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About instability in an elastic Cosserat body with pores

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Abstract

This article is centered on the study of the isotropic, porous Cosserat elastic media, realized by means of a parallel with the media of the same type, but anisotropic, by following the rewriting in a new form of the equations that govern this theory, respectively the equations of motion and the equation of equilibrated forces balance, dependent only on the displacements, respectively on the volume fraction, correlated to the pores, useful for future practical implementations.

1 Introduction

The deformation, in the continuous micropolar theory, is characterized both by the instrumentality of the displacement vector, as well as by the instrumentality of an independent vector of rotation, which particularizes the direction of the three vectors associated with each material point, a particle that can confront a microrotation unaccompanied by its subjection to a macrodisplacement.

The transmission of a force and a couple-vector by an infinitesimal element of surface leads to the appearance of a non-symmetric stress tensor, correlated with a non-symmetric strain tensor, and a couple-vector, correlated with a non-symmetric tensor of curvature, expressed as the gradient of the vector of rotation.

The elasticity theory in its classical form is inadequate to represent accurately the comportament of media equipped with an internal structure, see [4, 20, 21, 22, 24], precisely because it does not consider this particularity of materials composition, a fact that is included in the analysis of these media

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by the theory of micropolar elasticity, which gives it the appropriate character for their study.

The media with microstructure theories are based on the polar theories, where the material points are endowed with director vectors.

The identification of the polar nature of crystalline materials is attributed to Voigt, who explores in [34] their properties, elaborating the equilibrium equations for such crystals. Later, through a variational principle, the Cosserat brothers extended an elasticity theory, see [8]. The Cosserat pattern represents a reduced variant of the theory of micropolar media, more precisely, the coupled stress theory, which is based on the interdependence between the displacement and rotation vectors. The use of rigid director vectors, both by the Cosserat brothers and by the successors of that theory, led to the emergence of difficulties in the sphere of the development of symmetry rules, which belong to the characteristics of the materials and are directly correlated with the constitutive equations, see [13].

This theory was deepened by Mindlin and Tiersten in [26], by Nowacki in [27], who extends the heat conduction equation for isotropic media, obtains a fundamental system of equations characterizing coupled thermoelasticity and deduces the reciprocity theorem, by Eringen in [12], which renames Cosserat elasticity as micropolar elasticity and by Boschi and Iean in [1], which generalized the thermoelasticity linear theory introduced by Green and Lindsay to the case of continuous homogeneous micropolar media.

The theory of porous media is present in extremely varied areas of everyday life, such as geology, the drugs and medical devices industry or the fabrication process of porous materials, examples in this sense being ceramics, mineral wool or materials granulations superimposed on solid materials.

The theory foundation of the elastic media that present voids was made by Goodman and Cowin in [14], where they extend the concept of mass distribution to include granular media. The mass distribution must be correlated with the volume distribution of the granules, and for this purpose, an independent kinematic variable, namely the function of volume distribution, is considered.

The elastic materials theory, in its non-linear form, was researched and developed by Nunziato and Cowin in [28], work whose linear variant was presented by them in [9].

Porous media can be found everywhere in the environment, starting with construction materials, ceramics or foamed metal. The variety of materials that are endowed with voids have determined the research of their implementation in different senses, from the therapeutic prosthesses, which use these compositions, to engineering, where the modeling of soils sediments or renewable resources are some samples of the theory of media with voids use in everyday life.

The theory of elasticity and thermoelasticity research results of porous media various aspects are expressed by [2, 3, 5, 6, 18, 23, 32, 33].

The theories of micropolar and porous micropolar media are developed, under multiple regards, in numerous works, some examples being: [11, 30], respectively [7].

In this article, the study of the porous Cosserat media instability is pursued in the isotropic case, compared to the anisotropic one, the goal being that of reformulating the equations of motion and that of the balance of the equilibrated forces equation depending only on the displacements, respectively on the volume fraction, necessary for ulterior numerical applications.

2 Couple-stress elastodynamics

Considering an open region occupying the domain D in the three- dimensional Euclidean space $\mathbb{R}^3$, it corresponds, in the reference configuration, to an elastic, anisotropic and homogeneous Cosserat media with voids.

Utilizing a fixed orthogonal Cartesian system of axes $Ox_i, i = \overline{1,3}$, three orthogonal coordinates characterize each domain D point, with the mention that the notation x will be used for (x_1, x_2, x_3) and t for time.

In what follows, the functions will be considered as functions of (x, t), these arguments being omitted when no confusion can be made.

Throughout this article we will use the Einstein summation convention if an index is repetitive inside a monomial and the Latin indices values will be $1, 2, 3$.

A dot above the function represents the partial derivative of this function with respect to time, i.e. $\dot{f} = \dfrac{\partial f}{\partial t}$, and a comma followed by an index will represent the partial derivative with respect to the corresponding Cartesian coordinate, i.e. $f_{,i} = \dfrac{\partial f}{\partial x_i}$.

Bold symbol will be utilized to represent vectors, tensors and matrices.

Using the notations $v_l(x, t)$ and $\varphi_l(x, t)$ for the displacement vector components, respectively for the rotation vector components at the material point x and at the time t, the kinematic energy, in accordance with the adopted model, which differs from the classical form by the presence of the structural symmetric tensor $\mathcal{I}^2_{lm}$, can be written in the following form, see [15]:

$$K = \frac{\rho}{2}\dot{v}_l\dot{v}_l + \frac{\rho}{6}\mathcal{I}^2_{lm}\dot{\varphi}_l\dot{\varphi}_m. \tag{2.1}$$

Along with the displacement and the rotation vectors $\boldsymbol{v} = v_l(x, t)$, respectively $\boldsymbol{\varphi} = \varphi_l(x, t)$, the volume fraction $\boldsymbol{\nu} = \nu(x, t)$, associated with the pores, is the third independent variable that designates a porous Cosserat media.

The fundamental system which governs the elasticity linear theory of the Cosserat media with voids consists of, see [15, 18]:

– the equations of motion:

$$\begin{aligned} t_{kl,k} + F_l &= \rho \ddot{v}_l, \text{ in } D \times (0, \infty) \\ e_{lkr} t_{kr} + m_{kl,k} + L_l &= \frac{\rho \mathcal{I}_{kl}^2}{3} \ddot{\varphi}_k, \text{ in } D \times (0, \infty); \end{aligned} \tag{2.2}$$

– the balance of the equilibrated forces:

$$\sigma_{l,l} + \chi + G = \rho \mathcal{K} \ddot{\nu}, \text{ in } D \times (0, \infty) \tag{2.3}$$

–the geometric equations:

$$\begin{aligned} \varepsilon_{lk} &= v_{k,l} + e_{klr} \varphi_r, \\ \gamma_{lk} &= \varphi_{k,l}, \\ \Phi_l &= \nu_{,l}. \end{aligned} \tag{2.4}$$

The notations used in the above equations are as follows:

• t_{kl} and m_{kl} represent the components of the stress, respectively the couple- stress tensors, both asymmetric,

• ρ is the mass density in the reference configuration,

• σ_l are the equilibrated stress vector components,

• F_l and L_l are the body force and the couple body force vectors components, these being measured per volume unit,

• $\mathcal{K}$ is the coefficient of the inertia,

• χ is the intrinsic equilibrated body force,

• G is the extrinsic equilibrated body force, associated to the voids,

• the tensors $\varepsilon_{lk}, \gamma_{lk}$ and the vector Φ_l are the kinematic strain characteristics,

• e_{lkr} represent the Levi-Civita symbol.

In the linear theory of the anisotropic Cosserat media, in the absence of the initial stress and couple-stress in the case of the centrosymmetric materials, we have the form of the free energy as follows:

$$U = \frac{1}{2} A_{lksp} \varepsilon_{lk} \varepsilon_{sp} + \frac{1}{2} C_{lksp} \gamma_{lk} \gamma_{sp} + \frac{1}{2} A_{lk} \Phi_l \Phi_k + \frac{1}{2} g \nu^2. \tag{2.5}$$

The coefficients $A_{lksp}, C_{lksp}, A_{lk}$ and g, which represent the functions characteristics of the material, are prescribed functions of class $C^1(D)$ and are presumed to verify the following symmetry relations:

$$\begin{aligned} A_{lksp} &= A_{splk} = A_{klps}, \\ C_{lksp} &= C_{splk}, \\ A_{lk} &= A_{kl}. \end{aligned} \tag{2.6}$$

The constitutive equations are obtained from the relations (2.5), applying an usual procedure, so we have:

$$\begin{aligned} t_{lk} &= \frac{\partial U}{\partial \varepsilon_{lk}} = A_{lksp}\varepsilon_{sp}, \\ m_{lk} &= \frac{\partial U}{\partial \gamma_{lk}} = C_{lksp}\gamma_{sp}, \\ \chi &= -\frac{\partial U}{\partial \nu} = -g\nu, \\ \sigma_l &= \frac{\partial U}{\partial \Phi_l} = A_{lk}\Phi_k. \end{aligned} \tag{2.7}$$

Following the method described by Gourgiotis and Bigoni in [15], the further aim is to rewrite the two equations of motion in the form of a single displacements-dependent equation of motion.

The stress tensor can be written in relation to its symmetric part, respectively its antisymetric part $t_{kl} = \varsigma_{kl} + \zeta_{kl}$, where $\varsigma_{kl} = \varsigma_{lk}$ is the symmetric part, and $\zeta_{kl} = -\zeta_{lk}$ is the antisymmetric part.

Theorem 2.1. *The antisymmetric part components of the stress tensor, related to an anisotropic, homogeneous, Cosserat porous media, can be expressed in the form of the following relation*

$$\zeta_{kl} = -\frac{1}{2}e_{klr}\left(m_{pr,p} + L_r\right) + \frac{\rho e_{klr}\mathcal{I}^2_{pr}}{6}\ddot{\varphi}_p. \tag{2.8}$$

Proof. By decomposing the stress tensor according to the symmetric and antisymmetric parts, it is well known that the equalities below are fulfilled:

$$\begin{aligned} \varsigma_{kl} &= \frac{1}{2}(t_{kl} + t_{lk}), \\ \zeta_{kl} &= \frac{1}{2}(t_{kl} - t_{lk}). \end{aligned} \tag{2.9}$$

Using the previous relation $(2.9)_2$, along the second equation of motion $(2.2)_2$ and the properties of the Levi-Civita symbol, the above relation (2.8) is obtained. □

Theorem 2.2. *The symmetric part components of the stress tensor, associated with an anisotropic, homogeneous, Cosserat porous media, verify the following unitary equation of motion:*

$$\varsigma_{kl,k} - \frac{1}{2}e_{klr}m_{pr,pk} + F_l - \frac{1}{2}e_{klr}L_{r,k} + \frac{\rho e_{klr}\mathcal{I}^2_{pr}}{6}\ddot{\varphi}_{p,k} = \rho\ddot{v}_l. \tag{2.10}$$

Proof. The demonstration is immediate, by using the first equation of motion $(2.2)_1$, together with the previous relation (2.8), and the stress tensor representation in relation to its symmetric and antisymmetric parts, obtaining an unitary equation of motion, verified by the stress tensor symmetrical part components and represented by the relation (2.10). □

In the case of smooth boundary surface, referring to the traction boundary conditions at any point x, Koiter emphasizes in [19] that only five independent surface tractions can be specified because the rotation vector φ_l depends on the displacement vector v_l in the couple-stress elasticity, as can be seen in the relation presented below, regarding the connection between the rotation and the displacement vectors.

These surface tractions are represented by three reduced force – tractions and two tangential couple – tractions, see also [26],

$$\begin{aligned} P_k^{(n)} &= t_{lk}n_l - \frac{1}{2}e_{klr}n_l m_{(nn),r}, \\ R_k^{(n)} &= m_{lk}n_l - m_{(nn),k}, \end{aligned} \tag{2.11}$$

where n_l represents the unit normal to the surface and $m_{(nn)}$ indicates the couple - stress tensor m_{lk} components, $m_{(nn)} = m_{lk}n_l n_k$.

Theorem 2.3. *The equations of motion, related to a centrosymmetric, anisotropic, homogeneous, Cosserat porous media, can be represented in the form of a single equation of motion, depending on the displacements, as follows:*

$$A_{klsp}v_{p,sk} - \frac{1}{4}e_{klr}e_{tmn}C_{prst}v_{n,mspk} + F_l - \frac{1}{2}e_{klr}L_{r,k} = \rho\ddot{v}_l - \frac{\rho e_{klr}e_{pmn}\mathcal{I}_{pr}^2}{12}\ddot{v}_{n,mk}. \tag{2.12}$$

Proof. Based on the first constitutive equation $(2.7)_1$, the stress tensor expression as the sum of the symmetric and the antisymmetric parts, and the properties of these two parts, the relation $(2.9)_1$ can be rewritten in the form:

$$\varsigma_{kl} = \frac{1}{2}(A_{klsp} + A_{lksp})\varepsilon_{sp}. \tag{2.13}$$

Using the constitutive equations $(2.7)_1$ and $(2.7)_2$ into the equation (2.10) we deduce

$$\frac{1}{2}(A_{klsp} + A_{lksp})\varepsilon_{sp,k} - \frac{1}{2}e_{klr}C_{prst}\gamma_{st,pk} + F_l - \frac{1}{2}e_{klr}L_{r,k} + \frac{\rho e_{klr}\mathcal{I}_{pr}^2}{6}\ddot{\varphi}_{p,k} = \rho\ddot{v}_l. \tag{2.14}$$

Taking into account the centrosymmetric media property, that refers to the overlap between the stress tensor and its symmetrical part, see [25], from the relations (2.13) and $(2.7)_1$, we have:

$$A_{klsp}\varepsilon_{sp} = \frac{1}{2}(A_{klsp} + A_{lksp})\varepsilon_{sp},$$

therefore, the following equality is evident:

$$A_{klsp} = A_{lksp}, \tag{2.15}$$

from where, the relation (2.13) becomes:

$$\varsigma_{kl} = A_{klsp}\varepsilon_{sp}. \tag{2.16}$$

Considering that

$$\varphi_k = \frac{1}{2}e_{kmn}v_{n,m}, \tag{2.17}$$

along with the relation (2.15) and the geometric equations (2.4), the relation (2.14) leads to obtaining the unitary equation of motion (2.12), depending on the displacements . □

Theorem 2.4. *The equilibrated forces balance equation, associated with a centrosymmetric, anisotropic, homogeneous, Cosserat porous media, can be expressed in the following form, depending on the volume fraction ν, related to the pores:*

$$A_{lk}\nu_{,kl} - g\nu + G = \rho\mathcal{K}\ddot{\nu}. \tag{2.18}$$

Proof. Introducing the constitutive equations $(2.7)_3$ and $(2.7)_4$ into the balance of equilibrated forces (2.3), we get its new form (2.18). □

The previous relation can be rewritten in detail, also using the relation $(2.6)_3$, as follows:

$$\begin{aligned} &A_{11}\frac{\partial^2\nu}{\partial x_1^2} + A_{22}\frac{\partial^2\nu}{\partial x_2^2} + A_{33}\frac{\partial^2\nu}{\partial x_3^2} + 2A_{12}\frac{\partial^2\nu}{\partial x_1\partial x_2} + 2A_{13}\frac{\partial^2\nu}{\partial x_1\partial x_3} + \\ &+ 2A_{23}\frac{\partial^2\nu}{\partial x_2\partial x_3} - g\nu(x_1,x_2,x_3,t) + G = \rho\mathcal{K}\ddot{\nu}(x_1,x_2,x_3,t). \end{aligned} \tag{2.19}$$

3 Couple-stress elastodynamics in isotropic media

Considering an isotropic and homogeneous material, the constitutive coefficients will have the forms, see [29]:

$$\begin{aligned} A_{lksp} &= \lambda\delta_{lk}\delta_{sp} + (\mu+\eta)\delta_{ls}\delta_{kp} + \mu\delta_{lp}\delta_{ks}, \\ C_{lksp} &= \alpha\delta_{lk}\delta_{sp} + \beta\delta_{lp}\delta_{ks} + \gamma\delta_{ls}\delta_{kp}, \\ A_{lk} &= a\delta_{lk}, \end{aligned} \tag{3.1}$$

where λ and μ represent elastic constants, η, α, β and γ are micropolar parameters, a represents void parameter and δ_{lk} represents the Kronecker delta, about which it is well known that if the indices l, k are equal, its value is 1 and when they are inequal, it takes the value 0. In this context, the free energy (Helmholtz potential) has the form:

$$2U = \lambda\varepsilon_{ll}\varepsilon_{kk} + (\mu+\eta)\varepsilon_{lk}\varepsilon_{lk} + \mu\varepsilon_{lk}\varepsilon_{kl} + \alpha\gamma_{ll}\gamma_{kk} + \beta\gamma_{lk}\gamma_{kl} + \gamma\gamma_{lk}\gamma_{lk} + a\Phi_l\Phi_l + g\nu^2 \tag{3.2}$$

We will assume that the energy U is a positive definite quadratic form, and in the case of isotropic media, for this assumption to be fulfilled, see [17], it is necessary and suficient that the following conditions are imposed:

$$\begin{aligned} &3\lambda + 2\mu + \eta > 0, && 2\mu + \eta > 0, && \eta > 0, \\ &3\alpha + \beta + \gamma > 0, && \beta + \gamma > 0, && \gamma - \beta > 0, \end{aligned} \tag{3.3}$$

recalling the fact that λ and μ represent the Lamé coefficients for isotropic media.

The use of the previous relation (3.2) into the relations (2.7) leads to the following constitutive equations:

$$\begin{aligned} t_{lk} &= \lambda\varepsilon_{ss}\delta_{lk} + (\mu+\eta)\varepsilon_{lk} + \mu\varepsilon_{kl}, \\ m_{lk} &= \alpha\gamma_{ss}\delta_{lk} + \beta\gamma_{kl} + \gamma\gamma_{lk}, \\ \chi &= -g\nu, \\ \sigma_l &= a\Phi_l. \end{aligned} \tag{3.4}$$

These constitutive equations, together with the geometric equations (2.4), give us the components of the stress and couple-stress tensors, as well as the components of the equilibrated stress vector:

$$\begin{aligned} t_{lk} &= \lambda v_{s,s}\delta_{lk} + \mu(v_{k,l} + v_{l,k}) + \eta(v_{k,l} + e_{klr}\varphi_r), \\ m_{lk} &= \alpha\varphi_{s,s}\delta_{lk} + \beta\varphi_{l,k} + \gamma\varphi_{k,l}, \\ \sigma_l &= a\nu_{,l}. \end{aligned} \tag{3.5}$$

If we consider $\mathcal{I}^2_{lm} = \mathcal{I}\delta_{lm}$ along with the above relations (3.5), the equations of motion acquire a new form, see [17], namely:

$$\begin{aligned} (\mu+\eta)v_{l,kk} + (\lambda+\mu)v_{k,kl} + \eta e_{lkr}\varphi_{r,k} + F_l &= \rho\ddot{v}_l, \\ \eta e_{lkr}v_{r,k} - 2\eta\varphi_l + \gamma\varphi_{l,kk} + (\alpha+\beta)\varphi_{k,kl} + L_l &= \frac{\rho\mathcal{I}\delta_{kl}}{6}\ddot{\varphi}_k. \end{aligned} \tag{3.6}$$

The micropolar media with the property that the tensor $\mathcal{I}^2_{lm}$ has only one single component, see [29], is called microisotropic or spin-isotropic.

Theorem 3.1. *The stress tensor antisymmetric part components, related to an isotropic, homogeneous, Cosserat porous media can be represented in the following form:*

$$\zeta_{kl} = -\frac{1}{2}e_{klr}\left(\gamma\varphi_{r,pp} + (\alpha+\beta)\varphi_{p,pr} + L_r\right) + \frac{\rho e_{klr}\mathcal{I}\delta_{pr}}{6}\ddot{\varphi}_p. \tag{3.7}$$

Proof. Using the relations $(3.5)_2$ within the relations (2.8) leads to obtaining the new antisymmetric part representation (3.7). □

Theorem 3.2. *The stress tensor symmetric part components, associated with an isotropic, homogeneous, Cosserat porous media, verify the following unitary equation of motion:*

$$\varsigma_{kl,k} - \frac{1}{2}e_{klr}(\gamma\gamma_{pr,pk} + (\alpha+\beta)\gamma_{pp,rk}) + F_l - \frac{1}{2}e_{klr}L_{r,k} + \frac{\rho e_{klr}\mathcal{I}\delta_{pr}}{6}\ddot{\varphi}_{p,k} = \rho\ddot{v}_l. \tag{3.8}$$

Proof. The previous relations (3.7), by the instrumentality of the equations of motion, give us a single equation of motion, represented by the relation (3.8), through the symmetrical part of the stress tensor, equation which corresponds to the equation (2.10), related to an anisotropic micropolar media. □

From the first constitutive equation $(3.4)_1$, along with the relation $(2.9)_1$, we deduce the form of the stress tensor symmetric part as follows:

$$\varsigma_{kl} = \lambda\varepsilon_{ss}\delta_{kl} + \frac{1}{2}(2\mu+\eta)(\varepsilon_{kl}+\varepsilon_{lk}) = \lambda v_{s,s}\delta_{kl} + \frac{1}{2}(2\mu+\eta)(v_{l,k}+v_{k,l}). \tag{3.9}$$

Theorem 3.3. *Related to an isotropic, homogeneous, Cosserat porous media, the following unitary equation of motion is obtained:*

$$\begin{aligned}
&\lambda v_{s,sk}\delta_{kl} + \frac{1}{2}(2\mu+\eta)(v_{l,kk}+v_{k,lk}) - \frac{1}{2}e_{klr}(\gamma\varphi_{r,ppk} + (\alpha+\beta)\varphi_{p,prk}) + \\
&+ F_l - \frac{1}{2}e_{klr}L_{r,k} + \frac{\rho e_{klr}\mathcal{I}\delta_{pr}}{6}\ddot{\varphi}_{p,k} = \rho\ddot{v}_l,
\end{aligned} \tag{3.10}$$

which is equivalent to the unitary equation of motion related to this specific media, in the anisotropic case (2.14).

Proof. Taking into account the above relation (3.9), the relation (3.8) takes the new form (3.10). To verify that this relation (3.10), corresponding to the isotropic media, is equivalent to the relation (2.14), related to the anisotropic media, we calculate, by means of the relations $(3.1)_1$, and $(3.1)_2$, the expression of the symmetric part, represented by the formula (2.13), respectively of the antisymmetric part, given by the relations (2.8), and we demonstrate the equivalence of this form (2.14) with the one previously obtained, represented by (3.10).

We will present below only the calculations for determining the expression of the stress tensor symmetric part, so that:

$$\begin{aligned}\varsigma_{kl} &= \frac{1}{2}(A_{klsp} + A_{lksp})\varepsilon_{sp} = \frac{1}{2}(\lambda\delta_{kl}\delta_{sp} + (\mu+\eta)\delta_{ks}\delta_{lp} + \mu\delta_{kp}\delta_{ls} + \lambda\delta_{lk}\delta_{sp} + \\ &(\mu+\eta)\delta_{ls}\delta_{kp} + \mu\delta_{lp}\delta_{ks})\varepsilon_{sp} = (\lambda\delta_{kl}\delta_{sp} + \frac{1}{2}(2\mu+\eta)(\delta_{ks}\delta_{lp} + \delta_{kp}\delta_{ls}))\varepsilon_{sp} = \\ &= \lambda\varepsilon_{ss}\delta_{kl} + \frac{1}{2}(2\mu+\eta)(\varepsilon_{kl} + \varepsilon_{lk}) = \lambda v_{s,s}\delta_{kl} + \frac{1}{2}(2\mu+\eta)(v_{l,k} + v_{k,l}),\end{aligned} \tag{3.11}$$

which is exactly the relation (3.9), fact that proves the equivalence of this relation with the one given by (2.14). □

Taking into account that we are in the case of a centrosymmetric media, the property (2.15) occurs, which in the isotropic case, through the relation $(3.1)_1$, is transposed in the next form:

$$\lambda\delta_{kl}\delta_{sp} + (\mu+\eta)\delta_{ks}\delta_{lp} + \mu\delta_{kp}\delta_{ls} = \lambda\delta_{lk}\delta_{sp} + (\mu+\eta)\delta_{ls}\delta_{kp} + \mu\delta_{lp}\delta_{ks}, \tag{3.12}$$

the previous equality leading to the conclusion that

$$\eta = 0. \tag{3.13}$$

Based on the previous relation (3.13), the symmetric part of the stress tensor expression (3.11) becomes:

$$\varsigma_{kl} = \lambda v_{s,s}\delta_{kl} + \mu(v_{l,k} + v_{k,l}). \tag{3.14}$$

Theorem 3.4. *Related to a centrosymmetric, isotropic, homogeneous, Cosserat porous media, the equations of motion can be represented in the form of a single displacements-dependent equation of motion, as follows:*

$$\begin{aligned}&\lambda v_{s,sl} + \mu(v_{l,kk} + v_{k,lk}) - \frac{1}{4}e_{klr}e_{rmn}\gamma v_{n,mppk} - \frac{1}{4}e_{klr}e_{pmn}(\alpha + \\ &+ \beta)v_{n,mprk} + F_l - \frac{1}{2}e_{klr}L_{r,k} = \rho\ddot{v}_l - \frac{\rho e_{klr}e_{pmn}\mathcal{I}\delta_{pr}}{12}\ddot{v}_{n,mk}.\end{aligned} \tag{3.15}$$

Proof. Considering the relation (2.17), the relation (3.10) leads to determining the expression of the motion equations in the form a single equation, dependent on the displacements, represented by (3.15).

It is noted that this relation (3.15), related to the isotropic case, corresponds to the relation (2.12), from the anisotropic case. □

Theorem 3.5. *The equilibrated forces balance equation, associated with a centrosymmetric, isotropic, homogeneous, Cosserat porous media can be represented, depending on the volume function* ν*, in the following form:*

$$a\nu_{,ll} - g\nu + G = \rho\mathcal{K}\ddot{\nu}. \tag{3.16}$$

Proof. Introducing the constitutive equation $(3.4)_3$ and the relation $(3.5)_3$ into the equation of the equilibrated forces balance (2.3) we obtain the relation (3.16). □

The relation (3.15) can be written in the form:

$$a\left(\frac{\partial^2\nu}{\partial x_1^2} + \frac{\partial^2\nu}{\partial x_2^2} + \frac{\partial^2\nu}{\partial x_3^2}\right) - g\nu(x_1, x_2, x_3, t) + G = \rho\mathcal{K}\ddot{\nu}(x_1, x_2, x_3, t), \tag{3.17}$$

or, using the Laplacian, in the following equivalent form:

$$a \bigtriangledown^2 \nu - g\nu + G = \rho\mathcal{K}\ddot{\nu}. \tag{3.18}$$

4 Orthotropic couple – stress materials with voids subjected to the antiplane deformations

4.1 Governing equations

In the following, see [15], it is assumed that the body occupies a region in the plane (x_1, x_2), in the context of the antiplane strain conditions influence, the form of the displacement vector components being given by:

$$v_1 \equiv 0,\ v_2 \equiv 0,\ v_3 = w(x_1, x_2, t) \text{ and } \nu = \vartheta(x_1, x_2, t). \tag{4.1}$$

In the case of a centrosymmetric orthotropic media, overlaying the Cartesian rectangular system and the orthotropy axes, the tensors $\varepsilon_{ij}, \gamma_{ij}$ components, and the vectors $\boldsymbol{\varphi}$ and $\boldsymbol{\Phi}$ components, by using the geometric equations (2.4) along with the relation (2.17), are the following:

$$\begin{aligned}
\varepsilon_{13} &= v_{3,1} + e_{312}\varphi_2 = v_{3,1} + \frac{1}{2}e_{312}e_{213}v_{3,1} = \frac{1}{2}v_{3,1} = \frac{1}{2}\frac{\partial w}{\partial x_1},\\
\varepsilon_{23} &= v_{3,2} + e_{321}\varphi_1 = v_{3,2} + \frac{1}{2}e_{321}e_{123}v_{3,2} = \frac{1}{2}v_{3,2} = \frac{1}{2}\frac{\partial w}{\partial x_2},
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
\varphi_1 &= \frac{1}{2}e_{123}v_{3,2} = \frac{1}{2}\frac{\partial w}{\partial x_2},\\
\varphi_2 &= \frac{1}{2}e_{213}v_{3,1} = -\frac{1}{2}\frac{\partial w}{\partial x_1},
\end{aligned} \tag{4.3}$$

$$\begin{aligned}\gamma_{11}&=\varphi_{1,1}=\frac{1}{2}\frac{\partial^2 w}{\partial x_1\partial x_2},\\ \gamma_{22}&=\varphi_{2,2}=-\frac{1}{2}\frac{\partial^2 w}{\partial x_1\partial x_2}=-\gamma_{11},\\ \gamma_{12}&=\varphi_{2,1}=-\frac{1}{2}\frac{\partial^2 w}{\partial x_1^2},\\ \gamma_{21}&=\varphi_{1,2}=\frac{1}{2}\frac{\partial^2 w}{\partial x_2^2},\end{aligned}\tag{4.4}$$

$$\begin{aligned}\varPhi_1&=\nu_{,1}=\frac{\partial\vartheta}{\partial x_1},\\ \varPhi_2&=\nu_{,2}=\frac{\partial\vartheta}{\partial x_2}.\end{aligned}\tag{4.5}$$

Regarding the components A_{klsp}, in the case of an orthotropic media, there are additional coefficients that become zero, see [31], and being, at the same time, a centrosymmetric material, these lead to the following form of the constitutive equation $(2.7)_1$, see [31, 15]:

$$\begin{aligned}\varsigma_{13}&=a_{55}\frac{\partial w}{\partial x_1}=\varsigma_{31},\\ \varsigma_{23}&=a_{44}\frac{\partial w}{\partial x_2}=\varsigma_{32},\end{aligned}\tag{4.6}$$

a_{55} and a_{44} representing the shear moduli specific to an orthotropic Cauchy media, under the influence of antiplane conditions.

Also relating to an orthotropic material, the couple – stress elasticity tensor C_{lksp} is under the influence of the symmetry properties $(2.6)_2$, along with the condition

$$C_{lkss}=C_{sslk}=0,\tag{4.7}$$

equality that characterizes a centrosymmetric material, is deduced from the fact that the curvature tensor is deviatoric $\gamma_{ll}=0$, and converts after the following relation

$$C_{lksp}=Q_{lm}Q_{kn}Q_{si}Q_{pj}C_{mnij},\tag{4.8}$$

where Q_{lm} are orthogonal tensors, see [10].

Taking these into account, we can represent the constitutive equation $(2.7)_2$

in the form of a matrix system, see [16], as follows:

$$\begin{pmatrix} m_{11}\\ m_{12}\\ m_{13}\\ m_{21}\\ m_{22}\\ m_{23}\\ m_{31}\\ m_{32}\\ m_{33}\end{pmatrix}=\begin{pmatrix} C_{1111} & 0 & 0 & 0 & C_{1122} & 0 & 0 & 0 & \widetilde{C}_{1133}\\ & C_{1212} & 0 & C_{1221} & 0 & 0 & 0 & 0 & 0\\ & & C_{1313} & 0 & 0 & 0 & C_{1331} & 0 & 0\\ & & & C_{2121} & 0 & 0 & 0 & 0 & 0\\ & & & & C_{2222} & 0 & 0 & 0 & \widetilde{C}_{2233}\\ & & & & & C_{2323} & 0 & C_{2332} & 0\\ & & symm & & & & C_{3131} & 0 & 0\\ & & & & & & & C_{3232} & 0\\ & & & & & & & & \widetilde{C}_{3333}\end{pmatrix}\begin{pmatrix}\gamma_{11}\\ \gamma_{12}\\ 0\\ \gamma_{21}\\ \gamma_{22}\\ 0\\ 0\\ 0\\ 0\end{pmatrix} \tag{4.9}$$

where

$$\begin{aligned}\widetilde{C}_{1133} &= -(C_{1111}+C_{1122}) = -C_{1111},\\ \widetilde{C}_{2233} &= -(C_{1122}+C_{2222}) = -C_{2222},\\ \widetilde{C}_{3333} &= -(\widetilde{C}_{1133}+\widetilde{C}_{2233}).\end{aligned} \tag{4.10}$$

We make the assumption that the couple-stress body holds the principal torsional stiffness in the directions x_1 and x_2 and the secondary torsional stiffness is null, which means $C_{1122} = 0$, hence, the constitutive equations $(2.7)_2$ can be written in the form, see [16]:

$$\begin{aligned} m_{11} &= C_{1111}\gamma_{11} + C_{1122}\gamma_{22} = C_{1111}\gamma_{11},\\ m_{12} &= C_{1212}\gamma_{12} + C_{1221}\gamma_{21},\\ m_{21} &= C_{1221}\gamma_{12} + C_{2121}\gamma_{21},\\ m_{22} &= C_{1122}\gamma_{11} + C_{2222}\gamma_{22} = C_{2222}\gamma_{22},\\ m_{33} &= \widetilde{C}_{1133}\gamma_{11} + \widetilde{C}_{2233}\gamma_{22} = -C_{1111}\gamma_{11} - C_{2222}\gamma_{22} =\\ &\quad - C_{1111}\gamma_{11} + C_{2222}\gamma_{11} = -(C_{1111} - C_{2222})\gamma_{11}.\end{aligned} \tag{4.11}$$

Noting by

$$c_1 = C_{1111} = C_{2222}, \quad c_2 = C_{1212}, \quad c_3 = C_{1221} \text{ and } c_4 = C_{2121}, \tag{4.12}$$

the couple-stress orthotropic moduli, the previous constitutive equations (4.11) get the form

$$\begin{aligned} m_{11} &= c_1\gamma_{11} = \frac{c_1}{2}\frac{\partial^2 w}{\partial x_1 \partial x_2},\\ m_{12} &= c_2\gamma_{12} + c_3\gamma_{21} = -\frac{c_2}{2}\frac{\partial^2 w}{\partial x_1^2} + \frac{c_3}{2}\frac{\partial^2 w}{\partial x_2^2},\\ m_{21} &= c_3\gamma_{12} + c_4\gamma_{21} = -\frac{c_3}{2}\frac{\partial^2 w}{\partial x_1^2} + \frac{c_4}{2}\frac{\partial^2 w}{\partial x_2^2},\\ m_{22} &= c_1\gamma_{22} = -\frac{c_1}{2}\frac{\partial^2 w}{\partial x_1 \partial x_2} = -m_{11}. \end{aligned} \tag{4.13}$$

In the common orthotropic case, the systemic microinertia tensor $\mathcal{I}^2$ has three independent components $\{\mathcal{I}^2_{11}, \mathcal{I}^2_{22}, \mathcal{I}^2_{33}\}$, corresponding to the orthotropy axes.

The form of the stress tensor antisymmetric part, as well as the new form of the equations of motion, presented in [15], are given by the next two theorems.

Theorem 4.1. *In the case of a centrosymmetric, orthotropic, anisotropic, homogeneous, Cosserat porous media with zero body couples, the stress tensor antisymmetric part components can be written in the following new form:*

$$\begin{aligned} \zeta_{13} &= -\frac{1}{4}\left(c_2\frac{\partial^3 w}{\partial x_1^3} + (c_1 - c_3)\frac{\partial^3 w}{\partial x_1 \partial x_2^2}\right) + \frac{\rho}{12}\mathcal{I}^2_{22}\frac{\partial \ddot{w}}{\partial x_1} = -\zeta_{31},\\ \zeta_{23} &= -\frac{1}{4}\left(c_4\frac{\partial^3 w}{\partial x_2^3} + (c_1 - c_3)\frac{\partial^3 w}{\partial x_1^2 \partial x_2}\right) + \frac{\rho}{12}\mathcal{I}^2_{11}\frac{\partial \ddot{w}}{\partial x_2} = -\zeta_{32}. \end{aligned} \tag{4.14}$$

Proof. Using the relations (2.8), we obtain:

$$\begin{aligned} \zeta_{13} &= -\frac{1}{2}e_{132}m_{p2,p} + \frac{\rho e_{132}\mathcal{I}^2_{22}}{6}\ddot{\varphi}_2 = \frac{1}{2}(m_{12,1} + m_{22,2}) - \frac{\rho \mathcal{I}^2_{22}}{6}\ddot{\varphi}_2,\\ \zeta_{23} &= -\frac{1}{2}e_{231}m_{p1,p} + \frac{\rho e_{231}\mathcal{I}^2_{11}}{6}\ddot{\varphi}_1 = -\frac{1}{2}(m_{11,1} + m_{21,2}) + \frac{\rho \mathcal{I}^2_{11}}{6}\ddot{\varphi}_1, \end{aligned} \tag{4.15}$$

from where, by means of the relations (4.3) and (4.13), along with the antisymmetric stress tensor part property, the desired relations (4.14) are obtained. □

If the strain energy density is positively defined, the material moduli must verify the following conditions:

$$a_{44} > 0, a_{55} > 0; \tag{4.16}$$

$$c_1 > 0,\ c_2 > 0,\ c_4 > 0,\ c_2c_4 - c_3^2 > 0. \tag{4.17}$$

Theorem 4.2. *The equations of motion, corresponding to an orthotropic, centrosymmetric, anisotropic, homogeneous, Cosserat porous media, with zero body couples, can be rendered in the form of a single displacements-dependent equation as:*

$$\begin{aligned} &a_{55}\frac{\partial^2 w}{\partial x_1^2}+a_{44}\frac{\partial^2 w}{\partial x_2^2}-\frac{1}{4}\left(c_2\frac{\partial^4 w}{\partial x_1^4}+2c\frac{\partial^4 w}{\partial x_1^2\partial x_2^2}+c_4\frac{\partial^4 w}{\partial x_2^4}\right)+F_3= \\ &\rho\ddot{w}-\frac{\rho}{12}\left(\mathcal{I}_{22}^2\frac{\partial^2\ddot{w}}{\partial x_1^2}+\mathcal{I}_{11}^2\frac{\partial^2\ddot{w}}{\partial x_2^2}\right), \end{aligned} \tag{4.18}$$

where $c=c_1-c_3, c_2, c_4$ *represent a material specifications.*

Proof. We take into account that the form of the first equation of motion, in our specific case, becomes

$$t_{13,1}+t_{23,2}+F_3=\rho\ddot{w}. \tag{4.19}$$

Based on the relations (4.6) and (4.15), the stress tensor components can be expressed as follows:

$$\begin{aligned} t_{13}&=\varsigma_{13}+\zeta_{13}=a_{55}\frac{\partial w}{\partial x_1}-\frac{1}{4}\left(c_2\frac{\partial^3 w}{\partial x_1^3}+(c_1-c_3)\frac{\partial^3 w}{\partial x_1\partial x_2^2}\right)+\frac{\rho}{12}\mathcal{I}_{22}^2\frac{\partial\ddot{w}}{\partial x_1}, \\ t_{23}&=\varsigma_{23}+\zeta_{23}=a_{44}\frac{\partial w}{\partial x_2}-\frac{1}{4}\left(c_4\frac{\partial^3 w}{\partial x_2^3}+(c_1-c_3)\frac{\partial^3 w}{\partial x_1^2\partial x_2}\right)+\frac{\rho}{12}\mathcal{I}_{11}^2\frac{\partial\ddot{w}}{\partial x_2}, \end{aligned} \tag{4.20}$$

relations that lead to the rewriting of the motion equation in the single displacements-dependent form equation (4.18). □

For a positive definite kinetic energy density, the microinertia moduli must verify

$$\mathcal{I}_{11}^2>0,\ \mathcal{I}_{22}^2>0. \tag{4.21}$$

Theorem 4.3. *The equation of equilibrated forces balance, related to an anisotropic, orthotropic, centrosymmetric, homogeneous, Cosserat porous media, can be represented, depending on the volume function corresponding to the pores, as:*

$$A_{11}\frac{\partial^2\vartheta}{\partial x_1^2}+2A_{12}\frac{\partial^2\vartheta}{\partial x_1\partial x_2}+A_{22}\frac{\partial^2\vartheta}{\partial x_2^2}-g\vartheta(x_1,x_2,t)+G=\rho\mathcal{K}\vartheta(x_1,x_2,t). \tag{4.22}$$

Proof. Using the relations (4.5), the constitutive equation $(2.7)_4$ are presented as follows:

$$\begin{aligned} \sigma_1&=A_{11}\Phi_1+A_{12}\Phi_2=A_{11}\nu_{,1}+A_{12}\nu_{,2}=A_{11}\frac{\partial\vartheta}{\partial x_1}+A_{12}\frac{\partial\vartheta}{\partial x_2}, \\ \sigma_2&=A_{21}\Phi_1+A_{22}\Phi_2=A_{21}\nu_{,1}+A_{22}\nu_{,2}=A_{21}\frac{\partial\vartheta}{\partial x_1}+A_{22}\frac{\partial\vartheta}{\partial x_2}. \end{aligned} \tag{4.23}$$

The previous relation (4.22) is obtained immediately by using the relations (2.3) and $(2.6)_3$, the constitutive equations $(2.7)_3$ and the relations (4.23). □

4.2 The isotropic case

Studying the case of an isotropic material, by using the relations $(3.1)_1$ along with the symmetry relations $(2.6)_1$, we determine the elastic coefficients A_{klsp}, the elastic tensor $\mathbf{A}$ being defined by 21 components, as follows:

$$\begin{aligned}
&A_{1111} = A_{2222} = A_{3333} = \lambda + 2\mu + \eta,\\
&A_{1122} = A_{1133} = A_{2211} = A_{2233} = A_{3311} = A_{3322} = \lambda,\\
&A_{1212} = A_{1313} = A_{2323} = A_{2121} = A_{3131} = A_{3232} = \mu + \eta,\\
&A_{1221} = A_{1331} = A_{2332} = A_{2112} = A_{3113} = A_{3223} = \mu,
\end{aligned} \tag{4.24}$$

the rest of the coefficients $A_{1112}, A_{1113}, \ldots, A_{3323}$ being all zero.

In the case of a centrosymmetric material, the properties (2.15) and (3.13) are valid, a fact that transforms (4.24) into the following relations:

$$\begin{aligned}
&A_{1111} = A_{2222} = A_{3333} = \lambda + 2\mu,\\
&A_{1122} = A_{1133} = A_{2211} = A_{2233} = A_{3311} = A_{3322} = \lambda,\\
&A_{1212} = A_{1221} = A_{1313} = A_{1331} = A_{2323} = A_{2332} = A_{2121} = A_{2112} =\\
&\qquad = A_{3131} = A_{3113} = A_{3232} = A_{3223} = \mu.
\end{aligned} \tag{4.25}$$

The free energy form and the related conditions are presented in the previous section 3 in the form of the relations (3.2) respectively (3.3). We can express the first constitutive equation $(3.4)_1$ in the form of a matrix system shown below:

$$\begin{pmatrix} t_{11}\\ t_{12}\\ t_{13}\\ t_{21}\\ t_{22}\\ t_{23}\\ t_{31}\\ t_{32}\\ t_{33} \end{pmatrix} = \begin{pmatrix}
\lambda+2\mu & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & \lambda\\
 & \mu & 0 & \mu & 0 & 0 & 0 & 0 & 0\\
 & & \mu & 0 & 0 & 0 & \mu & 0 & 0\\
 & & & \mu & 0 & 0 & 0 & 0 & 0\\
 & & & & \lambda+2\mu & 0 & 0 & 0 & \lambda\\
 & & & & & \mu & 0 & \mu & 0\\
 & & symm & & & & \mu & 0 & 0\\
 & & & & & & & \mu & 0\\
 & & & & & & & & \lambda+2\mu
\end{pmatrix} \begin{pmatrix} 0\\ 0\\ \varepsilon_{13}\\ 0\\ 0\\ \varepsilon_{23}\\ 0\\ 0\\ 0 \end{pmatrix} \tag{4.26}$$

from where we deduce the non-vanishing components of the stress tensor:

$$\begin{aligned}
t_{13} &= t_{31} = \mu\varepsilon_{13},\\
t_{23} &= t_{32} = \mu\varepsilon_{23}.
\end{aligned} \tag{4.27}$$

Therefore, on the one hand, we have the relations (4.27) and, on the other hand, these relations are also checked by means of the common method offered by the relations (2.7), namely

$$\begin{aligned} t_{13} &= A_{13mn}\varepsilon_{mn} = A_{1313}\varepsilon_{13} + A_{1323}\varepsilon_{23} = A_{3113}\varepsilon_{13} + A_{3123}\varepsilon_{23} = \\ &= A_{31mn}\varepsilon_{mn} = t_{31} = \mu\varepsilon_{13}, \\ t_{23} &= A_{23mn}\varepsilon_{mn} = A_{2313}\varepsilon_{13} + A_{2323}\varepsilon_{23} = A_{3213}\varepsilon_{13} + A_{3223}\varepsilon_{23} = \\ &= A_{32mn}\varepsilon_{mn} = t_{32} = \mu\varepsilon_{23}, \end{aligned} \tag{4.28}$$

since $A_{1323} = A_{2313} = A_{3123} = A_{3213} = 0$.

The coefficients C_{lksp} will be determined by means of the relations $(3.1)_2$ and under the influence of the symmetry relations $(2.6)_2$ as follows:

$$\begin{aligned} &C_{1111} = C_{2222} = C_{3333} = \alpha + \beta + \gamma, \\ &C_{1122} = C_{1133} = C_{2211} = C_{2233} = C_{3311} = C_{3322} = \alpha, \\ &C_{1212} = C_{1313} = C_{2323} = C_{2121} = C_{3131} = C_{3232} = \gamma, \\ &C_{1221} = C_{1331} = C_{2332} = C_{2112} = C_{3113} = C_{3223} = \beta, \end{aligned} \tag{4.29}$$

all other coefficients $C_{1112}, C_{1113} \dots C_{3323}$ being null.

Since $C_{1122} = 0$, see the previous section 4.1, the relations (4.29) can be rewritten in the form:

$$\begin{aligned} &C_{1111} = C_{2222} = C_{3333} = \beta + \gamma, \\ &C_{1122} = C_{1133} = C_{2211} = C_{2233} = C_{3311} = C_{3322} = 0, \\ &C_{1212} = C_{1313} = C_{2323} = C_{2121} = C_{3131} = C_{3232} = \gamma, \\ &C_{1221} = C_{1331} = C_{2332} = C_{2112} = C_{3113} = C_{3223} = \beta. \end{aligned} \tag{4.30}$$

The previous relations lead to the transformation of the matrix system (4.9) into the following matrix system, corresponding to the isotropic case:

$$\begin{pmatrix} m_{11} \\ m_{12} \\ m_{13} \\ m_{21} \\ m_{22} \\ m_{23} \\ m_{31} \\ m_{32} \\ m_{33} \end{pmatrix} = \begin{pmatrix} \beta+\gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -(\beta+\gamma) \\ & \gamma & 0 & \beta & 0 & 0 & 0 & 0 & 0 \\ & & \gamma & 0 & 0 & 0 & \beta & 0 & 0 \\ & & & \gamma & 0 & 0 & 0 & 0 & 0 \\ & & & & \beta+\gamma & 0 & 0 & 0 & -(\beta+\gamma) \\ & & & & & \gamma & 0 & \beta & 0 \\ & & symm & & & & \gamma & 0 & 0 \\ & & & & & & & \gamma & 0 \\ & & & & & & & & -2(\beta+\gamma) \end{pmatrix} \begin{pmatrix} \gamma_{11} \\ \gamma_{12} \\ 0 \\ \gamma_{21} \\ \gamma_{22} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{4.31}$$

from where we conclude the components of the couple-stress tensor:

$$\begin{aligned}
m_{11} &= (\beta+\gamma)\gamma_{11} = \frac{\beta+\gamma}{2}\frac{\partial^2 w}{\partial x_1 \partial x_2},\\
m_{12} &= \gamma\gamma_{12} + \beta\gamma_{21} = -\frac{\gamma}{2}\frac{\partial^2 w}{\partial x_1^2} + \frac{\beta}{2}\frac{\partial^2 w}{\partial x_2^2},\\
m_{21} &= \beta\gamma_{12} + \gamma\gamma_{21} = -\frac{\beta}{2}\frac{\partial^2 w}{\partial x_1^2} + \frac{\gamma}{2}\frac{\partial^2 w}{\partial x_2^2},\\
m_{22} &= (\beta+\gamma)\gamma_{22} = -\frac{\beta+\gamma}{2}\frac{\partial^2 w}{\partial x_1 \partial x_2} = -m_{11},\\
m_{33} &= 0.
\end{aligned} \tag{4.32}$$

Theorem 4.4. *Considering an isotropic, orthotropic, centrosymmetric, homogeneous, Cosserat porous media, the stress tensor symmetric part components are written in the form:*

$$\begin{aligned}
\varsigma_{13} = \varsigma_{31} &= \frac{\mu}{2}\frac{\partial w}{\partial x_1},\\
\varsigma_{23} = \varsigma_{32} &= \frac{\mu}{2}\frac{\partial w}{\partial x_2}.
\end{aligned} \tag{4.33}$$

Proof. Using the relations (2.16) and (4.24), the previous relations are obtained. □

Theorem 4.5. *In the case of an isotropic, orthotropic, centrosymmetric, homogeneous, Cosserat porous media, with zero body couples, the stress tensor antisymmetric part components can be written in the form:*

$$\begin{aligned}
\zeta_{13} &= -\frac{\gamma}{4}\left(\frac{\partial^3 w}{\partial x_1^3} + \frac{\partial^3 w}{\partial x_1 \partial x_2^2}\right) + \frac{\rho\mathcal{I}}{12}\frac{\partial \ddot{w}}{\partial x_1} = -\zeta_{31},\\
\zeta_{23} &= -\frac{\gamma}{4}\left(\frac{\partial^3 w}{\partial x_2^3} + \frac{\partial^3 w}{\partial x_1^2 \partial x_2}\right) + \frac{\rho\mathcal{I}}{12}\frac{\partial \ddot{w}}{\partial x_2} = -\zeta_{32},
\end{aligned} \tag{4.34}$$

where $\mathcal{I}_{11}^2 = \mathcal{I}_{22}^2 = \mathcal{I}$.

Proof. Through the relations (4.12), (4.14) and (4.30), the relations (4.34) are easily determined. □

Theorem 4.6. *The motion equations, related to an isotropic, orthotropic, homogeneous, Cosserat porous media, with zero body couples, can be represented* *in the form of a single displacements-dependent equation of motion, as follows:*

$$\frac{\mu}{2}\left(\frac{\partial^2 w}{\partial x_1^2}+\frac{\partial^2 w}{\partial x_2^2}\right)-\frac{\gamma}{4}\left(\frac{\partial^4 w}{\partial x_1^4}+2\frac{\partial^4 w}{\partial x_1^2\partial x_2^2}+\gamma\frac{\partial^4 w}{\partial x_2^4}\right)+F_3= \rho\ddot{w}-\frac{\rho\mathcal{I}}{12}\left(\frac{\partial^2 \ddot{w}}{\partial x_1^2}+\frac{\partial^2 \ddot{w}}{\partial x_2^2}\right). \tag{4.35}$$

Proof. Based on the relations (4.33) and (4.34), we have:

$$\begin{aligned} t_{13}&=\frac{\mu}{2}\frac{\partial w}{\partial x_1}-\frac{\gamma}{4}\left(\frac{\partial^3 w}{\partial x_1^3}+\frac{\partial^3 w}{\partial x_1\partial x_2^2}\right)+\frac{\rho\mathcal{I}}{12}\frac{\partial \ddot{w}}{\partial x_1},\\ t_{23}&=\frac{\mu}{2}\frac{\partial w}{\partial x_2}-\frac{\gamma}{4}\left(\frac{\partial^3 w}{\partial x_2^3}+\frac{\partial^3 w}{\partial x_1^2\partial x_2}\right)+\frac{\rho\mathcal{I}}{12}\frac{\partial \ddot{w}}{\partial x_2}, \end{aligned} \tag{4.36}$$

these relations transforming the form of the first equation of motion into the required form (4.35). □

It is distinguished that the use of the Laplacian leads to the following form of the relation (4.35):

$$\frac{\mu}{2}\bigtriangledown^2 w-\frac{\gamma}{4}\bigtriangledown^4 w+F_3=\rho\ddot{w}-\frac{\rho\mathcal{I}}{12}\bigtriangledown^2\ddot{w}. \tag{4.37}$$

Theorem 4.7. *The equilibrated forces balance equation, related to an isotropic, homogeneous, centrosymmetric, Cosserat porous media, can be presented, depending on the volume function, as follows:*

$$a\left(\frac{\partial^2\vartheta}{\partial x_1^2}+\frac{\partial^2\vartheta}{\partial x_2^2}\right)-g\vartheta(x_1,x_2,t)-G=\rho\mathcal{K}\vartheta(x_1,x_2,t). \tag{4.38}$$

Proof. Using the relations (4.23) and $(3.1)_3$, we obtain the form of the coefficients A_{lk} as follows:

$$\begin{aligned} A_{11}&=A_{22}=a,\\ A_{12}&=A_{21}=0, \end{aligned} \tag{4.39}$$

consequently, the equilibrated stress vector components are:

$$\sigma_1=a\frac{\partial\vartheta}{\partial x_1},\ \sigma_2=a\frac{\partial\vartheta}{\partial x_2}, \tag{4.40}$$

which leads to obtaining the relation (4.38). □

It is noted that the relation (4.38) can be rewritten, by means of the Laplacian, in the form

$$a\bigtriangledown^2\vartheta-g\vartheta-G=\rho\mathcal{K}\vartheta. \tag{4.41}$$

5 Conclusions

Studying the instability of porous Cosserat media in the isotropic case, in parallel with the anisotropic one, leads to obtaining new forms of the equations that govern these types of media, writing the motion equations and the equilibrated forces balance equation in the form of an unitary displacement-dependent equation, respectively of a volume fraction-dependent equation, thereby achieving a theoretical support that will facilitate its implementation within subsequent numerical simulations.

References

[1] Boschi, E., Ieşan, D.: A generalized theory of linear micropolar thermoelasticity. Meccanica **8**(3), 154-157 (1973). https://doi.org/10.1007/BF02128724

[2] Ciarletta, M., Scalia, A.: Some Results in Linear Theory of Thermomicrostretch Elastic Solids. Meccanica **39**(3), 191-206 (2004). https://doi.org/10.1023/B:MECC.0000022843.48821.af

[3] Chandrasekharaiah, D.S.: A uniqueness theorem in the theory of elastic materials with voids. J. Elast. **18**(2),173-179 (1987). https://doi.org/10.1007/BF00127556

[4] Codarcea-Munteanu, L. F., Chirilă, A.N., Marin, M.: Modeling Fractional Order Strain in Dipolar Thermoelasticity. IFAC PapersOnLine **51**-2, 601-606 (2018). https://doi.org/10.1016/j.ifacol.2018.03.102

[5] Codarcea-Munteanu, L., Marin, M.: A study on the thermoelasticity of three-phase-lag dipolar materials with voids. Bound. Value Probl. **2019**(137), 1-24 (2019). https://doi.org/10.1186/s13661-019-1250-9

[6] Codarcea-Munteanu, L., Marin, M.: Influence of Geometric Equations in Mixed Problem of Porous Micromorphic Bodies with Microtemperature. Mathematics **8**(8) 1386, 1-16 (2020). https://doi.org/10.3390/math8081386

[7] Codarcea-Munteanu, L., Marin, M., Vlase, S.: The study of vibrations in the context of porous micropolar media thermoelasticity and the absence of energy dissipation. J. Comput. Appl. Math. (JCAMECH) **54**(3): 437-454 (2023). https://doi.org/10.22059/JCAMECH.2023.365634.881

[8] Cosserat, E., Cosserat, F.: Théorie des corps déformables. Librairie Scientifique A. Hermann et Fils, Paris (1909).

[9] Cowin, S.C., Nunziato, J.W.: Linear elastic materials with voids. J. Elast. **13** (2), 125-147 (1983). https://doi.org/10.1007/BF00041230

[10] Cowin, S.C.: Continuum Mechanics of Anisotropic Materials. Springer New York (2013). https://doi.org/10.1007/978-1-4614-5025_2

[11] Eringen, A.C.: Linear Theory of Micropolar Elasticity. J. Math. Mech. **15**(6), 909-923 (1966). https://www.jstor.org/stable/24901442

[12] Eringen, A.C.: Foundations of micropolar thermoelasticity. International Centre for Mechanical Sciences Udine, Courses and lectures **23**, Springer-Verlag, Wien GMBH (1970).

[13] Eringen, A.C.: Microcontinuum Field Theories: I. Foundations and Solids. Springer, New York (1999).

[14] Goodman, M.A., Cowin, S.C.: A continuum theory for granular materials. Arch. Ration. Mech. Anal. **44** (4), 249-266 (1972). https://doi.org/10.1007/BF00284326

[15] Gourgiotis, P.A., Bigoni, D.: The dynamics of folding instability in a constrained Cosserat medium. Phil. Trans. R. Soc. A**375**:20160159 (2017). http://doi.org/10.1098/rsta.2016.0159

[16] Gourgiotis, P.A., Bigoni, D.: Stress channelling in extreme couple-stress materials Part I: Strong ellipticity, wave propagations, ellipticity and discontinuity relations. J. Mech. Phys. Solids **88**, 150-168 (2016). https://doi.org//10.1098/j.jmps.2015.09.006

[17] Ieşan, D.: Mecanica generalizată a solidelor. Universitatea Al. I. Cuza, Centrul de multiplicare, Iaşi (1980).

[18] Ieşan, D.: A theory of thermoelastic materials with voids. Acta Mech. **60** (1-2), 67-89 (1986). https://doi.org/10.1007/BF01302942

[19] Koiter, W. T.: Couple stresses in the theory of elasticity, part I,II. Proc. K. Ned. Akad. Wet. **B**67: 17-44 (1964).

[20] Marin, M., Agarwal, R.P., Mahmoud, S.R.: Modeling a microstretch thermo-elastic body with two temperatures. Abstr. Appl. Anal. 2013, Art. ID 583464, 1-7 (2013).

[21] Marin, M., Agarwal, R.P., Codarcea, L.: A mathematical model for three-phase-lag dipolar thermoelastic bodies. J. Inequal. Appl. **2017**(109), 1-16 (2017). https://doi.org/10.1186/s13660-017-1380-5

[22] Marin, M., Chirilă, A., Codarcea, L., Vlase, S.: On vibrations in Green-Naghdi thermoelasticity of dipolar bodies. An. St. Univ. Ovidius Constanţa-Seria Matematică **27**(1), 125-140 (2019). https://doi.org/10.2478/auom-2019-0007

[23] Marin, M., Othman, M.I.A., Vlase, S., Codarcea-Munteanu, L.: Thermoelasticity of Initially Stressed Bodies with Voids: A Domain of Influence. Symmetry-Babel **11**(4) 573, 1-12, Multidisciplinary Digital Publishing Institute MDPI (2019). https://doi.org/10.3390/sym11040573

[24] Marin, M., Öchsner, A., Bhatti, M.M.: Some results in Moore-Gibson-Thompson thermoelasticity of dipolar bodies. ZAMM-Journal of Applied Mathematics and Mechanics 100(12)/e202000090 (2020). https://doi.org/10.1002/zamm.202000090

[25] Medeiros, R.A.R., Pitangueira, R.L.S., Gori, L.: Numerical evaluation of micropolar elastic parameters for a heterogeneous microstructure. XL CILAMCE Ibero-Latin-American Congress On Computational Metods In Engineering, Natal, RN-Brasil (2019)

[26] Mindlin, R.D., Tiersten, H.F.: Effects of couple-stresses in linear elasticity. Arch. Ration. Mech. Anal. **11**(1), 415-448 (1962). https://doi.org/10.1007/BF00253946

[27] Nowacki, W.: Couple-Stresses in the Theory of Thermoelasticity. Irreversible Aspects of Continuum Mechanics and Transfer of Physical Characteristics in Moving Fluids. IUTAM Symposia (International Union of Theoretical and Applied Mechanics), 259-278, Springer, Vienna (1968). https://doi.org/10.1007/978-3-7091-5581-3_17

[28] Nunziato, J.W., Cowin, S.C.: A nonlinear theory of elastic materials with voids. Arch. Ration. Mech. Anal. **72** (2), 175-201 (1979). https://doi.org/10.1007/BF00249363

[29] Pabst W.: Micropolar Materials. Ceramics-Silicáty **49**(3) 170-180 (2005). https://www.ceramics-silicaty.cz/2005/pdf/2005_03_170.pdf

[30] Passarella, F., Zampoli, V.: On the Theory of Micropolar Thermoelasticity without Energy Dissipation. J. Therm. Stresses **33** (4), 305-317 (2010). https://doi.org/10.1080/01495731003656907

[31] Sadd, H. M.: Elasticity: Theory, Applications, and Numerics. Elsevier Academic Press Butterworth-Heinemann, USA (2005).

[32] Vlase, S., Nastac,C., Marin, M., Mihalcica, M.: A method for the study of the vibration of mechanical bars systems with simmetries. Acta Tech. Napocensis, Ser. Appl. Math. Mech. Eng. 60(4), 539-544 (2017).

[33] Vlase , S., Marin, M., Scutaru, M.L., Munteanu, R.: Coupled transverse and torsional vibrations in a mechanical system with two identical beams. AIP Advances 7(6):065301 (2017). https://doi.org/10.1063/1.4985271

[34] Voigt, W.: Theoretische Studien ber die Elasticitätsverhältnisse der Krystalle. I. Abh. Gessel. Wiss. Götingen **34**, 3-52 (1887). http://eudml.org/doc/135896

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