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Approximation of fractional derivatives by Brass-Stancu operators

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Abstract

In this paper, we study uniform approximation of fractional derivatives in Caputo sense by Brass-Stancu operators. For the rate of approximation, we give some estimates in terms of the modulus of continuity.

1 Introduction

The fractional calculus knows a great extension in the last period due to the multiple applications in many fields, like engineering systems, diffusion phenomenon, dynamic systems, thermoelasticity and others, see for instance [8], [11], [16]. The appearance of the Caputo variant of the fractional derivative [7] greatly boosted the development of the field. As a general reference regarding the Caputo differential calculus, we mention the work [9], but there are many surveys on this subject.

Relatively recently the problem of approximation of the Caputo derivatives with the aid of positive linear operators raised. This a natural extension of the simultaneous approximation. We mention in this direction the papers [2], [3], [6], [14].

In the present paper we consider the approximation of Caputo derivatives with the aid of Brass-Stancu operators and this way we generalize the approximation by Bernstein operators.

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The paper is organized as follows. In Section 2 we present some general definitions and results, in Section 3 we obtain certain results concerning the Brass-Stancu operators and in Section 4 we give the main results.

2 Preliminaries

For $f \in C[0,1]$, a non-negative fixed integer s and $n \in \mathbb{N}$ such that $n > 2s$, Stancu [15] introduced the following Bernstein type positive linear operators

$$L_{n,s}(f;x) = \sum_{k=0}^{n-s} p_{n-s,k}(x) \left[(1-x) f\left(\frac{k}{n}\right) + x f\left(\frac{k+s}{n}\right) \right],$$

where $x \in [0,1]$ and $p_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k}$. The author examined uniform approximation of these operators and computed the order of approximation in terms of f and its first derivative. He also obtained a Voronovskaja type asymptotic formula for the remainder term in the approximation formula of f and investigated spectral properties of $L_{n,s}$. We note that, as stated in [15], for $s = 0$ or $s = 1$ these operators become the classical Bernstein operators.

We should note here that $L_{n,2}$ was constructed and studied earlier by Brass [4]. The operators $L_{n,s}$ are usually called as Stancu operators in the literature. Gonska [10] referred them as Brass-Stancu operators. We will adopt this name throughout the paper. Like Bernstein operators, Brass-Stancu operators have nice shape results such as preservation of a Lipschitz' constant and order of a Lipschitz continuous function and monotonicity of the sequence $\{L_{n,s}(f;x)\}_{n\geq 1}$ under convexity [17], monotonicity in terms of divided differences [1] and variation detracting property [5].

It is quite natural to extend the simultaneous approximation by positive linear operators to any fractional order. On the other hand, for the rate of that approximation, some quantitative results need to be measured with some convenient tools.

Khosravian-Arab and Torres [12] studied uniform approximation of fractional derivatives by the classical Bernstein operators in the sense of Caputo fractional derivatives and gave a history on the usage of Bernstein polynomials to approximate fractional derivatives and integrals and to approximate the solution of some fractional integro-differential equations and some other applications. Păltănea [14] obtained some estimates in terms of the first and the second order moduli of smoothness for the degree of approximation of Caputo fractional derivatives by Bernstein operators.

These papers have motivated us, by using similar techniques, to study uniform approximation of fractional derivatives in Caputo sense by Brass-Stancu operators and to give some quantitative estimates for the rate of the approximation via first order modulus of smoothness.

Now, we recall some definitions and results used throughout the paper. Suppose that $[a,b]$, $-\infty < a < b < \infty$, be a finite interval. Let $AC[a,b]$ denote the space of functions f which are absolutely continuous on $[a,b]$. For $p \in \mathbb{N}$, let $AC^p[a,b]$ denote the space of functions f with $f^{(p-1)} \in AC[a,b]$ and let $C^p[a,b]$ denote space of continuously differentiable functions on $[a,b]$ up to order p.

Throughout the paper $\|.\|$ stands for the sup-norm taken over $[0,1]$ and $\lceil . \rceil$ denotes the ceiling of a number.

From [17], we have the following simultaneous approximation result.

Theorem 2.1. *([17]) If $f \in C^p[0,1]$, $p \in \mathbb{N}$ and s is a non-negative fixed integer, then*

$$\lim_{n\to\infty} L_{n,s}^{(p)}(f) = f^{(p)}$$

uniformly on $[0,1]$.

Definition 2.1. *([9])Let $\alpha > 0$. The operator J_a^α on $L^1[a,b]$ defined by*

$$J_a^\alpha(f;x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\,dt$$

for $a \le x \le b$ is called the Riemann-Liouville fractional integral operator of order α. Here $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt$ is Euler's Gamma function.

For $\alpha = 0$, we set $J_a^0 := I$, the identity operator.

Definition 2.2. *([9])Let $\alpha \ge 0$, $p = \lceil \alpha \rceil$ and $f \in AC^p[a,b]$. The Caputo fractional derivative is defined by*

$$D_{*a}^\alpha f := D_a^\alpha[f - T_{p-1}[f;a]] = J_a^{p-\alpha} D^p f,$$

where $D_a^\alpha := D^p J_a^{p-\alpha}$ is the Riemann-Liouville fractional differential operator and $T_{p-1}[f;a]$ denotes the Taylor polynomial of degree $p-1$ of the function f, centered at a. In particular, for $p = 0$ we set $T_{p-1}[f;a] = 0$ and if $\alpha \in \mathbb{N}$, then $D_{*a}^\alpha f = D^p f$.

Theorem 2.2. *([9])If $f \in C^p[a,b]$, then the Caputo fractional derivative $D_{*a}^\alpha f$ is continuous on $[a,b]$.*

Let, as usual, we denote $e_j(x) := x^j$ for $x \in \mathbb{R}$ and $j \geq 0$. Then we have the following remarks.

Remark 2.1. *For each* $j \geq 0$, $\alpha > 0$ *and* $x \in [0,1]$, *we have* $J_0^\alpha(e_j;x) = \frac{\Gamma(j+1)}{\Gamma(\alpha+j+1)} x^{j+\alpha}$ *(see [14, Lemma 2]).*

Remark 2.2. *Let* $\alpha > 0$ *and* $p = \lceil \alpha \rceil$. *Then we have*

$$D_{*0}^\alpha e_j(x) = \begin{cases} 0, & j \in \mathbb{N}_0 \text{ and } j < p, \\ \frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)} x^{j-\alpha}, & j \in \mathbb{N}_0 \text{ and } j \geq p \text{ or } j \notin \mathbb{N} \text{ and } j > p-1 \end{cases}$$

for some $j \geq 0$ *(see [9, p.193]).*

3 Auxiliary results for Brass-Stancu operators

Let us consider the following p-dimensional sets

$$Q_{n,s} := \left[0, \frac{s}{n}\right] \times \left[0, \frac{1}{n}\right] \times \cdots \times \left[0, \frac{1}{n}\right] \text{ and } Q_n := \left[0, \frac{1}{n}\right] \times \left[0, \frac{1}{n}\right] \times \cdots \times \left[0, \frac{1}{n}\right].$$

We denote the integrals over $Q_{n,s}$ and Q_n by

$$\int\limits_{Q_{n,s}} d\overline{u} := \int_0^{\frac{1}{n}} \cdots \int_0^{\frac{1}{n}} \int_0^{\frac{s}{n}} du_1 du_2 \cdots du_p \text{ and } \int\limits_{Q_n} d\overline{u} := \int_0^{\frac{1}{n}} \cdots \int_0^{\frac{1}{n}} \int_0^{\frac{1}{n}} du_1 du_2 \cdots du_p,$$

where $d\overline{u} = du_1 du_2 \cdots du_p$, respectively. Let also $(n)_p$ denote the falling factorial given by

$$(n)_p = \begin{cases} n(n-1)\cdots(n-p+1), & p \geq 1, \\ 1, & p = 0. \end{cases}$$

As in the case of approximation of the Caputo fractional derivatives by Bernstein operators [14], we need the following usual derivatives of order p of $L_{n,s}(f;x)$ which was obtained by Yun and Xiang [17].

$$\begin{aligned} & L_{n,s}^{(p)}(f;x) \\ = \ & \frac{p(n-s)_p}{n-s-p+1} \sum_{k=0}^{n-s-p+1} p_{n-s-p+1,k}(x) \int\limits_{Q_{n,s}} f^{(p)}\left(\frac{k}{n} + \sum_{i=1}^p u_i\right) d\overline{u} \\ & + (n-s)_p \sum_{k=0}^{n-s-p} p_{n-s-p,k}(x) \\ & \times \int\limits_{Q_n} \left[(1-x) f^{(p)}\left(\frac{k}{n} + \sum_{i=1}^p u_i\right) + x f^{(p)}\left(\frac{k+s}{n} + \sum_{i=1}^p u_i\right)\right] d\overline{u}, \end{aligned}$$

where s is a non-negative integer, $n \in \mathbb{N}$ such that $n > 2s$ and $p \leq n-s$, $f \in C[0,1]$ and $x \in [0,1]$.

In order to investigate an approximation theorem for $L_{n,s}^{(p)}(f;x)$ the authors in [17] defined the positive linear operators $L_{n,s,p} : C[0,1] \to C[0,1]$ for $g \in C[0,1]$ as follows:

$$\begin{aligned} & L_{n,s,p}(g;x) \\ : & = \frac{p(n-s)_p}{n-s-p+1} \sum_{k=0}^{n-s-p+1} p_{n-s-p+1,k}(x) \int_{Q_{n,s}} g\left(\frac{k}{n} + \sum_{i=1}^{p} u_i\right) d\overline{u} \\ & + (n-s)_p \sum_{k=0}^{n-s-p} p_{n-s-p,k}(x) \\ & \times \int_{Q_n} \left[(1-x)\, g\left(\frac{k}{n} + \sum_{i=1}^{p} u_i\right) + xg\left(\frac{k+s}{n} + \sum_{i=1}^{p} u_i\right)\right] d\overline{u} \qquad (3.1) \end{aligned}$$

such that for $f \in C^p[0,1]$ $L_{n,s,p}(f^p;x) = L_{n,s}^{(p)}(f;x)$. Moreover,

$$L_{n,0,p} = L_{n,1,p} = \widetilde{L}_{n,p},$$

where the positive linear operators $\widetilde{L}_{n,p}$, obtained from the usual derivatives of Bernstein operators, given in [14] by

$$\widetilde{L}_{n,p}(g;x) := (n)_p \sum_{k=0}^{n-p} p_{n-p,k}(x) \int_{Q_n} g\left(\frac{k}{n} + \sum_{i=1}^{p} u_i\right) d\overline{u}.$$

We already have the first two moments and second central moment from the paper of Yun and Xiang [17].

Lemma 3.1. *([17]) Let $s \in \mathbb{N}_0$, $n, p \in \mathbb{N}$ such that $n > 2s$ and $p \leq n-s$. Then for $x \in [0,1]$, we have*

$$\begin{aligned} L_{n,s,p}(e_0;x) &= \frac{(n-s)_p}{n^p}\left[1 + \frac{ps}{n-s-p+1}\right], \\ L_{n,s,p}(e_1;x) &= \frac{(n-s)_p}{n^p}\left\{\left[1 + \frac{p(s-1)}{n}\right]x + \frac{p}{2n}\left[1 + \frac{s(s+p-1)}{n-s-p+1}\right]\right\} \end{aligned}$$

and

$$\begin{aligned} & L_{n,s,p}\left((e_1 - xe_0)^2 ; x\right) \\ = \quad & \frac{(n-s)_p}{n^p}\left\{\left[\frac{n+(p+s)(s-1)(p+1)}{n^2} - \frac{ps(s+p-1)}{n(n-s-p+1)}\right]x(1-x)\right. \\ & \left. + \frac{p}{n^2}\left[\frac{3p+1}{12} + \frac{s(p+1)(p-2)}{4(n-s-p+1)} + \frac{s\left(2s^2+3sp-3s+4-p\right)}{6(n-s-p+1)}\right]\right\}. \end{aligned}$$

It might be necessary to mention here that while we were verifying the calculations in Lemma 3.1, we encountered a sign misprint in the second central moment $L_{n,s,p}\left((e_1 - xe_0)^2 ; x\right)$. Here, the Lemma is reproduced by correcting this misprint. Now, in terms of Lemma 3.1 we can introduce the following Lemma.

Lemma 3.2. *Let* $s \in \mathbb{N}_0,\ n, p \in \mathbb{N}$ *such that* $n > 2s$ *and* $p \leq n - s$. *Then for* $x \in [0,1]$, *we have*

(i) $L_{n,s,p}(e_1 - xe_0; x) = \frac{(n-s)_p}{n^p}\frac{p}{2n}\left[1 + \frac{s(s+p-1)}{n-s-p+1}\right](1-2x)$.

(ii) $\|L_{n,s,p}(e_0)\| \leq \mu_{0,s,p}$,

where

$$\mu_{0,s,p} := \begin{cases} 1, & s = 0,1 \\ \frac{s+2+p(s-1)}{2s+1}, & s > 1 \end{cases}.$$

(iii) $\|L_{n,s,p}(e_0) - e_0\| \leq \frac{p(p-1)}{2n}\left(1 + \frac{2(|s-1|+s-1)}{p}\right)$.

(iv) $\|L_{n,s,p}(e_1 - \cdot e_0)\| \leq \frac{p}{2n}\mu_{1,s,p}$,

where

$$\mu_{1,s,p} = \begin{cases} 1, & s = 0,1 \\ \frac{s^2+p(s-1)+2}{2s+1}, & s > 1 \end{cases}.$$

(v) $\left\|L_{n,s,p}(e_1 - \cdot e_0)^2\right\| \leq \frac{1}{4n}\mu_{2,s,p}$,

where

$$\mu_{2,s,p} = \begin{cases} 1, & s = 0,1 \\ 1 + \frac{(p+s)(s-1)(p+1)+ps(p+s-1)}{2s+1} + \frac{3p^2+p}{3(2s+1)} \\ \quad + \frac{3ps\left(p^2-p\right)+2p^2s(3s-1)+2ps\left(2s^2-3s+1\right)}{3(2s+1)^2}, & s > 1 \end{cases}.$$

Proof. (i) It can be proved by direct computation.

(ii) From Lemma 3.1, one has

$$\begin{aligned}\left\|L_{n,s,p}\left(e_0\right)\right\| &= \frac{(n-s)_p}{n^p}\left[1+\frac{ps}{n-s-p+1}\right]\\ &= \frac{(n-s)_{p-1}}{n^{p-1}}\left(1+\frac{(s-1)(p-1)}{n}\right).\end{aligned} \tag{3.2}$$

Since $L_{n,s,p}=\widetilde{L}_{n,p}$ for $s=0,1$ and $\widetilde{L}_{n,p}\left(e_0\right)=\frac{(n)_p}{n^p}\leq 1$ [14], it follows that

$$\left\|L_{n,s,p}\left(e_0\right)\right\|\leq 1 \text{ for } s=0,1.$$

On the other hand, using the fact $n>2s$ which implies that $n\geq 2s+1$ and the inequality $\frac{(n-s)_{p-1}}{n^{p-1}}\leq 1$, from (3.2) we find

$$\left\|L_{n,s,p}\left(e_0\right)\right\|\leq 1+\frac{(s-1)(p-1)}{2s+1}=\frac{s+2+p(s-1)}{2s+1}$$

for $s>1$.

(iii) From Lemma 3.1, we have

$$\begin{aligned}&\left|L_{n,s,p}\left(e_0;x\right)-e_0\left(x\right)\right|\\ =\;&\left|\frac{(n-s)_p}{n^p}\left[1+\frac{ps}{n-s-p+1}\right]-1\right|\\ =\;&\left|\frac{(n-s)_{p-1}}{n^{p-1}}\left[1+\frac{(s-1)(p-1)}{n}\right]-1\right|\\ \leq\;&\left|\frac{(n-s)_{p-1}}{n^{p-1}}-1\right|+\frac{(n-s)_{p-1}}{n^{p-1}}\frac{|s-1|(p-1)}{n}.\end{aligned} \tag{3.3}$$

Since

$$\frac{(n-s)_{p-1}}{n^{p-1}}=\left(1-\frac{s}{n}\right)\left(1-\frac{s+1}{n}\right)\cdots\left(1-\frac{s+p-2}{n}\right)$$

for $p>1$ and $p\leq n-s$, using the well-known inequality

$$\prod_{J=1}^{k}\left(1-a_j\right)\geq 1-\sum_{j=1}^{k}a_j \quad \text{for } 0\leq a_j<1,\ j=1,2,\ldots,k$$

we can write

$$\begin{aligned}\frac{(n-s)_{p-1}}{n^{p-1}} &\geq 1-\left(\frac{s}{n}+\frac{s+1}{n}\cdots+\frac{s+p-2}{n}\right)\\ &= 1-\left[\frac{s(p-1)}{n}+\frac{(p-1)(p-2)}{2n}\right].\end{aligned}$$

Noting that $\frac{(n-s)_{p-1}}{n^{p-1}}=1$ for $p=1$, we obtain

$$1-\frac{(n-s)_{p-1}}{n^{p-1}}\leq\frac{s\,(p-1)}{n}+\frac{(p-1)\,(p-2)}{2n}$$

for $p\in\mathbb{N}$. Hence, considering $\frac{(n-s)_{p-1}}{n^{p-1}}\leq 1$ from (3.3) it follows that

$$\begin{aligned}\left\|L_{n,s,p}\left(e_0\right)-e_0\right\| &\leq \frac{s\,(p-1)}{n}+\frac{(p-1)\,(p-2)}{2n}+\frac{|s-1|\,(p-1)}{n}\\ &= \frac{p\,(p-1)}{2n}\left(1+\frac{2\left(|s-1|+s-1\right)}{p}\right).\end{aligned}$$

(iv) Since $L_{n,0,p}=L_{n,1,p}=\widetilde{L}_{n,p}$ and we have $\left\|\widetilde{L}_{n,p}\left(e_1-\cdot e_0\right)\right\|\leq\frac{p}{2n}$ from [14], we find $\left\|L_{n,s,p}\left(e_1-\cdot e_0\right)\right\|\leq\frac{p}{2n}$ for $s=0,1$.
On the other hand, by (i), one gets

$$\begin{aligned}\left\|L_{n,s,p}\left(e_1-\cdot e_0\right)\right\| &\leq \frac{p}{2n}\frac{(n-s)_p}{n^p}\left(1+\frac{s\,(s+p-1)}{n-s-p+1}\right)\\ &= \frac{p}{2n}\frac{(n-s)_{p-1}}{n^{p-1}}\left(1+\frac{(s-1)\,(s+p-1)}{n}\right)\\ &\leq \frac{p}{2n}\left(1+\frac{(s-1)\,(s+p-1)}{n}\right).\end{aligned}$$

In terms of the inequality $n\geq 2s+1$, we have

$$\begin{aligned}\left\|L_{n,s,p}\left(e_1-\cdot e_0\right)\right\| &\leq \frac{p}{2n}\left(1+\frac{(s-1)\,(s+p-1)}{n}\right)\\ &= \frac{p}{2n}\left(\frac{s^2+p\,(s-1)+2}{2s+1}\right)\end{aligned}$$

for $s>1$.

(v) Using the fact $L_{n,0,p}=L_{n,1,p}=\widetilde{L}_{n,p}$ and the inequality $\|\widetilde{L}_{n,p}\left(e_1-\cdot e_0\right)^2\|\leq\frac{1}{4n}$ given in [14], we find $\left\|L_{n,s,p}\left(e_1-\cdot e_0\right)^2\right\|\leq\frac{1}{4n}$ for $s=0,1$.

Now, we consider the case for $s>1$. From Lemma 3.1, one has

$$\begin{aligned}
&\left\|L_{n,s,p}\left((e_1-\cdot e_0)^2\right)\right\| \\
\leq\ & \frac{(n-s)_p}{n^p}\frac{1}{4n}\left\{\left|\frac{n+(p+s)(s-1)(p+1)}{n}-\frac{ps(s+p-1)}{n-s-p+1}\right|\right. \\
&+\frac{1}{3n(n-s-p+1)}\left|(3p^2+p)(n-s-p+1)+3ps(p+1)(p-2)\right. \\
&\left.\left.+2ps\left(2s^2+3sp-3s+4-p\right)\right|\right\} \\
\leq\ & \frac{(n-s)_p}{n^p}\frac{1}{4n}\left\{\frac{n+(p+s)(s-1)(p+1)}{n}+\frac{ps(s+p-1)}{n-s-p+1}\right. \qquad (3.4) \\
&+\frac{1}{3n(n-s-p+1)}\left|(3p^2+p)(n-s-p+1)+3ps(p+1)(p-2)\right. \\
&\left.\left.+2ps\left(2s^2+3sp-3s+4-p\right)\right|\right\}
\end{aligned}$$

Since $(3p^2+p)(n-s-p+1)>0$ and

$$\begin{aligned}
& 3ps(p+1)(p-2)+2ps\left(2s^2+3sp-3s+4-p\right) \\
=\ & 3ps\left(p^2-p\right)+2p^2s(3s-1)+2ps\left(2s^2-3s+1\right)>0
\end{aligned}$$

for $s>1$ and $p\in\mathbb{N}$, from (3.4) we have

$$\begin{aligned}
&\left\|L_{n,s,p}\left((e_1-\cdot e_0)^2\right)\right\| \\
\leq\ & \frac{(n-s)_p}{n^p}\frac{1}{4n}\left\{1+\frac{(p+s)(s-1)(p+1)}{n}+\frac{ps(p+s-1)}{n-s-p+1}\right. \\
&\left.+\frac{3p^2+p}{3n}+\frac{3ps\left(p^2-p\right)+2p^2s(3s-1)+2ps\left(2s^2-3s+1\right)}{3n(n-s-p+1)}\right\} \\
=\ & \frac{1}{4n}\left\{\frac{(n-s)_p}{n^p}\left[1+\frac{(p+s)(s-1)(p+1)}{n}+\frac{3p^2+p}{3n}\right]\right. \\
&+\frac{(n-s)_{p-1}}{n^{p-1}}\left[\frac{ps(p+s-1)}{n}\right. \\
&\left.\left.+\frac{3ps\left(p^2-p\right)+2p^2s(3s-1)+2ps\left(2s^2-3s+1\right)}{3n^2}\right]\right\} \\
\leq\ & \frac{1}{4n}\left\{1+\frac{(p+s)(s-1)(p+1)+ps(p+s-1)}{n}+\frac{3p^2+p}{3n}\right. \\
&\left.+\frac{3ps\left(p^2-p\right)+2p^2s(3s-1)+2ps\left(2s^2-3s+1\right)}{3n^2}\right\}.
\end{aligned}$$

Here we used the inequalities $\frac{(n-s)_p}{n^p} \leq 1$ and $\frac{(n-s)_{p-1}}{n^{p-1}} \leq 1$. Thus, taking into consideration of the inequality $n \geq 2s+1$, we obtain

$$\begin{aligned} & \left\| L_{n,s,p}\left((e_1 - \cdot e_0)^2 \right) \right\| \\ \leq \; & \frac{1}{4n}\left\{ 1 + \frac{(p+s)(s-1)(p+1) + ps(p+s-1)}{2s+1} + \frac{3p^2+p}{3(2s+1)} \right. \\ & \left. + \frac{3ps\left(p^2-p\right) + 2p^2 s(3s-1) + 2ps\left(2s^2-3s+1\right)}{3(2s+1)^2} \right\} \end{aligned}$$

which completes the proof. □

For our future correspondences, we need the following lemma.

Lemma 3.3. *Let* $s \in \mathbb{N}_0$, $n, p \in \mathbb{N}$ *such that* $n > 2s$ *and* $p \leq n-s$. *Then for* $x \in [0,1]$, *we have*

$$L_{n,s,p}\left(|e_1 - xe_0|; x\right) \leq \sqrt{2\left(\frac{sp}{n-s-p+1} + 1\right) L_{n,s,p}\left((e_1 - xe_0)^2; x\right)}.$$

Proof. By (3.1), one has

$$\begin{aligned} & \left(L_{n,s,p}\left(|e_1 - xe_0|; x\right)\right)^2 \\ = \; & \left\{ \sum_{k=0}^{n-s-p+1} p_{n-s-p+1,k}(x) \;\; p(n-s)_{p-1} \int_{Q_{n,s}} \left| \frac{k}{n} + \sum_{i=1}^{p} u_i - x \right| d\overline{u} \right. \\ & + \sum_{k=0}^{n-s-p} p_{n-s-p,k}(x) \;\; (n-s)_p \times \\ & \left. \times \int_{Q_n} \left[(1-x)\left| \frac{k}{n} + \sum_{i=1}^{p} u_i - x \right| + x \left| \frac{k+s}{n} + \sum_{i=1}^{p} u_i - x \right| \right] d\overline{u} \right\}^2 \\ \leq \; & 2\left\{ \left[\sum_{k=0}^{n-s-p+1} p_{n-s-p+1,k}(x) \;\; p(n-s)_{p-1} \int_{Q_{n,s}} \left| \frac{k}{n} + \sum_{i=1}^{p} u_i - x \right| d\overline{u} \right]^2 \right. \\ & + \left[\sum_{k=0}^{n-s-p} p_{n-s-p,k}(x) \;\; (n-s)_p \times \right. \\ & \left. \left. \int_{Q_n} \left[(1-x)\left| \frac{k}{n} + \sum_{i=1}^{p} u_i - x \right| + x \left| \frac{k+s}{n} + \sum_{i=1}^{p} u_i - x \right| \right] d\overline{u} \right]^2 \right\} \end{aligned}$$

$$=: 2(S_1 + S_2) \tag{3.5}$$

Applying the Cauchy-Schwarz inequality to each sum we get

$$S_1 \leq \sum_{k=0}^{n-s-p+1} p_{n-s-p+1,k}(x) \left[p(n-s)_{p-1}\right]^2 \left(\int_{Q_{n,s}} \left| \frac{k}{n} + \sum_{i=1}^{p} u_i - x \right| d\overline{u} \right)^2$$

and

$$S_2 \leq \sum_{k=0}^{n-s-p} p_{n-s-p,k}(x) \left[(n-s)_p\right]^2 \left(\int_{Q_n} C_{n,k,s,p}(u_i, x)\, d\overline{u} \right)^2,$$

where

$$C_{n,k,s,p}(u_i, x) := (1-x) \left| \frac{k}{n} + \sum_{i=1}^{p} u_i - x \right| + x \left| \frac{k+s}{n} + \sum_{i=1}^{p} u_i - x \right|.$$

Now, application of the Cauchy-Schwarz inequality to each iterated integral gives

$$\begin{aligned} S_1 \quad &\leq \quad \sum_{k=0}^{n-s-p+1} p_{n-s-p+1,k}(x) \left[p(n-s)_{p-1}\right]^2 \frac{s}{n^p} \int_{Q_{n,s}} \left(\frac{k}{n} + \sum_{i=1}^{p} u_i - x \right)^2 d\overline{u} \\ &= \quad \frac{sp(n-s)_{p-1}}{n^p} \times \qquad (3.6) \\ &\quad \times \left[\frac{p(n-s)_p}{n-s-p+1} \sum_{k=0}^{n-s-p+1} p_{n-s-p+1,k}(x) \int_{Q_{n,s}} \left(\frac{k}{n} + \sum_{i=1}^{p} u_i - x \right)^2 d\overline{u} \right] \end{aligned}$$

and

$$\begin{aligned} S_2 \quad &\leq \quad \sum_{k=0}^{n-s-p} p_{n-s-p,k}(x) \left[(n-s)_p\right]^2 \frac{1}{n^p} \int_{Q_n} \left[C_{n,k,s,p}(u_i, x)\right]^2 d\overline{u} \\ &= \quad \frac{(n-s)_p}{n^p} \left[(n-s)_p \sum_{k=0}^{n-s-p} p_{n-s-p,k}(x) \int_{Q_n} \left[C_{n,k,s,p}(u_i, x)\right]^2 d\overline{u} \right]. \end{aligned}$$

Moreover, since $f(t) = t^2,\ t \in [0, \infty)$ is convex, using Jensen's inequality for the term $C_{n,k,s,p}(u_i,x)$ in the last formula, we obtain

$$\begin{aligned} S_2 \leq & \frac{(n-s)_p}{n^p}\left\{(n-s)_p \sum_{k=0}^{n-s-p} p_{n-s-p,k}(x)\right. \\ & \left.\times \int\limits_{Q_n} \left[(1-x)\left(\frac{k}{n}+\sum_{i=1}^{p} u_i - x\right)^2 + x\left(\frac{k+s}{n}+\sum_{i=1}^{p} u_i - x\right)^2\right] d\overline{u}\right\}. \end{aligned} \tag{3.7}$$

Making use of (3.6) and (3.7) in (3.5) gives

$$\begin{aligned} & \left(L_{n,s,p}\left(\left|e_1 - xe_0\right|;x\right)\right)^2 \\ \leq & 2\left\{\frac{sp\,(n-s)_{p-1}}{n^p}\times\right. \\ & \times\left[\frac{p\,(n-s)_p}{n-s-p+1}\sum_{k=0}^{n-s-p+1} p_{n-s-p+1,k}(x)\int\limits_{Q_{n,s}}\left(\frac{k}{n}+\sum_{i=1}^{p}u_i - x\right)^2 d\overline{u}\right] \\ & +\frac{(n-s)_p}{n^p}\left\{(n-s)_p\sum_{k=0}^{n-s-p} p_{n-s-p,k}(x)\right. \\ & \left.\left.\times\int\limits_{Q_n}\left[(1-x)\left(\frac{k}{n}+\sum_{i=1}^{p}u_i - x\right)^2 + x\left(\frac{k+s}{n}+\sum_{i=1}^{p}u_i - x\right)^2\right]d\overline{u}\right\}\right\} \\ \leq & 2\left[\frac{sp\,(n-s)_{p-1}+(n-s)_p}{n^p}\right] L_{n,s,p}\left(\left(e_1 - xe_0\right)^2;x\right) \\ = & 2\left(\frac{sp}{n-s-p+1}\frac{(n-s)_p}{n^p}+\frac{(n-s)_p}{n^p}\right) L_{n,s,p}\left(\left(e_1 - xe_0\right)^2;x\right) \\ = & 2\frac{(n-s)_p}{n^p}\left(\frac{sp}{n-s-p+1}+1\right) L_{n,s,p}\left(\left(e_1 - xe_0\right)^2;x\right) \\ \leq & 2\left(\frac{sp}{n-s-p+1}+1\right) L_{n,s,p}\left(\left(e_1 - xe_0\right)^2;x\right). \end{aligned}$$

Hence, we arrive at

$$L_{n,s,p}\left(\left|e_1 - xe_0\right|;x\right) \leq \sqrt{2\left(\frac{sp}{n-s-p+1}+1\right) L_{n,s,p}\left(\left(e_1 - xe_0\right)^2;x\right)}$$

which completes the proof. □

4 Approximation of Caputo fractional derivatives

In this part, we study approximation of the Caputo fractional derivatives of Brass-Stancu operators. Using the binomial expansions of $(1-x)^{n-s-k}$ and $(1-x)^{n-s-k+1}$ Brass-Stancu operators can be expressed as

$$\begin{aligned} L_{n,s}(f;x) &= \sum_{k=0}^{n-s}\sum_{j=0}^{n-s-k+1}\binom{n-s}{k}\binom{n-s-k+1}{j} f\left(\frac{k}{n}\right)(-1)^j x^{k+j} \\ &+ \sum_{k=0}^{n-s}\sum_{j=0}^{n-s-k}\binom{n-s}{k}\binom{n-s-k}{j} f\left(\frac{k+s}{n}\right)(-1)^j x^{k+j+1}. \end{aligned}$$

According to Definition 2.2, for $\alpha = 0$ we have $D_{*0}^{0} = I$, the identity operator. So we consider the Caputo fractional derivatives of Brass-Stancu operators only for $\alpha > 0$. Applying the operator D_{*0}^{α} to $L_{n,s}(f;x)$ and by taking account of Remark 2.2 the Caputo fractional derivative of $L_{n,s}(f;x)$ can be obtained explicitly as

$$\begin{aligned} & D_{*0}^{\alpha} L_{n,s}(f;x) \\ = & \sum_{k=0}^{n-s}\sum_{j=\max\{0,\lceil\alpha\rceil-k\}}^{n-s-k+1}\binom{n-s}{k}\left[\binom{n-s-k+1}{j} f\left(\frac{k}{n}\right)\right. \\ & \left. -\binom{n-s-k}{j-1} f\left(\frac{k+s}{n}\right)\right](-1)^j \frac{\Gamma(k+j+1)}{\Gamma(k+j+1-\alpha)} x^{k+j-\alpha}, \end{aligned}$$

where $\alpha > 0$, $s \in \mathbb{N}_0$ $n \in \mathbb{N}$, such that $n > 2s$.

Now we prove the following uniform approximation theorem of the Caputo fractional derivatives by Brass-Stancu operators by using similar technique in [12].

Theorem 4.1. *If $\alpha > 0$, s is a nonnegative fixed integer and $p = \lceil\alpha\rceil$, then for $f \in C^p[0,1]$ we have*

$$\lim_{n\to\infty} D_{*0}^{\alpha} L_{n,s}(f) = D_{*0}^{\alpha} f$$

uniformly on $[0,1]$.

Proof. Let $\varepsilon > 0$ be a given number. By Theorem 2.1, for $f \in C^p[0,1]$ there exists an $n_0 \in \mathbb{N}$ such that for all $n > n_0$ it holds $n > 2s$, $n \geq p+s$ and

$$\left\| L_{n,s}^{(p)}(f) - f^{(p)} \right\| < \varepsilon.$$

By Definition 2.2, we can write

$$\begin{aligned} & \left|D_{*0}^{\alpha}L_{n,s}\left(f;x\right)-D_{*0}^{\alpha}f\left(x\right)\right| \\ = & \left|J_0^{p-\alpha}\left(L_{n,s}^{(p)}\left(f\right)-f^{(p)};x\right)\right| \\ \leq & J_0^{p-\alpha}\left(\left|L_{n,s}^{(p)}\left(f\right)-f^{(p)}\right|;x\right) \\ \leq & J_0^{p-\alpha}\left(\left\|L_{n,s}^{(p)}\left(f\right)-f^{(p)}\right\|e_0;x\right) \\ = & \left\|L_{n,s}^{(p)}\left(f\right)-f^{(p)}\right\|J_0^{p-\alpha}\left(e_0;x\right). \end{aligned} \tag{4.1}$$

From Remark 2.1, we get

$$\left\|J_0^{p-\alpha}\left(e_0;x\right)\right\|=\left\|\frac{x^{p-\alpha}}{\Gamma\left(p-\alpha+1\right)}\right\|\leq\frac{1}{\Gamma\left(p-\alpha+1\right)}. \tag{4.2}$$

Thus, (4.1) yields the desired result. □

Note that for the case $\alpha\in\mathbb{N}$ the above theorem reduces to Theorem 2.1.

Now, we present some quantitative estimates for the degree of the uniform approximation of the Caputo fractional derivatives by the Brass-Stancu operators $L_{n,s}$.

Recall the first order modulus of continuity of $g\in C[0,1]$ given by

$$\omega_1\left(g,\delta\right)=\sup_{x,t\in[0,1],\ |t-x|\leq\delta}\left|g\left(t\right)-g\left(x\right)\right|,\ \delta>0.$$

In [14], by using the following result of Mond [13], Păltănea obtained the subsequent result concerning quantitative estimate for the degree of approximation of Caputo fractional derivatives by Bernstein operators in terms of the first order modulus of continuity.

Theorem 4.2. *([13])Let I be an arbitrary interval and $V\subset C\left(I\right)$ be a linear subspace such that $e_j\in V,\ j=0,1,2$. If $L:V\rightarrow\mathcal{F}\left(I\right):=\left\{f\mid f:I\rightarrow\mathbb{R}\right\}$ is a positive linear operator, then*

$$\begin{aligned} \left|L\left(f;x\right)-f\left(x\right)\right| \leq & \left|f\left(x\right)\right|\left|L\left(e_0;x\right)-1\right| \\ & +\left(L\left(e_0;x\right)+\frac{1}{\delta^2}L\left(\left(e_1-xe_0\right)^2;x\right)\right)\omega_1\left(f,\delta\right) \end{aligned}$$

for any $f\in V$, any $x\in I$ and $\delta>0$.

Theorem 4.3. *([14]) For $\alpha\geq0$, $p=\lceil\alpha\rceil$, $f\in C^p[0,1]$, $\delta>0$ and $n\in\mathbb{N}$, we have*

$$\left\|D_{*0}^{\alpha}B_n\left(f\right)-D_{*0}^{\alpha}f\right\|\leq\frac{p\left(p-1\right)}{n}\left\|f^{(p)}\right\|+\left(1+\frac{1}{4n\delta^2}\right)\omega_1\left(f^{(p)},\delta\right).$$

For $\delta = \frac{1}{\sqrt{n}}$, *we have*

$$\left\| D_{*0}^{\alpha} B_n(f) - D_{*0}^{\alpha} f \right\| \leq \frac{p(p-1)}{n} \left\| f^{(p)} \right\| + \frac{5}{4} \omega_1 \left(f^{(p)}, \frac{1}{\sqrt{n}} \right).$$

For $\delta = \frac{1}{\sqrt{n}}$ *and* $0 \leq \alpha < 1$, *we have*

$$\left\| D_{*0}^{\alpha} B_n(f) - D_{*0}^{\alpha} f \right\| \leq \frac{5}{4} \omega_1 \left(f', \frac{1}{\sqrt{n}} \right).$$

Below, we extend this result to Brass-Stancu operators in a similar way that of [14].

Theorem 4.4. *If* $\alpha > 0$, s *is a nonnegative fixed integer and* $n, p \in \mathbb{N}$ *such that* $n > 2s$ *and* $p = \lceil \alpha \rceil \leq n - s$, *then for* $f \in C^p[0,1]$ *we have*

$$\begin{aligned} & \left\| D_{*0}^{\alpha} L_{n,s}(f) - D_{*0}^{\alpha} f \right\| \\ \leq \; & \left\{ \frac{p(p-1)}{2n} \left(1 + \frac{2(|s-1| + s - 1)}{p} \right) \left\| f^{(p)} \right\| \right. \\ & \left. + \left(\mu_{0,s,p} + \frac{\mu_{2,s,p}}{4n\delta^2} \right) \omega_1 \left(f^{(p)}, \delta \right) \right\} \frac{1}{\Gamma(p - \alpha + 1)}, \end{aligned} \tag{4.3}$$

where $\mu_{0,s,p}$ *and* $\mu_{2,s,p}$ *are defined as in Lemma 3.2.*

For $\delta = \frac{1}{\sqrt{n}}$ *we have*

$$\begin{aligned} & \left\| D_{*0}^{\alpha} L_{n,s}(f) - D_{*0}^{\alpha} f \right\| \\ \leq \; & \left\{ \frac{p(p-1)}{2n} \left(1 + \frac{2(|s-1| + s - 1)}{p} \right) \left\| f^{(p)} \right\| \right. \\ & \left. + \left(\mu_{0,s,p} + \frac{\mu_{2,s,p}}{4} \right) \omega_1 \left(f^{(p)}, \frac{1}{\sqrt{n}} \right) \right\} \frac{1}{\Gamma(p - \alpha + 1)}. \end{aligned} \tag{4.4}$$

Proof. Since $L_{n,s}^{(p)}(f) = L_{n,s,p}\left(f^{(p)}\right)$, from Theorem 4.2 and Lemma 3.2, one gets

$$\begin{aligned} & \left\| L_{n,s}^{(p)}(f) - f^{(p)} \right\| \\ = \; & \left\| L_{n,s,p}\left(f^{(p)}\right) - f^{(p)} \right\| \\ \leq \; & \left\| f^{(p)} \right\| \left\| L_{n,s,p}(e_0) - e_0 \right\| \\ & + \left(\left\| L_{n,s,p}(e_0) \right\| + \frac{1}{\delta^2} \left\| L_{n,s,p}\left((e_1 - \cdot e_0)^2\right) \right\| \right) \omega_1\left(f^{(p)}, \delta\right) \\ \leq \; & \frac{p(p-1)}{2n} \left(1 + \frac{2(|s-1| + s - 1)}{p} \right) \left\| f^{(p)} \right\| \\ & + \left(\mu_{0,s,p} + \frac{1}{4n\delta^2} \mu_{2,s,p} \right) \omega_1\left(f^{(p)}, \delta\right). \end{aligned} \tag{4.5}$$

Using (4.5) in (4.1) and passing to sup-norm over $[0,1]$, we reach to (4.3) with (4.2). Finally, choosing $\delta = \frac{1}{\sqrt{n}}$ in (4.3) we find (4.4). This completes the proof. □

With the help of Lemmas 3.2 and 3.3, we get the following result.

Theorem 4.5. *If s is a nonnegative fixed integer and $n, p \in \mathbb{N}$ such that $n > 2s$ and $p \leq n - s$, then for $f \in C^p[0,1]$ we have*

$$\begin{aligned} & \left\| L_{n,s}^{(p)}(f) - f^{(p)} \right\| \\ \leq \; & \left(\mu_{0,s,p} + \sqrt{2\left(\frac{sp}{n-s-p+1} + 1 \right)} \right) \left(1 + \frac{\sqrt{\mu_{2,s,p}}}{2} \right) \omega_1\left(f^{(p)}, \frac{1}{\sqrt{n}}\right) \\ + \; & \frac{p(p-1)}{2n} \left(1 + \frac{2|s-1| + s - 1}{p} \right) \left\| f^{(p)} \right\|, \end{aligned} \tag{4.6}$$

where $\mu_{0,s,p}$ and $\mu_{2,s,p}$ are defined as in Lemma 3.2.

Proof. Using the facts $L_{n,s}^{(p)}(f) = L_{n,s,p}\left(f^{(p)}\right)$ and $L_{n,s,p}\left(f^{(p)}(x)\, e_0; x\right) = f^{(p)}(x)\; L_{n,s,p}(e_0; x)$ the following well-known property of modulus of continuity of a function $g \in C[a,b]$,

$$|g(t) - g(x)| \leq \omega_1(g, |t-x|) \leq \left(1 + \frac{|t-x|}{\delta} \right) \omega_1(g, \delta), \;\; x, t \in [a,b],$$

where $\delta > 0$ is arbitrarily chosen, we can write

$$\begin{aligned}
&\left|L_{n,s}^{(p)}(f;x) - f^{(p)}(x)\right| \\
\leq\ & \left|L_{n,s,p}\left(f^{(p)};x\right) - L_{n,s,p}(e_0;x)\,f^{(p)}(x)\right| + \left|L_{n,s,p}(e_0;x)\,f^{(p)}(x) - f^{(p)}(x)\right| \\
\leq\ & L_{n,s,p}\left(\left|f^{(p)} - f^{(p)}(x)\,e_0\right|;x\right) + \left|L_{n,s,p}(e_0;x) - e_0(x)\right|\left|f^{(p)}(x)\right| \\
\leq\ & L_{n,s,p}\left(1 + \frac{|e_1 - xe_0|}{\delta};x\right)\omega_1\left(f^{(p)},\delta\right) + \left|L_{n,s,p}(e_0;x) - e_0(x)\right|\left|f^{(p)}(x)\right| \\
=\ & \left[L_{n,s,p}(e_0;x) + \frac{1}{\delta}L_{n,s,p}(|e_1 - xe_0|;x)\right]\omega_1\left(f^{(p)},\delta\right) \\
& + \left|L_{n,s,p}(e_0;x) - e_0(x)\right|\left|f^{(p)}(x)\right|. \qquad (4.7)
\end{aligned}$$

Using Lemma 3.3 and choosing $\delta = \sqrt{L_{n,s,p}\left((e_1 - xe_0)^2;x\right)}$, from (4.7) one has

$$\begin{aligned}
\left|L_{n,s}^{(p)}(f;x) - f^{(p)}(x)\right| &\leq \left|L_{n,s,p}(e_0;x) - e_0(x)\right|\left|f^{(p)}(x)\right| \\
&+ \left[L_{n,s,p}(e_0;x) + \sqrt{2\left(\frac{sp}{n-s-p+1} + 1\right)}\right] \times \\
&\times\omega_1\left(f^{(p)}, \sqrt{L_{n,s,p}\left((e_1 - xe_0)^2;x\right)}\right).
\end{aligned}$$

Passing to sup-norm over $[0,1]$ in the above inequality, from Lemma 3.2 it follows that

$$\begin{aligned}
&\left\|L_{n,s}^{(p)}(f) - f^{(p)}\right\| \\
\leq\ & \left(\mu_{0,s,p} + \sqrt{2\left(\frac{sp}{n-s-p+1} + 1\right)}\right)\omega_1\left(f^{(p)}, \frac{\sqrt{\mu_{2,s,p}}}{2}\frac{1}{\sqrt{n}}\right) \\
+\ & \frac{p(p-1)}{2n}\left(1 + \frac{2|s-1| + s - 1}{p}\right)\left\|f^{(p)}\right\| \\
\leq\ & \left(\mu_{0,s,p} + \sqrt{2\left(\frac{sp}{n-s-p+1} + 1\right)}\right)\left(1 + \frac{\sqrt{\mu_{2,s,p}}}{2}\right)\omega_1\left(f^{(p)}, \frac{1}{\sqrt{n}}\right) \\
+\ & \frac{p(p-1)}{2n}\left(1 + \frac{2|s-1| + s - 1}{p}\right)\left\|f^{(p)}\right\|.
\end{aligned}$$

Thus the proof is completed. □

Below, we present the Caputo fractional derivative extension of Theorem 4.5, which improves the direct application result obtained in Theorem 4.4.

Theorem 4.6. *If $\alpha > 0$, s is a nonnegative fixed integer and $n, p \in \mathbb{N}$ such that $n > 2s$ and $p = \lceil \alpha \rceil \leq n - s$, then for $f \in C^p[0,1]$ we have*

$$\begin{aligned} & \left\| D_{*0}^{\alpha} L_{n,s}(f) - D_{*0}^{\alpha} f \right\| \\ \leq & \left\{ \left(\mu_{0,s,p} + \sqrt{2\left(\frac{sp}{n-s-p+1} + 1\right)} \right) \left(1 + \frac{\sqrt{\mu_{2,s,p}}}{2}\right) \omega_1 \left(f^{(p)}, \frac{1}{\sqrt{n}}\right) \right. \\ & \left. + \frac{p(p-1)}{2n} \left(1 + \frac{2|s-1| + s - 1}{p}\right) \left\| f^{(p)} \right\| \right\} \frac{1}{\Gamma(p - \alpha + 1)} \end{aligned}$$

where $\mu_{0,s,p}$ and $\mu_{2,s,p}$ are defined as in Lemma 3.2.

Proof. Taking into consideration the fact $L_{n,s}^{(p)}(f) = L_{n,s,p}\left(f^{(p)}\right)$ if we use the estimate (4.6) in the inequality (4.1) and take to sup-norm over $[0,1]$, we arrive at the desired result with (4.2). □

Again, with the help of Lemmas 3.2 and 3.3, we get the following result..

Theorem 4.7. *If s is a nonnegative fixed integer and $n, p \in \mathbb{N}$ such that $n > 2s$ and $p \leq n - s$, then for $f \in C^{p+1}[0,1]$ we have*

$$\begin{aligned} & \left\| L_{n,s}^{(p)}(f) - f^{(p)} \right\| \\ \leq & \frac{p(p-1)}{2n} \left(1 + \frac{2(|s-1| + s - 1)}{p}\right) \left\| f^{(p)} \right\| + \frac{p}{2n} \mu_{1,s,p} \left\| f^{(p+1)} \right\| \\ & + \frac{3}{4\sqrt{n}} \mu_{2,s,p} \left[\frac{1}{2} + \sqrt{2\left(\frac{sp}{n-s-p+1} + 1\right)} \right] \omega_1 \left(f^{(p+1)}, \frac{1}{\sqrt{n}}\right) \end{aligned} \tag{4.8}$$

where $\mu_{1,s,p}$ and $\mu_{2,s,p}$ are defined as in Lemma 3.2.

Proof. Let $f \in C^{p+1}[0,1]$. Then for any $x, t \in [0,1]$ we have

$$f^{(p)}(t) = f^{(p)}(x) + (t - x) f^{(p+1)}(x) + \int_x^t \left[f^{(p+1)}(u) - f^{(p+1)}(x) \right] du.$$

Applying the operator $L_{n,s,p}$ on both sides of the above equality, we get

$$\begin{aligned} & L_{n,s,p}\left(f^{(p)};x\right)-f^{(p)}\left(x\right) \\ = & \left[L_{n,s,p}\left(e_0;x\right)-e_0\left(x\right)\right]f^{(p)}\left(x\right)+L_{n,s,p}\left(e_1-xe_0;x\right)f^{(p+1)}\left(x\right) \\ & +L_{n,s,p}\left(\int_x^t\left[f^{(p+1)}\left(u\right)-f^{(p+1)}\left(x\right)\right]du;x\right). \end{aligned} \tag{4.9}$$

Therefore, using the fact $L_{n,s}^{(p)}\left(f\right)=L_{n,s,p}\left(f^{(p)}\right)$ and the following inequality

$$\left|\int_x^t\left|f^{(p+1)}\left(u\right)-f^{(p+1)}\left(x\right)\right|du\right|\leq\left(\left|t-x\right|+\frac{\left(t-x\right)^2}{2\delta}\right)\omega_1\left(f^{(p+1)},\delta\right),\ \ \delta>0,$$

in (4.9) and taking account of Lemma 3.3 in the obtained result, we have

$$\begin{aligned} & \left|L_{n,s}^{(p)}\left(f;x\right)-f^{(p)}\left(x\right)\right| \\ \leq & \left|L_{n,s,p}\left(e_0;x\right)-e_0\left(x\right)\right|\left|f^{(p)}\left(x\right)\right|+\left|L_{n,s,p}\left(e_1-xe_0;x\right)\right|\left|f^{(p+1)}\left(x\right)\right| \\ & +\left\{\left[\sqrt{2\left(\frac{sp}{n-s-p+1}+1\right)}\right]\sqrt{L_{n,s,p}\left(\left(e_1-xe_0\right)^2;x\right)}\right. \\ & \left.+\frac{1}{2\delta}L_{n,s,p}\left(\left(e_1-xe_0\right)^2;x\right)\right\}\omega_1\left(f^{(p+1)},\delta\right). \end{aligned}$$

Choosing $\delta=\sqrt{L_{n,s,p}\left(\left(e_1-xe_0\right)^2;x\right)}$ in the above inequality, one gets

$$\begin{aligned} & \left|L_{n,s}^{(p)}\left(f;x\right)-f^{(p)}\left(x\right)\right| \\ \leq & \left|L_{n,s,p}\left(e_0;x\right)-e_0\left(x\right)\right|\left|f^{(p)}\left(x\right)\right|+\left|L_{n,s,p}\left(e_1-xe_0;x\right)\right|\left|f^{(p+1)}\left(x\right)\right| \\ & +\left[\frac{1}{2}+\sqrt{2\left(\frac{sp}{n-s-p+1}+1\right)}\right]\sqrt{L_{n,s,p}\left(\left(e_1-xe_0\right)^2;x\right)} \\ & \times\omega_1\left(f^{(p+1)},\sqrt{L_{n,s,p}\left(\left(e_1-xe_0\right)^2;x\right)}\right). \end{aligned}$$

Here, passing to the sup-norm over $[0,1]$ and using Lemma 3.2, we obtain

$$\begin{aligned}
&\left\| L_{n,s}^{(p)}(f) - f^{(p)} \right\| \\
\leq\; & \frac{p(p-1)}{2n}\left(1+\frac{2(|s-1|+s-1)}{p}\right)\left\| f^{(p)}\right\| + \frac{p}{2n}\mu_{1,s,p}\left\| f^{(p+1)}\right\| \\
&+\left[\frac{1}{2}+\sqrt{2\left(\frac{sp}{n-s-p+1}+1\right)}\right]\sqrt{\frac{\mu_{2,s,p}}{4n}}\omega_1\left(f^{(p+1)},\sqrt{\frac{\mu_{2,s,p}}{4n}}\right) \\
\leq\; & \frac{p(p-1)}{2n}\left(1+\frac{2(|s-1|+s-1)}{p}\right)\left\| f^{(p)}\right\| + \frac{p}{2n}\mu_{1,s,p}\left\| f^{(p+1)}\right\| \\
&+\frac{\sqrt{\mu_{2,s,p}}}{2\sqrt{n}}\left[\frac{1}{2}+\sqrt{2\left(\frac{sp}{n-s-p+1}+1\right)}\right]\left(1+\frac{\sqrt{\mu_{2,s,p}}}{2}\right)\times \\
&\quad\times\omega_1\left(f^{(p+1)},\frac{1}{\sqrt{n}}\right).
\end{aligned}$$

Now, making use of the fact $1 \leq \sqrt{\mu_{2,s,p}}$ in the last line of the above inequality, we reach to (4.8) which completes the proof. □

Caputo fractional extension of the above result is presented below.

Theorem 4.8. *If $\alpha > 0$, s is a nonnegative fixed integer and $n, p \in \mathbb{N}$ such that $n > 2s$ and $p = \lceil \alpha \rceil \leq n - s$, then for $f \in C^{p+1}[0,1]$ we have*

$$\begin{aligned}
&\left\| D_{*0}^{\alpha} L_{n,s}(f) - D_{*0}^{\alpha} f \right\| \\
\leq\; & \left\{ \frac{p(p-1)}{2n}\left(1+\frac{2(|s-1|+s-1)}{p}\right)\left\| f^{(p)}\right\| + \frac{p}{2n}\mu_{1,s,p}\left\| f^{(p+1)}\right\| \right. \\
&\left. +\frac{3}{4\sqrt{n}}\mu_{2,s,p}\left[\frac{1}{2}+\sqrt{2\left(\frac{sp}{n-s-p+1}+1\right)}\right]\omega_1\left(f^{(p+1)},\frac{1}{\sqrt{n}}\right)\right\}\times \\
&\times\frac{1}{\Gamma(p-\alpha+1)},
\end{aligned}$$

where $\mu_{1,s,p}$ and $\mu_{2,s,p}$ are defined as in Lemma 3.2.

Proof. Using the estimate (4.8) in the inequality (4.1) and passing to sup-norm over $[0,1]$, we find the required result with (4.2). □

References

[1] O. Agratini, Application of divided differences to the study of monotonicity of a sequence of D.D. Stancu polynomials, Rev. Anal. Numér. Théor. Approx., 25 (1996), nos, 1–2, 3–10.

[2] G. Anastassiou, Fractional Korovkin theory, Chaos, Solitons & Fractals, 42(4), (2009), 2080–2094.

[3] G. Anastassiou, Fractional Voronovskaya type asymptotic expansions for quasi-interpolation neural network operators, Cubo, 14(3) (2012), 71–83.

[4] H. Brass, Eine Verallgemeinerung der Bernsteinschen Operatoren, Abh. Math. Sem.Univ. Hamburg, 36 (1971), 111–122.

[5] T. Bostancı, G. Başcanbaz-Tunca, On Stancu operators depending on a non-negative integer, Filomat, 36(18) (2022), 6129–6138.

[6] M. Cantarini, D. Costarelli, G. Vinti, Approximation results in Sobolev and fractional Sobolev spaces by sampling Kantorovich operators, Frac. Calc. Appl. Anal., 26 (2023), 2493–2521.

[7] M. Caputo, Linear model of dissipation whose Q is almost frequency independent-II, Geophys. J. Roy. Astr. S., 13(5) (1967), 529–539.

[8] A. Chirilă, M. Marin, The theory of generalized thermoelasticity with fractional order strain for dipolar materials with double porosity, J. Mater. Sci., 53, (2018), 3470–3482.

[9] K. Diethelm, The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type, Lecture Notes in Mathematics, Springer-Verlag, Berlin Heildeberg, 2010.

[10] H. H. Gonska, On the composition and decomposition of positive linear operators, in: Kovtunets, V. V. (ed.) et al., Approximation Theory and its Applications. Proc. int. conf. ded. to the memory of Vl. K. Dzyadyk, Kiev, Ukraine, May 27–31, 1999. Pr. Inst. Mat. Nats. Akad. Nauk Ukr., Mat. Zastos. 31 (2000), 161–180.

[11] C. Ionescu, A. Lopes, D. Copot, J. A. T. Machado, J. H. T. Bates, The role of fractional calculus in modeling biological phenomena: a review, Commun. Nonlinear Sci. Numer. Simul. 51 (2017), 141–159.

[12] H. Khosravian-Arab, D. F. M. Torres, Uniform approximation of fractional derivatives and integrals with application to fractional differential equations. Nonlinear Stud., 20(4) (2013), 533–548.

[13] B. Mond, Note: On the degree of approximation by linear positive operators, J. Approx. Theory, 18 (1976), 304–306.

[14] R. Păltănea, Approximation of fractional derivatives by Bernstein operators, Gen. Math., 22(1) (2014), 91–98.

[15] D. D. Stancu, Approximation of functions by means of a new generalized Bernstein operator, Calcolo, 20(2) (1983), 211–229.

[16] V. E. Tarasov (Ed.) Mathematical Economics: Application of fractional calculus, MDPI, Basel, 2020.

[17] Lianying Yun, Xueyan Xiang, On shape-preserving properties and simultaneous approximation of Stancu operator, Anal. Theory Appl., 24(2) (2008), 195–204.

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