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On topological quotient hyperrings and α^*-relation

A. Zare and B. Davvaz

Abstract

In this research, we first introduce the concept of a topological Krasner hyperring and then proceed to investigate its properties. By applying relative topology to subhyperrings, we analyze the properties associated with them. In other words, the aim is to utilize specific topologies to identify the diverse substructural characteristics of this type of hyperring. Additionally, we examine the quotient topology resulting from an interesting relation on the discussed spaces to understand how this relation influences the topological structure of the hyperring. Finally, we demonstrate that the topological Krasner hyperring induced by τ_α, which is the finest and strongest topology on $\mathcal{H}$, ultimately forms a ring. In summary, this research not only analyzes the structural properties of these hyperrings but also examines, from a topological perspective, how different relations impact this structure, proving that the resulting topology is strong enough to form a ring.

1 Introduction and basic definitions

Hyperrings are a generalization of algebraic structures like rings, where hyperoperations are used instead of conventional operations. In a typical ring, addition and multiplication operate as functions that take two elements and produce a new element. However, in hyperrings, at least one of these operations (usually addition) is defined as a hyperoperation. This means that when

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you add two elements together, the result is not a single element but a set of elements.

In Krasner hyperrings, addition is defined as a hyperoperation. This means that when you add two numbers, instead of obtaining a specific number, a set of numbers appears as the result. Therefore, the addition structure in this system takes the form of a hypergroup, which is a generalization of groups.

In Krasner hyperrings, multiplication is defined as a regular operation, forming a semigroup structure. This means that multiplication always yields a specific result, whereas addition can yield multiple possible results. These types of algebraic structures are used in various fields of modern mathematics, particularly in cryptography, algebraic geometry, and dynamical systems.

In algebraic hyperstructure theory, a hyperoperation is a generalized operation defined as $+ : \mathrm{T} \times \mathrm{T} \longrightarrow \mathcal{P}^*(\mathrm{T})$, where $\mathcal{P}^*(\mathrm{T})$ represents the collection of all non-empty subsets of T. Unlike standard operations, a hyperoperation assigns a non-empty subset of T to each pair of elements from T. For any subsets $\mathbb{U}, \mathbb{D} \in \mathcal{P}^*(\mathrm{T})$ and an element $\mathbf{r} \in \mathrm{T}$, we define, $\mathbb{U} + \mathbb{D} = \bigcup_{a\in\mathbb{U}, b\in\mathbb{D}}(a + b)$ and $\mathbf{r}+\mathbb{U}$ is symbol of $\{\mathbf{r}\}+\mathbb{U}$ and $\mathbb{U}+\mathbf{r}$ is $\mathbb{U}+\{\mathbf{r}\}$. A structure $(\mathrm{T}, +)$ is called a semihypergroup if the hyperoperation is associative, i.e., for all $\mathbf{r}, \mathbf{s}, \mathbf{t} \in \mathrm{T}$, the following holds, $(\mathbf{r} + \mathbf{s}) + \mathbf{t} = \mathbf{r} + (\mathbf{s} + \mathbf{t})$. A semihypergroup is called a hypergroup if for any element $\mathbf{r} \in \mathrm{T}$, $\mathbf{r} + \mathcal{H} = \mathrm{T} + \mathbf{r} = \mathrm{T}$. This means adding any element to the whole set does not change the set. A non-empty subset $\mathcal{J} \subseteq \mathrm{T}$ is a subhypergroup if for every $k \in \mathcal{J}$, it holds that $k + \mathcal{J} = \mathcal{J} + k = \mathcal{J}$. This condition ensures closure of the subset under the hyperoperation.

A nonempty subset C of a hyperring $\mathcal{H}$ is said to be a complete part of $\mathcal{H}$ if for any nonzero natural number n and for all $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$ of $\mathcal{H}$, the following implication holds:

$$C \cap \sum_{j=1}^{n} \mathbf{r}_j \neq \emptyset \Rightarrow \sum_{j=1}^{n} \mathbf{r}_j \subseteq C.$$

2 Topological Krasner hyperrings

Krasner hyperrings generalize rings in mathematics, allowing flexible operations with outputs that can be sets of values. They share features with rings, such as an additive identity and the ability to perform addition and multiplication. Hyperrings have applications in number theory and algebra, leading to new insights and advancements in mathematics.

Definition 2.1. [6] A Krasner hyperring is an algebraic satisfying the following axioms:

(1) $(\mathcal{H},+)$ is canonical hypergroup, i.e, " $+$ " is a hyperoperation on $\mathcal{H}$ so that

(a) for every $\mathbf{r}$, $\mathbf{s}$ and $\mathbf{z}$ in $\mathcal{H}$, $\mathbf{r}+(\mathbf{s}+\mathbf{z})=(\mathbf{r}+\mathbf{s})+\mathbf{z}$,

(b) for all $\mathbf{r},\mathbf{s}\in\mathcal{H}$, $\mathbf{r}+\mathbf{s}=\mathbf{s}+\mathbf{r}$,

(c) there is $0\in\mathcal{H}$ so that $0+\mathbf{r}=\mathbf{r}$ for every $\mathbf{r}\in\mathcal{H}$,

(d) for all $\mathbf{r}\in\mathcal{H}$ is only a $\mathbf{s}\in\mathcal{H}$ so that $0\in\mathbf{s}+\mathbf{r}$,

(e) when $\mathbf{t}\in\mathbf{r}+\mathbf{q}$, then $\mathbf{q}\in-\mathbf{r}+\mathbf{t}$ and $\mathbf{r}\in\mathbf{t}-\mathbf{q}$, for any $t,r,q\in\mathcal{H}$

(2)$(\mathcal{H},\cdot)$ is a semigroup with zero as a two-way absorbing element ,i.e., $0\cdot\mathbf{r}=\mathbf{r}\cdot 0=0$ for all $\mathbf{r}\in\mathcal{H}$,

(3) the multiplication operation denoted by " $\cdot$ " distributes over the hyperoperation represented by " $+$ ".

Consider $(\mathcal{H},\tau)$ as a space that τ is a topology. Then we Consider a topology on power set $\mathcal{H}$ which is generated by $\mathcal{B}=\{S_{\mathcal{T}}\mid\mathcal{T}\in\tau\}$, where $S_{\mathcal{T}}=\{S\in\mathcal{P}^*(\mathcal{H})\mid S\subseteq\mathcal{T},S\in\tau\}$ [5].

In [3, 4], Heidari et al. defined the notion of topological polygroups and topological hypergroups. By considering the relative topology on subpolygroups they proved some properties of them. Nodehi et al. [9] and Singha and Das [10] studied topological hyperrings and presented some of their properties. Now, we investigate more results on topological Krasner hyperrings.

Definition 2.2. Let H be Krasner hyperring endowed with some topology τ. Then H is said to be topological Krasner hyperring, indicated by $(H,+,\cdot,\tau)$, if with respect to the product topology on $H\times H$ and the topology τ^* on $\mathcal{P}^*(H)$, the following maps

(1) $(\mathbf{h},\mathbf{h_1})\mapsto\mathbf{h}+\mathbf{h_1}$ from $H\times H$ to $\mathcal{P}^*(H)$;

(2) $\mathbf{h}\mapsto-\mathbf{r}$ from H to H;

(3) $(\mathbf{h},\mathbf{h_1})\mapsto\mathbf{h}\cdot\mathbf{h_1}$ from $H\times H$ to H;

are continuous.

Lemma 2.3. *Let $\mathcal{H}$ be a topological Krasner hyperring. Then, the hyperoperation $+:\mathcal{H}\times\mathcal{H}\longrightarrow\mathcal{P}^*(\mathcal{H})$ is continuous if and only if for all $\mathbf{r},\mathbf{s}\in\mathcal{H}$ and $S\in\tau$ so that $\mathbf{r}+\mathbf{s}\subseteq S$ then there are $\mathcal{T},\mathcal{Q}\in\tau$ so that $\mathbf{r}\in\mathcal{T}$ and $\mathbf{s}\in\mathcal{Q}$ and $\mathcal{T}+\mathcal{Q}\subseteq S$ [9].*

Example 1. Suppose $(\chi,+,\cdot)$ is a topological ring. where $\mathbf{r}\oplus\mathbf{s}=\{\mathbf{r},\mathbf{s}\}$ then, $(\chi,\oplus,\cdot)$ is a topological Krasner hyperring.

EXAMPLE 2. Consider the hyperring $(R, +, \cdot)$, where $R = \{q, s\}$, the hyperoperation " $+$ " and the binary operation " $\cdot$ " defined as follows :

$+$	q	s
q	$\{q\}$	$\{s\}$
s	$\{s\}$	$\{q, s\}$

$.$	q	s
q	q	q
s	q	s

Then, $(R, +, \cdot, \tau)$ is a topological Krasner hyperring, where $\tau = \{\varnothing, \{\text{ q }\}, R\}$.

EXAMPLE 3. Consider R as a set of real numbers. Consider the Krasner hyperring $(R, +, \cdot)$, where $\mathbf{r} + \mathbf{s} = R$ and $\mathbf{r} \cdot \mathbf{s}$ is common product. $(R, +, \cdot)$ is a topological Krasner hyperring by desired topology on R.

EXAMPLE 4. Consider $R = \{o, q, s\}$, where the hyperoperation " $+$ " and the binary operation " $\cdot$ " defined as follows:

$+$	o	q	s
o	$\{o\}$	$\{q\}$	$\{s\}$
q	$\{q\}$	$\{q\}$	R
s	$\{s\}$	R	$\{s\}$

$.$	o	q	s
o	o	o	o
q	o	q	s
s	o	s	s

For a Krasner hyperring $(R, +, \cdot)$, let R be topological with

$$\mathcal{T} = \{\varnothing, R, \{o\}, \{o, q\}, \{o, s\}\}.$$

Then $(R, +, \cdot, \mathcal{T})$ is a topological Krasner hyperring.

REMARK 1. In the above example $A = \{0, 1\}$ is a open set ($\{0, 1\} \in \tau$), but is not complete part, because

$$A \cap 1 + 2 \neq \ \emptyset \text{ but } 1 + 2 \nsubseteq A.$$

Lemma 2.4. *Let $\mathcal{H}$ be a topological Krasner hyperring. Then, the mappings*

$$\begin{aligned} {}_a\varrho &: \mathcal{H} \longrightarrow \mathcal{P}^*(\mathcal{H}) \ \textit{with} \ \mathbf{r} \mapsto a + \mathbf{r}, \\ \varrho_a &: \mathcal{H} \longrightarrow \mathcal{P}^*(\mathcal{H}) \ \textit{with} \ \mathbf{r} \mapsto \mathbf{r} + a, \end{aligned}$$

are continuous , for all $a \in \mathcal{H}$.

Lemma 2.5. *Let K is a subset of $\mathcal{H}$ that is topological Krasner hyperring and S be a member of topology that is defined on $\mathcal{H}$. Then, $K \subseteq -\mathbf{r} + S$ if and only if $\mathbf{r} + K \subseteq S$ for all $\mathbf{r} \in \mathcal{H}$.*

Proof. Consider $K \subseteq -\mathbf{r} + S$ and $t \in \mathbf{r} + a$ for some $a \in K$. Then, $a \in -\mathbf{r} + t \cap -\mathbf{r} + S$. So $a \in -\mathbf{r} + u$ for some $u \in S$. Thus, $u \in \mathbf{r} + a \cap S$ as a result $\mathbf{r} + a \subseteq S$. Thus, $\mathbf{r} + K \subseteq S$. Conversely, consider $\mathbf{r} \in \mathcal{H}$ and $\mathbf{r} + K$ is subset of $\mathcal{H}$. Then $(-\mathbf{r} + \mathbf{r}) + K = -\mathbf{r} + (\mathbf{r} + K) \subseteq -\mathbf{r} + S$. $\square$

Lemma 2.6. *Let S be a member of topology that is defined on a topological Krasner hyperring so that S is a complete part. For all $w \in \mathcal{H}$, $w + W$ is open subsets of $\mathcal{H}$, also is $W + w$.*

Proof. Consider W as a subset of $\mathcal{H}$ such that is open and $w \in \mathcal{H}$. Then, by Lemma 2.5, we have

$$\varrho_{-w}^{-1}(W) = \{\mathbf{r} \in \mathcal{H} \mid -w + \mathbf{r} \subseteq W\} = w + W$$

So, by Lemma 2.4, the mapping ϱ_{-w} is continuous; thus, $w + W$ is open. In a similar manner, is also $W + w$. □

Theorem 2.7. *Consider $\mathcal{H}$ as a topological Krasner hyperring. Also V and U be open subsets of $\mathcal{H}$. If V or U is a complete part, then $V + U$ is open.*

Proof. Consider V as a complete part. Based on 2.4, the set $V + b$ is open. Since the union of any collection of open sets remains open, this proves the proposition. □

Lemma 2.8. *Consider $\mathcal{H}$ as a topological Krasner hyperring so that every open subset of $\mathcal{H}$ is a complete part. Let $\mathcal{S}$ be an open basis at 0. Then, the collections $\{\mathbf{r} + S\}$ and $\{S + \mathbf{r}\}$, Where $\mathbf{r}$ varies over all elements of $\mathcal{H}$ and S ranges over all members of $\mathcal{S}$, they constitute an open basis for $\mathcal{H}$.*

Proof. Consider $\mathcal{Q}$ as an open subset of $\mathcal{H}$ and $a \in \mathcal{Q}$. Since $0 \in -a + \mathcal{Q}$, it implies that there is $S \in \mathcal{S}$ so that $0 \in S \subseteq -a + \mathcal{Q}$. Since $\mathcal{Q}$ is a complete part we conclude that $a \in a + S \subseteq \mathcal{Q}$. Thus, $\mathcal{Q}$ is a union of open subsets $a + S$. This means we have shown that $\{\mathbf{r} + S\}$ is a collection of open sets such that every open set in the space can be written as a union of sets from this collection. for $\mathcal{H}$. As the same way, the sets $\{S + \mathbf{r}\}$ is a basis for $\mathcal{H}$. □

Theorem 2.9. *Let $\mathcal{H}$ be a topological Krasner hyperring and $\mathcal{S}$ be a basis at 0. Then, the following assertions hold:*

(1) *for all S belonging to $\mathcal{S}$ and $\mathbf{r} \in S$ there is $\mathcal{T} \in \mathcal{S}$ so that $\mathbf{r} + \mathcal{T}$ is subset of S;*

(2) *for all S belonging to $\mathcal{S}$ there is $\mathcal{T} \in \mathcal{S}$ so that $\mathcal{T} + \mathcal{T}$ is subset of S;*

(3) *for all $S \in \mathcal{S}$ there is $\mathcal{T} \in \mathcal{S}$ so that $-\mathcal{T} \subseteq S$.*

In the context of topological spaces, neighborhoods refer to open sets. An open set S within a topological Krasner hyperring $\mathcal{H}$ is defined as a symmetric neighborhood if $-S = S$.

Theorem 2.10. *Each topological Krasner hyperring contains an open basis at 0, including a symmetric open basis at 0.*

Proof. Consider $\mathcal{S}$ as an open basis at 0. Then, for every $T \in \mathcal{S}$, put $A = T \cap -T$. Then, $A = -A$ and $A \subseteq S$. □

Theorem 2.11. *Let $\mathcal{H}$ be a topological Krasner hyperring so that every open subset of $\mathcal{H}$ is a complete part. Then, for all neighborhood S of 0 there is a neighborhood $\mathcal{T}$ of 0 so that $\overline{\mathcal{T}} \subseteq S$, where $\overline{\mathcal{T}}$ is the closure of $\mathcal{T}$.*

Proof. Consider $\mathcal{T}$ as a symmetric neighborhood of 0 so that $\mathcal{T}+\mathcal{T} \subseteq S$. Now, if $\mathbf{r} \in \overline{\mathcal{T}}$, then $\mathbf{r} + \mathcal{T} \cap \mathcal{T} \neq \emptyset$. So there is $v_1, v_2 \in \mathcal{T}$ so that $v_2 \in \mathbf{r} + v_1$. Thus, $\mathbf{r} \in v_2 - v_1 \subseteq \mathcal{T} + -\mathcal{T} = \mathcal{T} + \mathcal{T} \subseteq S$. □

Theorem 2.12. *Consider $\mathcal{H}$ as a topological Krasner hyperring so that every open subset of $\mathcal{H}$ is a complete part. Also S be a neighborhood of 0 and $\mathcal{C}$ be any compact subset of $\mathcal{H}$. Then, there is a neighborhood $\mathcal{T}$ of 0 so that $\mathbf{r} + \mathcal{T} - \mathbf{r} \subseteq S$ for all $\mathbf{r} \in \mathcal{C}$.*

Proof. Consider S as a neighborhood of 0 so by Theorem 2.9, there is a symmetric neighborhood I of 0 so that $I + I \subseteq S$. Applying Theorem 2.9 for I, there is a symmetric neighborhood $\mathcal{Q}$ of 0 so that $\mathcal{Q} + \mathcal{Q} \subseteq I$. So $\mathcal{Q} + \mathcal{Q} + \mathcal{Q} \subseteq I + I \subseteq S$. Since $\mathcal{C}$ is compact and $\mathcal{C} \subseteq \bigcup_{\mathbf{r}\in\mathcal{C}} \mathcal{Q} + \mathbf{r}$, it follows that there are $\mathbf{r}_1, \ldots, \mathbf{r}_n$ so that $\mathcal{C} \subseteq \bigcup_{j=1}^{n} \mathcal{Q} + \mathbf{r}_j$.

Let $\mathcal{T} = \cap_{i=1}^{n} -\mathbf{r}_i + \mathcal{Q} + \mathbf{r}_i$. We claim that $-\mathbf{r}_i + \mathcal{T} + \mathbf{r}_i \subseteq \mathcal{Q}$, for $i = 1, \ldots, n$. Since $\mathcal{Q}$ is a complete part and $\omega \in (\mathbf{r}_i + (-\mathbf{r}_i)) + \omega + (-\mathbf{r}_i + \mathbf{r}_i) \cap \mathcal{Q}$ for $i \in \{1, ..., n\}$ and $\omega \in \mathcal{Q}$, is obtained $(\mathbf{r}_i + (-\mathbf{r}_i)) + \omega + (-\mathbf{r}_i + \mathbf{r}_i) \subseteq \mathcal{Q}$. Now for The natural number k ranges from 1 to n we have

$$\begin{aligned}\mathbf{r}_k + \mathcal{T} + (-\mathbf{r}_k) &= \mathbf{r}_k + \left(\bigcap_{i=1}^{n} (-\mathbf{r}_i + \mathcal{Q} + \mathbf{r}_i)\right) + \mathbf{r}_k \\ &\subseteq \mathbf{r}_k + (-\mathbf{r}_k) + \mathcal{Q} + \mathbf{r}_k + (-\mathbf{r}_k) \subseteq \mathcal{Q}.\end{aligned}$$

Thus, for all $\mathbf{r} \in \mathcal{C}$ there is $\omega \in \mathcal{Q}$ and $1 \leq k \leq n$ so that $\mathbf{r} \in \omega + \mathbf{r}_k$. As a result we have $\mathbf{r} + \mathcal{T} + (-\mathbf{r})$ is in $(\omega + \mathbf{r}_k) + \mathcal{T} + (-\mathbf{r}_k + (-\omega))$ and this is subset of $\omega + (\mathbf{r}_k + \mathcal{T} + (-\mathbf{r}_k)) + (-\omega) \subseteq \omega + \mathcal{Q} + \omega \subseteq \mathcal{Q} + \mathcal{Q} + \mathcal{Q} \subseteq S$. □

Theorem 2.13. *Let $\mathcal{H}$ be a topological Krasner hyperring so that every open subset of $\mathcal{H}$ is a complete part, S be any neighborhood of 0 and $\mathcal{C}$ be any compact subset of $\mathcal{H}$ so that $\mathcal{C} \subseteq S$. Then there is a neighborhood $\mathcal{T}$ of 0 so that $(\mathcal{C} + \mathcal{T}) \cup (\mathcal{T} + \mathcal{C}) \subseteq S$.*

Proof. Consider $\mathcal{C}$ as a compact subset of $\mathcal{H}$ and S be a neighborhood of 0 so that $\mathcal{C} \subseteq S$. Then, for all $\mathbf{r} \in \mathcal{C}$ there is a neighborhood $\mathcal{Q}_{\mathbf{r}}$ of 0 so that $\mathbf{r} + \mathcal{Q}_{\mathbf{r}} \subseteq S$ and a neighborhood $\mathcal{T}_{\mathbf{r}}$ of 0 so that $\mathcal{T}_{\mathbf{r}} + \mathcal{T}_{\mathbf{r}} \subseteq \mathcal{Q}_{\mathbf{r}}$. Since $\mathcal{C}$ is compact and $\mathcal{C} \subseteq \bigcup_{\mathbf{r}\in\mathcal{C}} \mathbf{r} + \mathcal{T}_{\mathbf{r}}$, so there are $\mathbf{r}_1, \ldots, \mathbf{r}_n \in \mathcal{C}$ so that $\mathcal{C} \subseteq \bigcup_{j=1}^{n} \mathbf{r}_j + \mathcal{T}_{\mathbf{r}_j}$.

Let $\mathcal{T}_1 = \cap_{j=1}^{n} \mathbf{r}_j + \mathcal{T}_{\mathbf{r}_j}$. As a result, we have

$$\mathcal{C} + \mathcal{T}_1 \subseteq \left(\bigcup_{j=1}^{n} \mathbf{r}_j + \mathcal{T}_{\mathbf{r}_j} \right) + \mathcal{T}_1 \subseteq \bigcup_{j=1}^{n} \mathbf{r}_j + \mathcal{T}_{\mathbf{r}_j} + \mathcal{T}_{\mathbf{r}_j} \subseteq \bigcup_{j=1}^{n} \mathbf{r}_j + \mathcal{Q}_{\mathbf{r}_j} \subseteq S. \quad \square$$

3 Subhyperring of topological Krasner hyperring

In this paragraph, we present the notion of a subhyperring within a topological Krasner hyperring. We examine the relative topology on a subhyperring. A nonempty subset $\mathcal{J}$ of the hyperring H is considered a subhyperring of H if the structure $(\mathcal{J}, +, \cdot)$ forms a hyperring itself. The subset $\mathcal{J}$ qualifies as a hyperideal of H if, for every $h \in H$ and $k \in \mathcal{J}$, the products $h \cdot k$ and $k \cdot h$ are both in $\mathcal{J}$. Furthermore, $\mathcal{J}$ is termed a normal hyperideal in H if and only if, for all $h \in \mathcal{H}$, the set $h + \mathcal{J} - h$ is contained in $\mathcal{J}$.

Lemma 3.1. *Let M and M_1 and M_2 be subsets of a topological Krasner hyperring $\mathcal{H}$. Also every open subset of $\mathcal{H}$ is a complete part. Then, the following assertions hold:*

(1) $\overline{M_1} + \overline{M_2} \subseteq \overline{M_1 + M_2}$;

(2) $-\overline{M} = \overline{-M}$.

Proof. (1) Consider $t \in \overline{M}_1 + \overline{M}_2$. Then $t \in \mathbf{r} + \mathbf{s}$ for some $\mathbf{r} \in \overline{M}_1$ and $\mathbf{s} \in \overline{M}_2$. We prove that each neighborhood S of t has a non-empty intersection with $M_1 + M_2$. Since S is a complete part, it follows that $\mathbf{r} + \mathbf{s} \subseteq S$. Thus, there is neighborhoods $\mathcal{T}$ that $\mathbf{r} \in \mathcal{T}$ and $\mathcal{Q}$ that $\mathbf{s} \in \mathcal{Q}$ so that $\mathcal{T} + \mathcal{Q} \subseteq S$. From $\mathbf{r} \in \mathcal{T} \cap \overline{M}_1$ and $\mathbf{s} \in \mathcal{Q} \cap \overline{M}_2$ we conclude that there is $a \in \mathcal{T} \cap M_1$ and $d \in \mathcal{Q} \cap B$. Now, we have $a + d \subseteq S \cap M_1 + M_2$. Thus, $t \in \overline{M_1 + M_2}$.

(2) Consider $\mathbf{r} \in -\overline{M}$. Then , $-\mathbf{r} \in \overline{M}$. If $\mathbf{r} \in S \in \tau$, then $-\mathbf{r} \in -S$ so there is $\mathbf{s} \in M \cap -S$ thus $-\mathbf{s} \in -M \cap S$. As a result, $\mathbf{r} \in \overline{-M}$. Thus, $-\overline{M} \subseteq \overline{-M}$. As the same way, we can prove that $\overline{-M} \subseteq -\overline{M}$. Thus, $-(\overline{M}) = \overline{(-M)}$. $\square$

Theorem 3.2. *Let $\mathcal{H}$ be a topological Krasner hyperring so that every open subset of $\mathcal{H}$ is a complete part. Then, the following assertions hold:*

(1) *If $\mathcal{J}$ is a subsemihypergroup of $\mathcal{H}$,then $\overline{\mathcal{J}}$ is a subsemihypergroup of $\mathcal{H}$;*

(2) *If $\mathcal{J}$ is a subhyperring of $\mathcal{H}$,then $\overline{\mathcal{J}}$ is a subhyperring of $\mathcal{H}$.*

Proof. (1) Consider $\mathcal{J}$ as a subsemihypergroup of $\mathcal{H}$. Since $\mathcal{J}$ is a subsemihypergroup, it is closed under the semihypergroup operation "$+$"; then $\mathcal{J}$ plus $\mathcal{J}$ is subset of $\mathcal{J}$. Denote $\overline{\mathcal{J}}$ as the closure of $\mathcal{J}$ in $\mathcal{H}$, we have $\overline{\mathcal{J}}$ plus (hyperoperation) $\overline{\mathcal{J}}$ is subset of $\overline{\mathcal{J} + \mathcal{J}}$ and this set is subset of $\overline{\mathcal{J}}$; thus, The first part of the theorem is obtained.

(2) Consider $\mathcal{J}$ as a subhyperring of $\mathcal{H}$; According to the definition of a subhyperring for any $\underline{a, z} \in \mathcal{J}$; $\underline{a - z} \subseteq \mathcal{J}$ and $\underline{a.z} \in \mathcal{J}$, then $-b$ is member of $\mathcal{J}$ for any $b \in \mathcal{J}$. So $-\overline{\mathcal{J}} = \overline{-\mathcal{J}}$ and $\overline{-\mathcal{J}} \subseteq \overline{\mathcal{J}}$; thus, $\overline{\mathcal{J}}$ is a subhyperring of $\mathcal{H}$. □

Theorem 3.3. *Let $\mathcal{H}$ be a topological Krasner hyperring so that every open subset of $\mathcal{H}$ is a complete part. Then, every subhyperring $\mathcal{M}$ of $\mathcal{H}$ is open if and only if its interior is not empty.*

Proof. Consider $\mathcal{M}$ that its interior is non-empty, Then, there is point $\mathbf{p}$ and a open set S of 0 so that $\mathbf{p} + S \subseteq \mathcal{M}$. Now, for all $\mathbf{s} \in \mathcal{M}$ we have

$$\mathbf{s} + S \subseteq \mathbf{s} + (-\mathbf{p} + \mathbf{p}) + \mathcal{M} = (\mathbf{s} - \mathbf{p}) + (\mathbf{p} + \mathcal{M}) = (\mathbf{s} - \mathbf{p}) + \mathcal{M} = \mathcal{J}$$

Thus, $\mathbf{s}$ is a point inside $\mathcal{M}$. As a result, $\mathcal{M}$ is open. □

Theorem 3.4. *Let $\mathcal{H}$ be a topological Krasner hyperring so that every open subset of $\mathcal{H}$ is a complete part. Then, every open subhyperring is closed.*

Proof. Consider $\mathcal{J}$ as an open subhyperring of $\mathcal{H}$, then $\mathcal{H}$ equals the union of r (where r belongs to $\mathcal{H}$) plus $\mathcal{J}$, which amounts to $\mathcal{J}$ union the union of r (where r does not belong to $\mathcal{J}$) plus $\mathcal{J}$. So $\mathcal{J}^c = \bigcup_{\mathbf{r} \notin \mathcal{J}} \mathbf{r} + \mathcal{J}$. By assumption $\mathcal{J}$ is a complete part, it follows that $\mathbf{r} + \mathcal{J}$ is open. Thus, $\mathcal{J}^c$ is open and it conclude that $\mathcal{J}$ is closed. □

Theorem 3.5. *Let $\mathcal{G}$ be a family of neighborhood of* 0 *in a topological Krasner hyperring $\mathcal{H}$ so that*

(1) *for all $S \in \mathcal{G}$, there is $\mathcal{T} \in \mathcal{G}$ so that $\mathcal{T} + \mathcal{T}$ is subset of S;*

(2) *for all $S \in \mathcal{G}$, there is $\mathcal{T} \in \mathcal{G}$ so that $-\mathcal{T}$ is subset of S;*

(3) *for every $S, \mathcal{T} \in \mathcal{G}$, there is $\mathcal{Q} \in \mathcal{G}$ so that $\mathcal{Q} \subseteq S + \mathcal{T}$.*

Let $\mathcal{J} = \cap\{S \mid S \in \mathcal{G}\}$. Then, the desired result is obtained.

Proof. Consider $\mathbf{r}, \mathbf{s} \in \mathcal{J}$ and $S \in \mathcal{G}$. Then, by (1) there are $\mathcal{T} \in \mathcal{G}$ so that $\mathcal{T} + \mathcal{T}$ is subset of S. Therefor $\mathbf{r}$ and $\mathbf{s}$, members of $\mathcal{T}$ there are such that $\mathbf{r} + \mathbf{s} \subseteq \mathcal{T} + \mathcal{T} \subseteq S$. As a result, $\mathbf{r} + \mathbf{s} \subseteq \mathcal{J}$. So we can prove that if $\mathbf{r} \in \mathcal{J}$, then $-\mathbf{r} \in \mathcal{J}$. Thus, $\mathcal{J}$ is a subhyperring of $\mathcal{H}$ .Now we prove that $\mathcal{J}$ is closed. Let $\mathbf{r} \in \mathcal{H} \setminus \mathcal{J}$. Then, $\mathbf{r} \notin S$ for some $S \in \mathcal{G}$. So by (1),(2) and (3) there are $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T} \in \mathcal{G}$ so that $\mathcal{T}_1 + \mathcal{T}_1 \subseteq S, -\mathcal{T}_2 \subseteq \mathcal{T}_1$ and $\mathcal{T} \subseteq \mathcal{T}_1 \cap \mathcal{T}_2$. Thus, $\mathcal{T} + (-\mathcal{T}) \subseteq S$. As a result, if $\mathbf{r} + \mathcal{T} \cap \mathcal{T} \neq \emptyset$, then we have $\mathbf{r} \in \mathcal{T} + (-\mathcal{T}) \subseteq S$ so $\mathbf{r} \in \mathcal{J}$, and it is a cotradiction. Thus, $\mathbf{r} \in \mathcal{H} + \mathcal{T} \subseteq \mathcal{H} \setminus \mathcal{J}$. Thus, $\mathcal{H} \setminus \mathcal{J}$ is open, that is, $\mathcal{J}$ is closed. □

Theorem 3.6. *Consider S as a symmetric neighborhood of 0 in a topological Krasner hyperring $\mathcal{H}$. Also every open subset of $\mathcal{H}$ is a complete part. Then, if S is a hyperideal the set $L = \bigcup_{n=1}^{\infty} \sum_{j=1}^{n} S$ acts as a subhyperring that is both open and closed within $\mathcal{H}$ for all $n \in \mathbb{N}$.*

Proof. Consider $\mathbf{e}, \mathbf{f} \in L$, so there are $\mathbf{e} \in \sum_{j=1}^{t} S$ and $\mathbf{f} \in \sum_{j=1}^{k} S$ for some $k, t \in \mathbb{N}$. Then $-\mathbf{f} \in \sum_{j=1}^{k}(-S) = \sum_{j=1}^{k} S$ and so $\mathbf{e} - \mathbf{f} \subseteq \sum_{j=1}^{m} S$ for some $m \in \mathbb{N}$ and it is easy that $\mathbf{e}.\mathbf{f} \in L$. As a result, L is a subhyperring of $\mathcal{H}$. By considering the results of the previous theorems, the desired result is obtained. $\square$

Theorem 3.7. *Let $\mathcal{H}$ be a topological Krasner hyperring so that every open subset of $\mathcal{H}$ is a complete part. Then, a subhyperring $\mathcal{J}$ of $\mathcal{H}$ is closed if and only if there is an open subset S of $\mathcal{H}$ so that $S \cap \mathcal{J} = S \cap \overline{\mathcal{J}} \neq \emptyset$.*

Proof. If $\mathcal{J}$ is closed subhyperring of $\mathcal{H}$, then it is sufficient to consider S as a neighborhood of 0. Conversely, suppose there is an open subset S of $\mathcal{H}$ so that $S \cap \mathcal{J} = S \cap \overline{\mathcal{J}}$ and $S \cap \mathcal{J} \neq \emptyset$. Let $\mathbf{r} \in \overline{\mathcal{J}}$ and $\mathbf{s} \in S \cap \mathcal{J}$. So there is $h \in \mathcal{J} \cap \mathbf{r} - \mathbf{s} + S$. Thus, $h \in \mathbf{r} - \mathbf{s} + u$ for some $u \in S$. As a result $u \in \mathbf{s} - \mathbf{r} + h$. So $u \in S \cap \overline{\mathcal{J}}$, since by Theorem 3.3, $\overline{\mathcal{J}}$ is a subhyperring of $\mathcal{H}$, then $u \in S \cap \mathcal{J}$. As a result $\mathbf{r} \in h - u + \mathbf{s} \subseteq \mathcal{J}$. Thus, $\overline{\mathcal{J}} = \mathcal{J}$, that is, $\mathcal{J}$ is closed subhyperring of $\mathcal{H}$. $\square$

Definition 3.8. We say that a subset A in the space T, which has a topology, is dense in T if and only if for any point $\mathbf{r}$ in T , any neighborhood of $\mathbf{r}$ contain at least one point from A. Equivalently, A is dense in T if and only if the only closed subset of T including A is T itself.

Theorem 3.9. *Let $\mathcal{H}$ be a topological Krasner hyperring and every open subset of $\mathcal{H}$ is a complete part. If $\mathcal{J}$ is a non-closed subhyperring of $\mathcal{H}$, Then, for $\overline{\mathcal{J}} \cap \mathcal{J}^c$, It satisfies the conditions of Definition 3.8 in $\overline{\mathcal{J}}$.*

Proof. Let $\mathcal{J}$ be considered as a non-closed subhyperring of $\mathcal{H}$. According to Theorem 3.7, for any open subset S of $\mathcal{H}$, we have either $S \cap \mathcal{J} = \emptyset$ or $S \cap \mathcal{J} \neq \emptyset$ and $S \cap \mathcal{J} \subsetneq S \cap \overline{\mathcal{J}}$. Now, take $\mathbf{r} \in \overline{\mathcal{J}}$ and let S be a neighborhood of $\mathbf{r}$. This implies that $S \cap \mathcal{J} \neq \emptyset$. Consequently, there exists some $u \in S \cap \overline{\mathcal{J}} \setminus S \cap \mathcal{J}$. Therefore, u lies in $S \cap (\overline{\mathcal{J}} \cap \mathcal{J}^c)$. Hence, for $\overline{\mathcal{J}} \cap \mathcal{J}^c$, It meets the requirements of the previous definition in $\overline{\mathcal{J}}$. $\square$

4 Topological quotient hyperrings

In this section we study topological quotient hyperring and state isomorphism theorem for topological Krasner hyperring.

Definition 4.1. Let $\mathcal{Q}$ represent a hyperideal within a Krasner hyperring denoted as $(\mathcal{H}, +, \cdot)$. The quotient $\mathcal{H}/\mathcal{Q} = \{\mathbf{b} + \mathcal{Q} : \mathbf{b} \in \mathcal{H}\}$ is classified as a Krasner hyperring, specifically termed the quotient Krasner hyperring formed by $\mathcal{H}$ and $\mathcal{Q}$. The operations for this hyperring are defined in the following manner: for any $\mathbf{b}, \mathbf{s} \in \mathcal{H}$,

$$(\mathbf{b} + \mathcal{Q}) \oplus (\mathbf{s} + \mathcal{Q}) = \{\mathbf{t} + \mathcal{Q} : \mathbf{t} \in \mathbf{b} + \mathbf{s}\}$$

and

$$(\mathbf{b} + \mathcal{Q}) \odot (\mathbf{s} + \mathcal{Q}) = (\mathbf{b} \cdot \mathbf{s}) + \mathcal{Q}.$$

Now, consider a hyperideal $\mathcal{Q}$ associated with a topological Krasner hyperring $(\mathcal{H}, +, \cdot, \tau)$. We define a function $\phi : \mathcal{H} \longrightarrow \mathcal{H}/\mathcal{Q}$ by setting $\phi(\mathbf{r}) = \mathbf{r} + \mathcal{Q}$ for each $\mathbf{r}$ in $\mathcal{H}$. To establish a topology on $\mathcal{H}/\mathcal{Q}$, we assert that ϕ is a quotient map. This indicates that a subset $\mathcal{U}$ of $\mathcal{H}/\mathcal{Q}$ is considered open if the preimage $\phi^{-1}(\mathcal{U})$ is open in the topology that has been defined for $\mathcal{H}$ [1, 10].

Lemma 4.2. *Let $\mathcal{H}$ be a topological Krasner hyperring and $\mathcal{Q}$ be a normal hyperideal of $\mathcal{H}$. Let $\pi : \mathcal{H} \longrightarrow \mathcal{H}/\mathcal{Q}$ such that $\pi(r) = \mathcal{Q} + \mathbf{r}$. Thus $\pi^{-1}(\{\mathcal{Q}+m : m \in \mathcal{N}\}) = \mathcal{Q}+\mathcal{N}$ for every $\mathcal{N} \subseteq \mathcal{H}$ and If every open subset of $\mathcal{H}$ is a complete part, then the natural mapping π is open.*

Proof. Suppose that $y \in \mathcal{Q} + \mathcal{N}$. Then there exsist $n \in \mathcal{N}$ and $q \in \mathcal{Q}$ such that $y \in q + n$ and $\pi(y) \in \pi(q + n) = \{\pi(z) : z \in q + n\} = \{\mathcal{Q} + z : z \in q + n\}$ is subset of $\{\mathcal{Q} + n : z \in n + q\}$. Now let $y \in \pi^{-1}(\{\mathcal{Q} + n \mid n \in \mathcal{N}\})$, thus $\pi(y) = \mathcal{Q} + y \in \{\mathcal{Q} + n \mid n \in \mathcal{N}\}$. Then for some $n \in \mathcal{N}$, $\mathcal{Q} + y = \mathcal{Q} + n$ and $\mathcal{Q}$ is a normal hyperideal, thus $n - q \cap \mathcal{Q} \neq \emptyset$. Thus, there is $q \in \mathcal{Q}$ such that $q \in y - n$, thus $y \in q + n \subseteq \mathcal{Q} + \mathcal{N}$. Therefore, the proof is complete. Now in next part If S is a subset of $\mathcal{H}$ that has theorem's condition, by previous part we have $\pi^{-1}(\pi(S)) = \mathcal{Q} + S$. Since S is a complete part, it follows that $\mathcal{Q} + S$ is open in $\mathcal{H}$. Thus, the proof is finished. $\square$

Theorem 4.3. *Let $\mathcal{H}$ be a topological Krasner hyperring and every open subset of $\mathcal{H}$ is a complete part. Then $(\mathcal{H}/\mathcal{M}, \oplus, \odot)$ is a topological Krasner hyperring.*

Proof. We prove that the hyperoperation $\oplus$ and operation $\odot$ and the map $\mathbf{r} + \mathcal{M} \to -(\mathbf{r} + \mathcal{M})$ are continuous. Consider $\mathbf{r} + \mathcal{M}, \mathbf{s} + \mathcal{M} \in \mathcal{H}/\mathcal{M}$, and $\mathcal{J}$ is an subset of $\mathcal{H}/\mathcal{M}$ so that has necessary condition and $\mathbf{r} + \mathcal{M} \oplus \mathbf{s} + \mathcal{M} \subseteq \mathcal{J}$. Then, $\mathbf{r} + \mathbf{s} \subseteq \pi^{-1}(\mathcal{J})$. Since $\pi^{-1}(\mathcal{J})$ is open in $\mathcal{H}$, there is open subset $\mathcal{T}$ and $\mathcal{Q}$ of $\mathcal{H}$ including $\mathbf{r}$ and $\mathbf{s}$, respectively, so that $\mathcal{T} + \mathcal{Q} \subseteq \pi^{-1}(\mathcal{J})$. It follows that $\pi(\mathcal{T})$ and $\pi(\mathcal{Q})$ are open in $\mathcal{H}/\mathcal{M}$ including $\mathbf{r} + \mathcal{M}$ and $\mathbf{s} + \mathcal{M}$, respectively, so that $\pi(\mathcal{T}) \oplus \pi(\mathcal{Q}) \subseteq \mathcal{J}$. Thus, the hyperoperation $\oplus$ is continuous.

Consider $-(\mathbf{r} + \mathcal{M}) = -\mathbf{r} + \mathcal{M} \in \mathcal{J}$. Then, $-\mathbf{r} \in \pi^{-1}(\mathcal{J})$. Thus, there is an open subset S in $\mathcal{H}$ so that $-\mathbf{r} \in -S \subseteq \pi^{-1}(\mathcal{J})$ so $\pi(-\mathbf{r}) = -\mathbf{r} + \mathcal{M} \in \pi(-S) \subseteq \mathcal{J}$ and $\pi(-S)$ is open in $\mathcal{H}/\mathcal{M}$. $\square$

In the continue, consider $\mathcal{H}$ as a topological Krasner hyperring and $\mathcal{Q}$ be a hyperideal of $\mathcal{H}$. Then, $\mathcal{H}/\mathcal{Q}$ is a ring if and only if $\mathcal{Q}$ is a hyperideal of $\mathcal{H}$ and also is normal. Thus $(\mathcal{H}/\mathcal{Q}, \oplus, \odot)$ is a topological Krasner ring [1].

Now, we state the isomorphism theorems for topological Krasner hyperrings. the proofs of theorems are in [10].

Definition 4.4. Let $(\mathcal{H}, +, \cdot, \tau_1)$ and $(\mathcal{L}, \oplus, \bullet, \tau_2)$ be topological Krasner hyperrings. A mapping ϱ from $\mathcal{H}$ into $\mathcal{L}$ is known as a good topological homomorphism if

(1) $\varrho(0_{\mathcal{H}}) = 0_{\mathcal{L}}$;

(2) $\varrho(h_1 + h_2) = \varrho(h_1) \oplus \varrho(h_2)$ when $h_1, h_2 \in \mathcal{H}$;

(3) $\varrho(h_1 \cdot h_2) = \varrho(h_1) \bullet \varrho(h_2)$ when $h_1, h_2 \in \mathcal{H}$;

(4) ϱ is continuous;

(5) ϱ is open.

Theorem 4.5. *Let $(\mathcal{H}, +, \cdot, \tau_1)$ and $(\mathcal{L}, \oplus, \bullet, \tau_2)$ be topological Krasner hyperrings and $\boldsymbol{\Upsilon} : \mathcal{H} \longrightarrow \mathcal{L}$ be a homomorphism. If the map $\boldsymbol{\Upsilon}$ at $0_{\mathcal{H}}$ is continuous, then is continuous and opposite is correct.*

Proof. Specifically, if $\boldsymbol{\Upsilon}$ is continuous, then $\boldsymbol{\Upsilon}$ is continuous at $0_{\mathcal{H}}$. Conversely, if $\boldsymbol{\Upsilon}$ is continuous at $0_{\mathcal{H}}$ and $\boldsymbol{\Upsilon}(\mathbf{r}) \in S_2$ for some $\mathbf{r} \in \mathcal{H}$ and open subset S_2 of $\mathcal{L}$. Now, we have $\boldsymbol{\Upsilon}(0) \in \boldsymbol{\Upsilon}(\mathbf{r} + (\mathbf{r}^*)) = \boldsymbol{\Upsilon}(\mathbf{r}) +' \boldsymbol{\Upsilon}(\mathbf{r}^*) \subseteq S_2 +' \boldsymbol{\Upsilon}(\mathbf{r}^*)$, (so that $0 \in \mathbf{r} + (\mathbf{r}^*)$) , so there is an open subset S_1 of $\mathcal{H}$ including $0_{\mathcal{H}}$ so that $\boldsymbol{\Upsilon}(S_1) \subseteq S_2 +' \boldsymbol{\Upsilon}(-\mathbf{r})$. As a result, by Lemma 2.5, we have $\boldsymbol{\Upsilon}(S_1 + \mathbf{r}) = \boldsymbol{\Upsilon}(S_1) +' f(\mathbf{r}) \subseteq S_2$. Thus, $\boldsymbol{\Upsilon}$ is continuous at $\mathbf{r}$. $\square$

Theorem 4.6. *Let $(\mathcal{H}, +, \cdot, \tau_1)$ and $(\mathcal{L}, \oplus, \bullet, \tau_2)$ be topological Krasner hyperrings so that every open subset of $\mathcal{H}$ is a complete part. Let ϱ be an open and continuous good topological homomorphism from $\mathcal{H}$ onto $\mathcal{L}$ so that $N = ker\,\varrho$ is a normal hyperideal of $\mathcal{H}$. Then, $\mathcal{H}/N$ and $\mathcal{L}$ are topologically isomorphic.*

Theorem 4.7. *Let $\mathcal{K}$ and N be hyperideals of topological Krasner hyperring $\mathcal{H}$ and $\mathcal{K}$ open in $\mathcal{H}$ and every subset of $\mathcal{H}$ that is open, is a complete part. Then, $\mathcal{K}/(N \cap \mathcal{K})$ and $(N + \mathcal{K})/N$ are topological isomorphic.*

Theorem 4.8. *Let $\mathcal{J}$ and N be hyperideals of topological Krasner hyperring $\mathcal{H}$ so that every subset of $\mathcal{H}$ that is open, also is a complete part and $N \subseteq \mathcal{J}$. Then, $(\mathcal{H}/N)/(\mathcal{J}/N)$ and $\mathcal{H}/\mathcal{J}$ are topologically equivalent and have topological isomorphism.*

Theorem 4.9. *Consider hyperideals N, K of topological Krasner hyperrings $\mathcal{H}$ and $\mathcal{M}$, respectively. $N \times K$ is a normal hyperideal of $\mathcal{H} \times \mathcal{M}$ and $(\mathcal{H} \times \mathcal{M})/(N \times K)$ and $\mathcal{H}/N \times \mathcal{M}/K$ are topologically equivalent.*

Proof. It is straightforward. $\square$

5 The α^*- relation on topological Krasner hyperrings

In [11, 12], Vougiouklis introduced the fundamental relation γ^* on a hyperring R as the smallest equivalence relation on R such that the quotient R/γ^* is a fundamental ring. Then, in [2], Davvaz and Vougiouklis defined the relation α^* as the smallest equivalence relation on R such that the quotient R/α^* is a commutative ring, also see [8]. In [7], Mirvakili and Davvaz applied this relation to Krasner hyperrings.

This relation is one of the important and interesting relations that appears in hyperrings, and through it is possible to create a commutative ring, which we need, meaning the definition of the α relation that is mentioned in the references [1].

Let α^* be defined as the transitive closure of the relation α. This means that α^* encompasses not only the original pairs in α but also all pairs that can be reached through a finite sequence of applications of α. As a result, this relation naturally establishes a strongly regular relation on both $(\mathcal{H}, +)$ and $(\mathcal{H}, \cdot)$. A strongly regular relation has the property that the equivalence classes partition the set in such a way that the structure of the operation remains well-defined across these classes. Furthermore, the quotient $\mathcal{H}/\alpha^*$ forms a commutative ring, meaning that it supports both addition and multiplication operations that are commutative and associative, and that there exists an additive identity and a multiplicative identity.

Notably, α^* is the smallest equivalence relation that guarantees $\mathcal{H}/\alpha^*$ is a commutative ring. This property is significant because it allows for the construction of the quotient structure while maintaining the necessary algebraic properties that define a ring. The equivalence relation α^* allows us to identify elements of $\mathcal{H}$ that are related in a way that preserves the operations defined on the set.

In the context of a semihypergroup $(H, \circ)$ with a strongly regular relation R defined on H, each equivalence class associated with an element $\mathbf{t}$ in H constitutes a complete subset of H. This means that for every element within a given equivalence class, the operation $\circ$ produces results that remain within the same class, highlighting the internal consistency and closure properties of the operation relative to the equivalence relation.

Now, consider $(\mathcal{H}, +, \cdot, \tau)$ as a topological Krasner hyperring, which integrates both algebraic and topological structures. Here, α^* serves as the fundamental relation that organizes the elements of $\mathcal{H}$ based on the equivalences defined by α. The structure $(\mathcal{H}/\alpha^*, \overline{\tau})$ then forms a topological space, where $\overline{\tau}$ denotes the quotient topology induced by the natural mapping $\pi : \mathcal{H} \longrightarrow \mathcal{H}/\alpha^*$.

In this context, the mapping π plays a crucial role by associating each element in $\mathcal{H}$ with its corresponding equivalence class in $\mathcal{H}/\alpha^*$. Specifically, a subset A of $\mathcal{H}/\alpha^*$ is deemed open if and only if the preimage $\pi^{-1}(A)$ is open in $\mathcal{H}$. This condition ensures that the topological properties of the quotient space $(\mathcal{H}/\alpha^*, \overline{\tau})$, thus allowing for a coherent interplay between the algebraic structure and the topological structure.

Theorem 5.1. *Let $(\mathcal{H}, +, \cdot, \tau)$ be a topological Krasner hyperring so that every open subset of $\mathcal{H}$ is a complete part. Then, $(\mathcal{H}/\alpha^*, \oplus, \odot, \overline{\tau})$ is a topological ring.*

Theorem 5.2. *Consider $(\mathcal{H}, +, \cdot, \tau)$ as a topologica Krasnerl hyperring and $W \in \tau$ such that W is a complete part. If $\mathcal{H}$ be a commutative hyperring then $W = \bigcup_{w \in W} \alpha^*(w)$.*

Proof. Obviously, W is subset of $\bigcup_{w \in W} \alpha^*(w)$. Suppose that $w \in W$ and $x \in \alpha^*(w)$. Then, $\exists n \in \mathbb{N}$ and there exists $(\nu_1, \ldots, \nu_n) \in \mathbb{N}^n, \exists \delta \in \mathbb{S}_n$ and $\exists (x_{i1}, \ldots, x_{i\nu_i}) \in \mathcal{H}^{\nu_i}, \exists \delta_i \in \mathbb{S}_{\nu_i}, (i \in \{1, \ldots, n\}$ such that

$$x \in \sum_{i=1}^{n} \left(\prod_{j=1}^{\nu_i} x_{ij} \right) \text{ and } w \in \sum_{i=1}^{n} A_{\delta(i)},$$

where A be defined as the product of $x_{i\delta_i(j)}$ for j ranging from 1 to ν_i. Since W is complete part, it follows that $x \in \sum_{i=1}^{n} \left(\prod_{j=1}^{\nu_i} x_{ij} \right) \subseteq W$ and so $\alpha^*(w) \subseteq W$. Therefor, $W = \bigcup_{w \in W} \alpha^*(w)$. $\square$

Lemma 5.3. *Let $(\mathcal{H}, +, \cdot)$ be a Krasner hyperring. The set $\mathcal{B} = \{\alpha^*(\mathbf{r}) \mid \mathbf{r} \in \mathcal{H}\}$ is a base for a topology on $\mathcal{H}$ and this topology display by τ_α. Also every open subset of $\mathcal{H}$ is a complete part.*

Proof. Since by the definition $\mathcal{H} = \bigcup_{\mathbf{r} \in \mathcal{H}} \alpha^*(\mathbf{r})$, it follows that $\mathcal{B}$ is a base for a topology on $\mathcal{H}$. It is easy to see that every open subset of $\mathcal{H}$ is a complete part. $\square$

Theorem 5.4. *Let $(\mathcal{H}, +, \cdot)$ be a commutative Krasner hyperring and α^* be the fundamental relation on $\mathcal{H}$. Then, τ_α is the a topology on $\mathcal{H}$ so that $\mathcal{H}$ becomes a topological Krasner hyperring.*

Proof. We analyze Definition 2.2. Consider $\mathfrak{a}, \mathfrak{d} \in \mathcal{H}$ so that $\mathfrak{a} + \mathfrak{d} \subseteq S$ for some open subset S of $\mathcal{H}$. So by Theorem 5.2, $S = \bigcup_{r \in S} \alpha^*(r)$. Thus, there is $r \in S$ so that $\mathfrak{a} + \mathfrak{d} \subseteq \alpha^*(r)$. As a result, $\alpha^*(\mathfrak{a}) \oplus \alpha^*(\mathfrak{d})$ is a subset of $\alpha^*(r) \subseteq S$ and $\alpha^*(\mathfrak{a})$ and $\alpha^*(\mathfrak{d})$ are open subsets of $\mathcal{H}$ including $\mathfrak{a}$ and $\mathfrak{d}$. Thus, the hyperoperation " $+$ " is continuous.

Consider $\mathfrak{a}, \mathfrak{d} \in \mathcal{H}$ so that $\mathfrak{a} \cdot \mathfrak{d} \subseteq S$ for some open subset S of $\mathcal{H}$. $S = \bigcup_{r \in S} \alpha^*(r)$. Thus, there is $r \in S$ so that $\mathfrak{a} \cdot \mathfrak{d} \subseteq \alpha^*(r)$. So $\alpha^*(\mathfrak{a}) \cdot \alpha^*(\mathfrak{d})$ is subset of $\alpha^*(r) \subseteq S$ and $\alpha^*(\mathfrak{a})$ and $\alpha^*(\mathfrak{d})$ are open subsets of $\mathcal{H}$ including $\mathfrak{a}$ and $\mathfrak{d}$. Thus, the operation " $\cdot$ " is continuous. Now, Consider τ as a topology on $\mathcal{H}$ so that every open subset of $(\mathcal{H}, \tau)$ is a complete part and $(\mathcal{H}, +, ., \tau)$ is a topological Krasner hyperring. Let $\mathfrak{a} \in S$ and $S \in \tau$. Then, by Theorem **??**, we have $S = \bigcup_{r \in S} \alpha^*(r)$. Thus, $\alpha^*(\mathfrak{a}) \subseteq S$ and $\alpha^*(\mathfrak{a})$ is an open subset of $(\mathcal{H}, \tau_\alpha)$. Thus, τ_α is the finest topology on $\mathcal{H}$ so that $\mathcal{H}$ becomes a topological Krasner hyperring. □

By using Theorem 5.4, let $(\mathcal{H}, +, \cdot)$ be a commutative Krasner hyperring and α^* be the fundamental relation on $\mathcal{H}$. Then, topology τ_α on $\mathcal{H}$ is defined as the collection of all possible unions of sets $\alpha^*(u)$ for subsets S of $\mathcal{H}$ and every element u of S, along with the empty set, and $(\mathcal{H}, +, \cdot, \tau_\alpha)$ is a topological Krasner hyperring.

REMARK 2. A T_0 represent topological space where for any two different points, there is at least one open set that contains one point but not the other. In simpler terms, in a T_0 space, distinct points can be differentiated topologically. That means, for any pair of distinct points, there is always an open set that includes one while excluding the other.

Theorem 5.5. *Let $(\mathcal{H}, +, \cdot, \tau_\alpha)$ be a T_0 topological Krasner hyperring. Then, $\mathcal{H}$ is a ring.*

Proof. We prove that $|\mathbf{r} + \mathbf{s}| = 1$ for all $\mathbf{r}, \mathbf{s} \in \mathcal{H}$. Assume for the contradiction that $q, v \in \mathbf{r} + \mathbf{s}$ and $q \neq v$. Then according to the assumption, there is an open subset S of $\mathcal{H}$ including exactly one of v or q. Let $q \in S$ and $v \notin S$. Then, $q \in \alpha^*(z)$ for some $z \in S$. Thus, $v \in \alpha^*(v) = \alpha^*(q) = \alpha^*(z)$. As a result, $v \in S$, and it is a contradiction. So, we have $|\mathbf{r} + \mathbf{s}| = 1$. Thus, $\mathcal{H}$ is a ring. □

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A. ZARE,
Department of Mathematical Sciences,
Yazd University,
Yazd, Iran.
Email: azam59zare@gmail.com

B. DAVVAZ,
Department of Mathematical Sciences,
Yazd University,
Yazd, Iran.
Email: davvaz@yazd.ac.ir