X+3)$
$2(2X+1)\left(4X^3+14X^2+15X+4\right)$	$(2X+1)(2X+3)$	$2(2X+3)$
$(2X+1)\left(18X^3+45X^2+36X+8\right)$	$3(X+1)(2X+1)$	$6(X+1)$
$2(2X+1)\left(16X^3+32X^2+21X+4\right)$	$(2X+1)(4X+3)$	$2(4X+3)$
$(2X+1)\left(2X^3+17X^2+40X+15\right)$	$(X+4)(2X+1)$	$2(X+4)$
$(2X+1)\left(8X^3+36X^2+48X+15\right)$	$2(X+2)(2X+1)$	$4(X+2)$

Table 1: Examples of $D(2X+1)$-pairs s.t. $P^2-abT^2=1$ is solvable in $\mathbb{Z}[X]$

In Table 1 we give some examples of $D(2X+1)$-pairs (where $\deg ab \leq 4$) that can be extended to triples of the type (3) or (5), i.e. such that Pell's equation (18) is solvable. Most of them are examples of pairs $\{a,b\}$ such that $2X+1 \mid ab$ and only a few do not meet this condition (marked red, with $(*)$).

Example 1. *According to Corollary 9, $D(2X+1)$-pairs:*

(a) $\{1, X^2+2X+3\}$,

(b) $\{1, X^2\left(X^2-2X-1\right)\}$,

(c) $\{X^2, X^2-2X-1\}$,

(d) $\{1, X^2-2X-1\}$

can be extended to $D(2X+1)$-triple of the type (3), for instance with c_1:

(a) $(2X^2+5X+4)(2X^4+13X^3+36X^2+50X+30)$,

(b) $(4X^4-16X^3+16X^2+2X-5)(4X^8-24X^7+52X^6-54X^5+35X^4-12X^3-5x^2+4X-3)$,

(c) $(2X+1)(4X^4-16X^3+16X^2+2X-5)(8X^5-44X^4+80X^3-48X^2-4X+7)$,

(d) $X(2X-3)(2X^4-3X^3-4X^2+2X+2)$.

Also, in each of the cases

$$c_n c_m + 2X + 1 \neq \square \quad \text{in} \quad \mathbb{Z}[X].$$

Indeed, we have

(a) for $X=-1$:

$$P_n(1) = 1,\ T_n(1) = 0,\ c_n(1) = 5,\ \forall n \geq 0$$

and $c_n c_m(2) + 2 \cdot (-1) + 1 = 24 \neq \square$;

(b) for $X=1$:

$$P_n(1) = 1,\ T_n(1) = 0,\ c_n(1) = -3,\ \forall n \geq 0$$

and $c_n c_m(2) + 2 \cdot 1 + 1 = 12 \neq \square$;

(c) same as (b);

(d) for $X = 2$:

$$P_n(2) = \begin{cases} (-1)^{k+1}, & n = 2k \\ 0, & n = 2k+1 \end{cases}, \quad T_n(2) = \begin{cases} 0, & n = 2k \\ (-1)^k, & n = 2k+1 \end{cases},$$

$$c_n(2) = (-1)^n 4,$$

for all $n \geq 0$, and $c_n c_m(2) + 2 \cdot 2 + 1 = \pm 16 + 5 \neq \square$ in $\mathbb{Z}$.

Example 2. *Note that many pairs $\{a, b\}$ satisfy the conditions of Lemma 10 and Proposition 11, but some pairs do not. For instance, if*

$$ab = 2(2X+1)\left(4X^3 - 18X^2 + 15X + 12\right)$$

and $q = 2X - 5$ then, for

$$a = 2,\ b = (2X+1)(4X^3 - 18X^2 + 15X + 12),$$

we get

$$(a+b)^2 + 2X + 1 \equiv (X+1)^2 \pmod 4.$$

The same is obtained for

$$a = 2(2X+1),\ b = 4X^3 - 18X^2 + 15X + 12.$$

But, for

$$a = 1,\ b = 2(2X+1)(4X^3 - 18X^2 + 15X + 12)$$

or

$$a = 2(2X+1),\ b = 4X^3 - 18X^2 + 15X + 12,$$

we have

$$(a+b)^2 + 2X + 1 \equiv 2(X+1) \pmod 4$$

and, according to Lemma 10, these pairs cannot be extended to $D(2X+1)$-quadruples.

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